Analysis of the 12 run Plackett-Burman design

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December 19, 1997

Abstract

Plackett-Burman designs whose run sizes are not a power of two, have lately received a lot of attention. Traditionally these designs have been recommended as main effects plans, since their complex alias pattern may create confusing results when standard methods of analysis are applied. This paper is concerned with the 12 run Plackett-Burman design. We present a new approach to the analysis of this design where both performance and implementation of methods are discussed. Under the assumption of effect sparsity we demonstrate how its very structured alias pattern leads to a new interpretation of half-normal plots, and how the good projection properties may be exploited in searching for active factors. The suggested procedures seem to overcome some of the shortcomings reported by existing methods and make the 12 run PB design a most valuable screening design also when interactions are present.

Key words: Alias structure; Design projectivity; Effect sparsity; Factor screening.

1 Introduction

In the initial stage of experimentation, when usually incomplete knowledge about the system under study is available, an efficient screening ought to identify the active factors using as few experimental runs as possible. By an active factor is understood one that individually or interactively produces a change in the response. Factors that are not active are said to be inert. The most popular designs for screening purposes are the unreplicated 2^{k-p} fractional factorial designs (Finney, 1945; Box and Hunter, 1961), and the Plackett-Burman designs (Plackett and Burman, 1946). While the fractional factorials restrict the number of runs, n, to be a power of 2, the PB designs allow n to be a multiple of 4 and thereby open up for an economical screening of many factors in few runs. For $n = 2^k$, $k = 1, 2, \dots$ these two types of designs coincide and are called *geometric* designs. PB designs with the number of runs not a power of two are called *non-geometric*. Used to their complete capacity these designs allow screening of $n-1$ factors in n runs at the expense of possibly highly biased estimated effects.

The designs are written as columns with an equal number of plus and minus signs in each. They allow simple calculations of data contrasts by taking differences of the averaged data opposite plus and minus signs for each of the $n-1$ columns. An attractive property of the geometric designs is that all possible contrast columns are orthogonal. Thus standard methods for analysing 2^{k-p} fractional factorial designs are directed to identify those contrasts that are too large to be explained by noise alone. Such methods include normal probability plots (Daniel, 1959; 1976) and Bayes plots (Box and Meyer, 1986). However, due to the fully confounding of effects there are often problems in allocating the large contrasts to the individual effects, and active factors can easily be overlooked. For the non-geometric designs all contrast columns associated with main effects are mutual orthogonal but can be correlated with contrast columns associated with interactions. Thus analysis with standard methods can be seriously distorted in the sense that active factors may wrongly appear as inert and vice versa. However, despite their compounded alias structure they also have the advantage that effects are not fully confounded, and thereby in certain circumstances and in contrast to the fractional factorials allow interactions to be estimated simultaneously.

An appropriate assumption at the early stage of an experimental investigation is factor sparsity (Box and Meyer, 1985) which states that only a few factors are responsible for most of the variation in the data. Based on this and the principle of effect heredity, which says that for an interaction to be significant, at least one of the corresponding factor main effects also need to be significant, Hamada and Wu (1992) suggested a strategy for analysing designs with complex alias structure. Their method starts with entertaining main effects and two-factor interactions orthogonal to these found by standard methods such as half-normal plots. Iterative use of a forward selection procedure is then used for including terms in the model. As is pointed out in their paper this method may have problems in situations with large two-factor interactions. Hynén (1996) also found that if the number of active factors is not small, spurious two-factor interactions may appear due to the aliasing between a two-factor interaction and other main effects and interaction effects. A different strategy is proposed in Box and Meyer (1993). Here all possible sets of active factors are investigated. Each of these subsets constitutes different models, which include main effects and interactions of the factors in the subset. A Bayesian method is developed in which the marginal posterior probability that a factor is active is computed.

In this paper we present some alternative methods for analysing designs with complex alias structure. In illustrating their performance we have striven to use examples and data analysed by others so that methods can be compared. The motivation for our investigation is three-fold. First each design has its special alias structure. If this structure could be brought up-front in some convenient manner at the early stage of an analysis, this should give useful guiding in reaching at the most plausible models. In Section 2 we suggest a way of doing this under certain assumptions. Secondly, it may be argued that the appropriate rationale for the use of designs for factor screening are their projections properties (Box, Hunter and Hunter, 1978, p. 388). Therefore a reasonable requirement will be that the analysis is consistent with the projection properties of the design. There has lately been published many results on projection properties of non-geometric designs (Box and Bisgaard, 1993; Lin and Draper, 1992; Draper and Lin, 1995; Cheng, 1995; Box and Tyssedal, 1996), but so far these results have not been incorporated into the analysis of these designs. This is the subject of Section 3 and to some extent also Section 4. Thirdly, both of the two earlier mentioned methods are reported to fail for a 12 run PB design in an example with four active factors (where both main effects and two-factor interactions are included) and reasonably low noise added (Hamada and Wu, 1992; Hynén, 1996). It is therefore a certain challenge to investigate the possibility to identify four active factors in a 12 run PB design. In Section 4 we reanalyse this example and give alternative ways of searching for four active factors. Each of the non-geometric designs have their special alias structure and projection properties. When these in particular are taken into consideration, there will necessarily be differences in the analysis of each design and a complete description for all of them will be too extensive. Therefore we have concentrated on the analysis of the 12 run PB design, but some extensions of the methods to the 20 run PB design is discussed in Section 5 together with a characteristic that one should be special aware of. Having analysed quite a lot of simulated examples with the 12 run PB design, we are completely convinced that this design is a very important screening design and well worthy an analysis in considerable depth.

2 Plotting absolute values of main effect and two-factor interaction contrasts

Even though estimated contrasts may be highly biased in non-geometric designs, use of plots may still facilitate the detection of active factors if correctly interpreted. This is due to the fact that bias caused by interaction terms may have very systematic influence on the estimated contrasts.

The general way of obtaining the alias pattern for any design was given by Box and Wilson (1951). Suppose we try to fit the regression model

$$E(\boldsymbol{y}) = \boldsymbol{X\beta} \tag{2.1}$$

by least squares, but that the true expectation is given by

$$E(\boldsymbol{y}) = \boldsymbol{X\beta} + \boldsymbol{X}_1\boldsymbol{\beta}_1. \tag{2.2}$$

Then the estimator of the regression coefficient vector is given by $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$, and it follows that

$$E(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta} + \boldsymbol{A\beta}_1 \tag{2.3}$$

where $\boldsymbol{A} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}_1$. $\boldsymbol{A}$ is called the alias matrix or bias matrix. $\boldsymbol{X}_1$ and thereby $\boldsymbol{A}$ is dependent on the true model which in general is unknown.

Assuming all interactions of order three and higher to be zero, the alias matrix for the 12 run PB design shown in Table 1 possibly consists of $\binom{11}{2} = 55$ two-factor interaction columns, and each main effect contrast may theoretically be partially aliased with the 45 two-factor interactions for which it is not involved with an amount of $\pm 1/3$, see Lin and Draper (1993). For real situations, however, under the assumption of effect sparsity, typically only a few of these two-factor interactions are active, and the alias matrix will consist of only a few columns. In such cases the influence on the estimated main effect contrasts may be very systematic.

Let us assume that the factors are labelled $A, B, C, \ldots, K$, and that $\boldsymbol{X}_1$ consists of only one two-factor interaction column, say the AB interaction column. Since this interaction column is orthogonal to the columns for factor A and factor B, the one column matrix $\boldsymbol{A}$ has zeroes in the positions corresponding to factor A and factor B, and hence the estimated main effects of these factors are not biased. The rest of the entries in $\boldsymbol{A}$ will consist of $\pm 1/3$. Thus all the main effect contrasts of the inert factors are biased with the same amount in absolute value. In a plot of the absolute values of these contrasts they should all be in a group with the same expected value. In the following we will call such a plot a main effect contrast plot.

Similar effects may be observed if more than one two-factor interaction is active. If $\boldsymbol{X}_1$ consists of the two interaction columns AB and AC, $\boldsymbol{A}$ will be a 11×2 matrix with two zeroes

Table 1: The 12 run Plackett-Burman design.

run	A	B	C	D	E	F	G	H	I	J	K
1	+	-	+	-	-	-	+	+	+	-	+
2	+	+	-	+	-	-	-	+	+	+	-
3	-	+	+	-	+	-	-	-	+	+	+
4	+	-	+	+	-	+	-	-	-	+	+
5	+	+	-	+	+	-	+	-	-	-	+
6	+	+	+	-	+	+	-	+	-	-	-
7	-	+	+	+	-	+	+	-	+	-	-
8	-	-	+	+	+	-	+	+	-	+	-
9	-	-	-	+	+	+	-	+	+	-	+
10	+	-	-	-	+	+	+	-	+	+	-
11	-	+	-	-	-	+	+	+	-	+	+
12	-	-	-	-	-	-	-	-	-	-	-

in row 1, with one zero in row 2 and row 3 and all the other entries equal to $\pm 1/3$. This means that the main effect contrast of factor A is the only unbiased one, while all the main effect contrasts of the inert factors should in absolute value group in two groups, each with the same expected value. If the two two-factor interaction columns in $\boldsymbol{X}_1$ are not involved with the same main effect, the pattern will be similar, but now all main effect contrasts are biased.

In the 12 run PB design each factor is involved with ten two-factor interactions. The ten two-factor interaction columns associated with a factor, say A, are all mutually orthogonal and orthogonal to the main effect column for factor A, and will together with this column make a new essentially equivalent PB design. The alias matrix for this design then potentially consists of the 45 additional two-factor interaction columns together with the ten other main effect columns and the amount of confounding is still either $\pm 1/3$ or 0. A plot of the absolute value of these contrasts will if only one or two of the terms in the alias matrix are active, in the same way as explained above reveal useful information. In particular if none of the additional two-factor interactions are active, they may tell us how many of the additional main effects that should be considered active. We will call such a plot a two-factor interaction contrast plot. If the alias matrix $\boldsymbol{A}$ consists of three or more columns these plots will normally be hard to interpret.

It is worth mentioning that these plots can be done with very little additional effort. Let us call the design obtained by multiplying entrywise each column in $\boldsymbol{X}$ with the column for factor A for $A\boldsymbol{X}$. Then from $(A\boldsymbol{X})'\boldsymbol{y}/6$, where $\boldsymbol{y}$ is the vector of response values, we will get an estimate of the ten two-factor interactions involved with factor A and $2\bar{y}$ since the column for factor A in $A\boldsymbol{X}$ now will contain only pluses. But a simpler operation, which gives us exactly the same result, is just to multiply the vector of response values, $\boldsymbol{y}$, entrywise by A to obtain $A\boldsymbol{y}$ and compute $\boldsymbol{X}'(A\boldsymbol{y})/6$. Further, if the main effect contrasts and two-factor interaction contrasts are all plotted in separated half-normal plots, the inert main effects and two-factor interactions will (with no random error present) all lie on parallel vertical lines. In practice, due to error, the lines will be close to parallel having a slope determined by the standard deviation of the noise.

Several authors (Box and Meyer, 1993; Hynén, 1996) have commented on that clustering

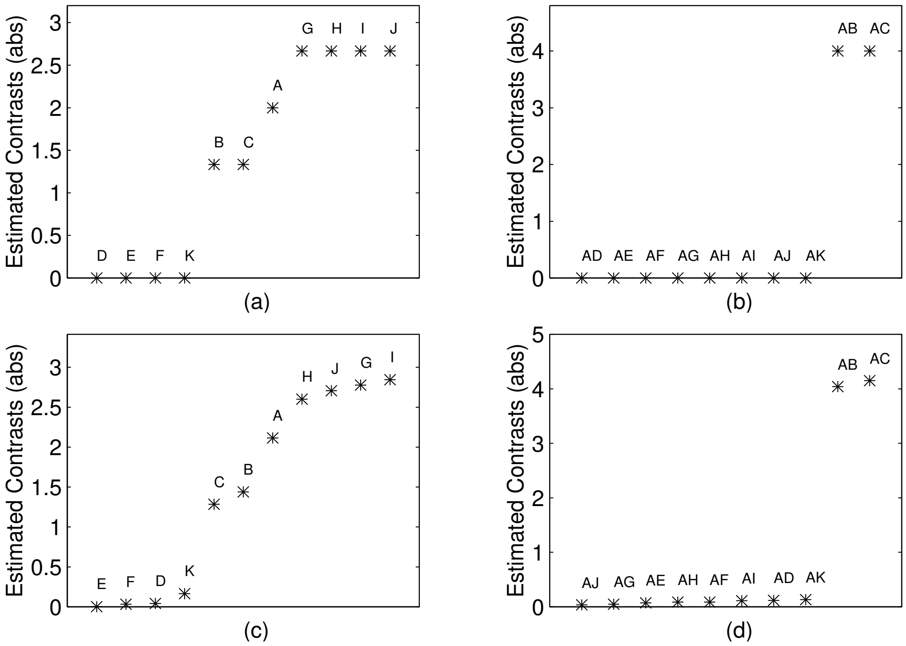

Figure 1: Main effect and two-factor interaction contrast plots for model (2.4): (a) and (c) Main effect contrast plots without and with noise in the vector of response values, (b) and (d) Contrast plots of two-factor interaction involved with factor A without and with noise in the vector of response values.

may appear in normal plots when these are used to analyse non-geometric designs and interactions are present. An example on use of what we call main effect and two-factor interaction contrast plots can be found in Hamada and Wu (1991). However, to our knowledge none has pointed out that the influence of two-factor interactions on main effect contrasts (and vice versa) has an explicit structure which can be exploited to identify active factors. The point of looking for parallel lines in normal and half-normal plots has earlier been advocated by Daniel (1976) in order to detect outliers in an experiment. The same idea may be used to discriminate active from inert factors in designs with certain alias structures. The idea is best illustrated with some examples.

Hamada and Wu (1992) report on two simulated examples where their suggested strategy has problems due to large two-factor interactions. In the first (their example 4) the true model is given by

$$y = A + 2AB + 2AC + \epsilon \tag{2.4}$$

where ϵ is normally distributed with mean 0 and standard deviation 0.25. Figure 1 shows the main effect contrast plot together with the two-factor interaction contrast plot of interactions involved with factor A.

In Figure 1a and 1b the response values are generated without noise and thereafter with the error term given above added, Figure 1c and 1d. From these plots it is observed that the main effect contrasts D, E, F and K cluster in one group with about the same absolute value, and that the main effect contrasts G, H, I and J cluster in an other group with a different absolute value. Hence these factors are most likely to be inert, and since they split into two groups the presence of two two-factor interactions is suggested. This is further confirmed by the two-factor interaction contrast plots which suggest these interactions to be AB and AC and that no other main effects or interactions need to be considered since all the other estimated two-factor interaction contrasts are zero (Figure 1b) or seemingly due to noise (Figure 1d). We know that C is confounded with AB and that B is confounded with AC each with an amount of $\pm 1/3$ in absolute value which may explain their main effect contrast values. Therefore from these plots it is clear that a very plausible and simple explanation of the data is given by factor A and its two-factor interactions with factor B and C.

In Figure 2 the half-normal plots for the same main effect contrasts and two-factor interaction contrasts are given together with half-normal plots for the two-factor interaction contrasts involved with factors B, C, D and G. The information from Figure 2a and 2b is clearly the same as in Figure 1. The half-normal plot for the two-factor interaction contrasts involved with factor B, Figure 2c, suggests two terms in addition to the AB interaction since the rest of the interaction contrasts split into two groups. For factor C it is similar. The half-normal plots for the two-factor interaction contrasts involved with factors that are inert, Figure 2e and 2f, are considerable more complex and difficult to interpret as should be expected since the alias matrix now consists of three columns.

In the other example of Hamada and Wu (their example 5) there are four active factors, and the true model is given by

$$y = 2A + 4C + 2BC - 2CD + \epsilon \tag{2.5}$$

where ϵ is normally distributed with mean 0 and standard deviation 0.5. This example has later been reanalysed in Hynén (1996) using the method by Box and Meyer reaching at about the same conclusions as Hamada and Wu that additional runs are needed to solve ambiguities. We are using the same simulated data as Hynén. Thus our vector of response values is:

$$\boldsymbol{y} = (6.266\ \text{-}3.205,\ 5.513,\ 1.936,\ \text{-}2.481,\ 10.352,\ 2.472,\ \text{-}2.835,\ \text{-}2.927,\ \text{-}1.192,\ \text{-}10.090,\ \text{-}5.975)'.$$

A half-normal plot of estimated contrasts as shown in Figure 3a puts H, F and K in one group and D, E, G, I and J in another suggesting that these factors most likely are inert and that two two-factor interactions are present. Figure 3b suggests to put D off the list of inert factors and also that the two active two-factor interactions are BC and CD. Under these assumptions a plausible explanation of Figure 3b is that in addition to factor C and the interactions above there is only one additional term in the model. A very qualified guess for this term is the main effect of factor A, since all the rest of the two-factor interactions, except AC which is close to zero, lie on one line. In Figure 3 is also shown half-normal plots of two-factor interactions contrasts involved with the three active factors A, B and D and the inert factor E. In these cases the alias matrix consists of at least three terms, and the plots are considerable more difficult to interpret. We will return to this example in Section 4.

As is illustrated in these two examples one should always look for contrast plots and half-normal plots with simple structures. These will be the ones that are easiest to interpret since simple structure in such plots means that few terms are contained in the alias matrix.

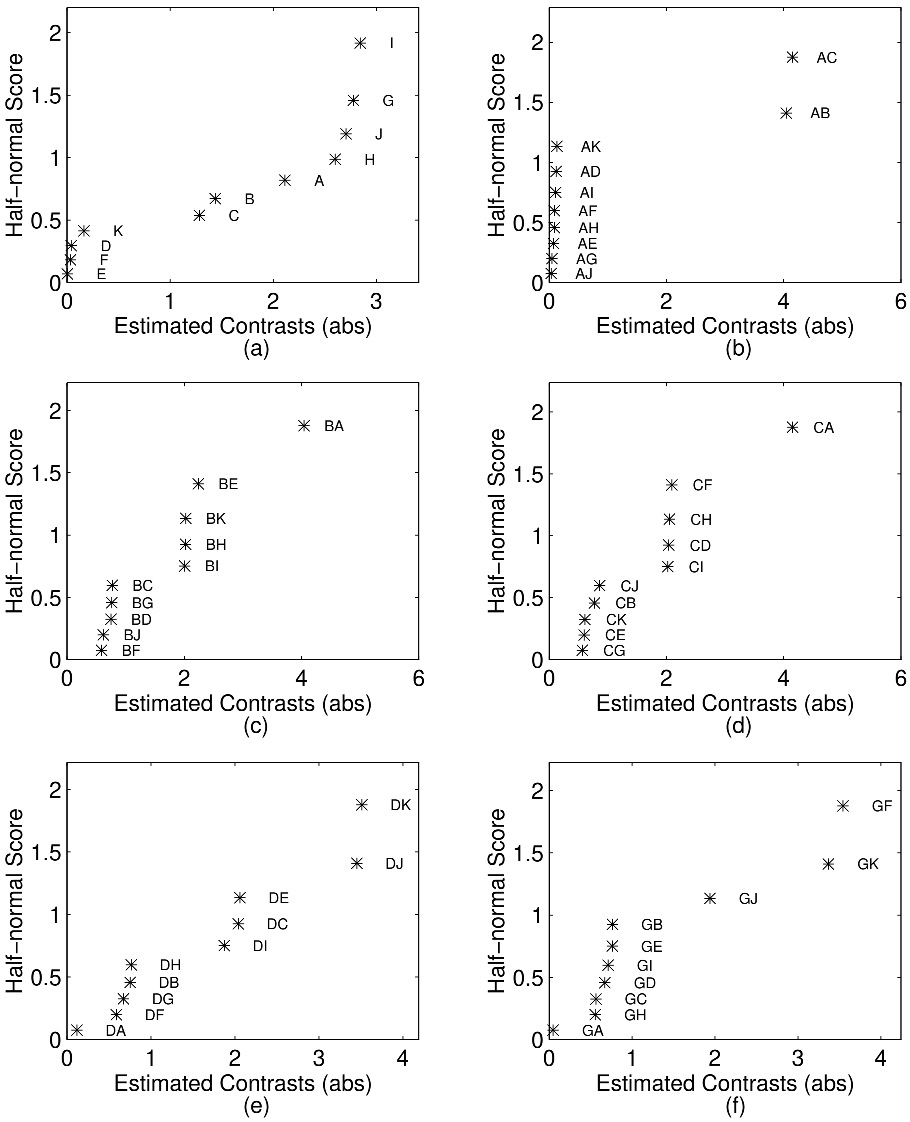

Figure 2: Half-normal plot of main effects and two-factor interactions for model (2.4): (a) Half-normal plots of main effects, (b) - (f) Half-normal plots of two-factor interactions involved with factors A, B, C, D and G.

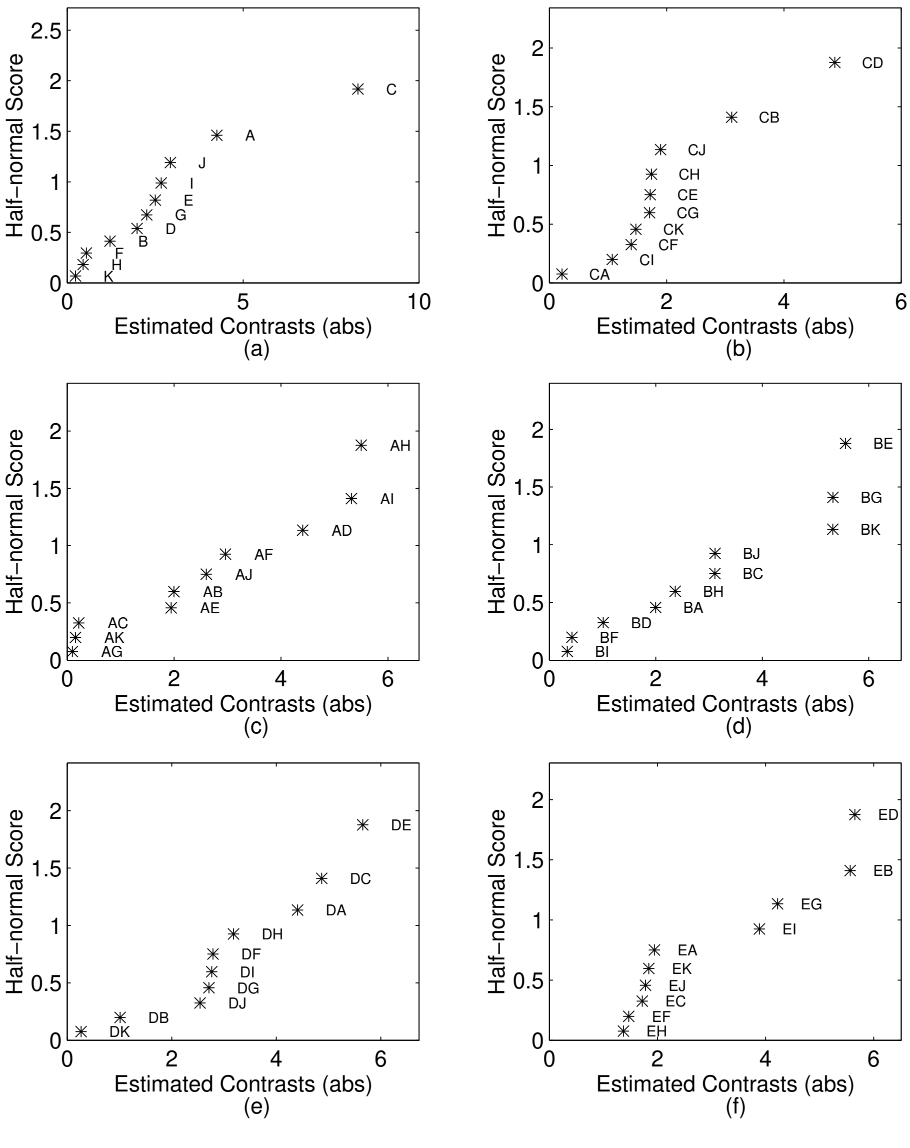

Figure 3: Half-normal plots of main effects and two-factor interactions for model (2.5): (a) Half-normal plots of main effects, (b) - (f) Half-normal plots of two-factor interactions involved with factors A, B, C, D and E.

3 Analysis based on Projection Properties

In searching for active factors in non-geometric designs one preferably should perform an all subset selections procedure on a set consisting of all main effect and interaction terms. This is a burdensome work even for the 12 run PB design. Box and Meyer (1993) use a factor based approach where in investigating a subset of factors one also include all interaction terms up to a given order among these factors in the potential model. A problem with this approach is the large number of terms in each candidate model as the number of active factors become large which may lead to over-fitting. However, if this works satisfactorily, it reduces the models to be investigated substantially, especially when many factors have to be screened. It also turns out that taking advantage of the projection properties of a design the efficiency of the procedure may be increased, and further improvement is available if the design is constructed by cyclic generation.

The projection properties of orthogonal two level designs, at least up to 3 dimensions, are now fairly well understood, see Lin and Draper (1992), Cheng (1995) and Box and Tyssedal (1996). Especially for the 12 run PB design we know that it projects into 6 replicates of a 2^1 design in 1 dimension, 3 replicates of a 2^2 design in every two dimensions and $1\frac{1}{2}$ replicate of a 2^3 design in every 3 dimensions.

Assuming just one active factor we have for each of the eleven factors six replicated points on the low level of the factor and six on the high level. From each of this two groups of points, two estimates of σ^2 with 5 degrees of freedom may be calculated and pooled to get an estimate of σ^2 with ten degrees of freedom. In this way we get altogether 11 estimates of σ^2, one for each factor, and the factor associated with the lowest $\hat{\sigma}^2$ would be judged most capable of explaining the variation in the data.

Similarly assuming two active factors we have for each of the $\binom{11}{2} = 55$ possibilities, 3 replicates for each of the four level combinations of the factors. Four estimates of σ^2 may be pooled to give a $\hat{\sigma}^2$ with 8 degrees of freedom. The two factors associated with the smallest $\hat{\sigma}^2$ would be considered to be the most capable of explaining the variation in the data.

Assuming three active factors, only four of the eight combinations are replicated, and our pooled estimate of σ^2 will for all of the possible $\binom{11}{3} = 165$ possibilities have 4 degrees of freedom. Again the three factors associated with the smallest $\hat{\sigma}^2$ would be considered to be active.

The cyclic nature of the design makes it very easy to implement an algorithm to calculate these estimates of σ^2. For instance if the design is as given in Table 1, the six pluses in column 1 is in run number 1,2,4,5,6 and 10. Increasing each of these numbers by l, the new numbers modulus 11 gives us the position of pluses in column number $l+1$ (modulus 11). Hence only the vector of response values and the position of pluses in the first column is needed in order to create the 11 estimates of σ^2, assuming only one factor is active. The same idea may be exploited in investigating more than than one active factor. In particular run s for any set of k factors will be identical to run $s+l$ (modulus 11) in a different set of k factors if this set is obtained from the first one by shifting each factor to the right an equal amount l. Run number 12 remains unchanged.

For three active factors only eight of the level combinations are needed. For the three first columns the four repeated runs are as given in Table 2. Once the four replicated runs are detected for any set of three columns, we also get the position of the replicated runs for any three columns shifted to the right with the same amount.

Table 2: Run numbers for repeated runs in the three first columns in Table 1.

(+ - +)	(+ + -)	(- + +)	(- - -)
1	2	3	9
4	5	7	12

Now the following is true

$$\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i:\text{unrepeated}} (y_i - \hat{y}_i)^2 + \sum_{i:\text{repeated}} (y_i - \hat{y}_i)^2$$

$$= \sum_{i:\text{unrepeated}} (y_i - \hat{y}_i)^2 + \sum_{i=1}^{m}\sum_{j=1}^{m_i}(y_{ij} - \hat{y}_i)^2 \tag{3.1}$$

where $\hat{y}_i$ is the fitted value for y_i, m is the number of repeated runs and m_i is the number of observations at each repeated run. From this it follows that $\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$ is minimised when $\hat{y}_i$ is chosen equal to y_i for all runs that are unrepeated and equal to $\bar{y}_i$, the average over the m_i observations at each repeated run, otherwise. Applying the procedure suggested above for the 12 run PB design is therefore equivalent to successively assuming that the design matrix in the regression model

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{3.2}$$

contains (in addition to a column of only 1's): i) only one main effect column, ii) two main effect columns and their interaction column, iii) three main effect columns, their three two-factor interaction columns and their three-factor interaction column.

Hence this corresponds to the factor based treatment of the model space used by Box and Meyer (1993) forcing all interactions into the model. Only in the two first cases will the design matrix $\boldsymbol{X}$ be orthogonal. In the last case we know that, see Box and Tyssedal (1996), any three main effect columns, say A, B or C, have defining generator $I = \pm ABC$ for eight experiments and $I = \mp ABC$ for the four others. In particular this means that the three-factor interaction column is correlated with the column of only $+1$'s and we also have that each main effect column is correlated with the two-factor interaction column for which it is not involved. It follows that each column in $\boldsymbol{X}$ is correlated with exactly one other column with an amount of either $1/3$ or $-1/3$.

Now assume that only main effects and two-factor interactions are active. If these respective columns are contained in $\boldsymbol{X}$, the estimation of main effects and interactions are unbiased and the estimated three-factor interaction term is only due to noise. If on the other side not all columns corresponding to the active factors are contained in $\boldsymbol{X}$, the estimation of the three-factor interactions may be biased and therefore have some impact on the estimate of σ^2 as will be illustrated in examples that follows. Therefore it may be considered worthwhile to remove the three-factor interaction column from the design matrix and thereby obtain an estimate of σ^2 with 5 degrees of freedom. It is straightforward to obtain that the extra sum of squares due to the three-factor interaction term is given by

$$\left(-\delta_{ABC}\frac{1}{32}\sum_i y_i + \frac{3}{32}ABC'\boldsymbol{y}\right)^2 \frac{32}{3} \tag{3.3}$$

Table 3: Estimated variances for the *Weld Repair Cast Fatigue Experiment* assuming 1, 2 and 3 active factors.

1 active		2 active		3 active		3 active	
Factor	$\hat{\sigma}^2$	Factors	$\hat{\sigma}^2$	Factors	$\hat{\sigma}^2$	Factors	$\hat{\sigma}_c^2$
G	0.31	F, G	0.06	E, F, G	0.02	E, F, G	0.03
I	0.48	G, I	0.29	D, F, G	0.02	F, G, I	0.03
D	0.50	E, G	0.30	F, G, I	0.03	C, F, G	0.04
E	0.50	D, G	0.31	C, F, G	0.04	D, F, G	0.06
A	0.51	A, H	0.31	B, F, G	0.04	B, F, G	0.06

where δ_{ABC} is the sign in the three-factor interaction column for the replicated runs.

The corrected estimate of σ^2 is then given by

$$\hat{\sigma}_c^2 = \frac{4\hat{\sigma}^2 + \left(-\delta_{ABC}\frac{1}{32}\sum_i y_i + \frac{3}{32}ABC'\boldsymbol{y}\right)^2 \frac{32}{3}}{5} \tag{3.4}$$

The following two examples illustrate the performance of the algorithm above. The first one is the *Weld Repaired Cast Fatigue Experiment* which is analysed both in Hamada and Wu (1992) and in Box and Meyer (1993), with fairly good agreement. The experiment contains 7 factors and four of the columns will represent dummy variables. Due to a different cycling procedure the factor labelling in Table 1 is not the same as the one used by the authors above, except for factor A. The correspondence may be obtained relabelling their columns to the left instead of to the right. Hence their column 11 is the column for our factor B, their column 10 is the column for our factor C and so on. This means that for the design given in Table 1 their factor F correspond to our factor G and vice versa, that their factor D correspond to our factor I and that B, C, D and E in our design represents the dummy variables. The vector of response values is given by

$$\boldsymbol{y} = (6.058, 4.733, 4.625, 5.899, 7.000, 5.752, 5.682, 6.607, 5.818, 5.917, 5.863, 4.809)'.$$

The estimated variances are shown in Table 3. Assuming only two active factors, F and G are the obvious candidates. If we open up for three active factors, we observe that many candidate models have very low $\hat{\sigma}^2$ and $\hat{\sigma}_c^2$, a typical pattern of over-fitting. The only serious alternative to the two-factor model, is the model with the three active factors F, G and I which is in agreement with the analysis in Hamada and Wu (1992). We observe that several models containing one dummy factor equally well explain the variation in the data. This is also an indication that a model with three active factors is over-fitted. In Figure 4 is shown a half-normal plot of main effects and also a half-normal plot of two-factor interactions involved with factor G. These plots strongly support that F and G are active factors.

Our second example is taken from Box and Meyer (1993). Twelve response values are extracted from the 2^5 *Reactor example* given in Box et al. (1978, p. 377). The design was analysed by the Box-Meyer method, arriving at the conclusion that the 12 run experiment in this case has the potential to identify the same set of active factors B, D and E as the full 2^5 design. In this example A, B, C, D, E are the five design factors while F to K here are dummy factors. We have run the algorithm presented above on the same set of data allowing up to three active factor. The twelve extracted response values are given by

$$y = (56\ 93\ 67\ 60\ 77\ 65\ 95\ 49\ 44\ 63\ 63\ 61)'.$$

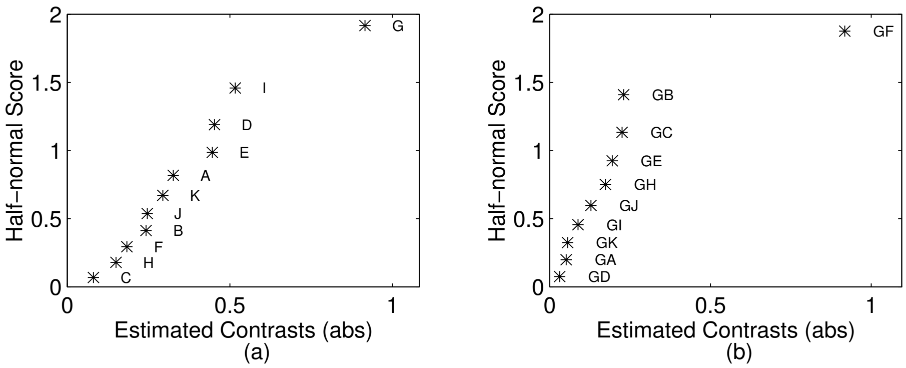

Figure 4: Half-normal plots for main effects and two-factor interactions in the *Weld Repair Cast Fatigue Experiment*: (a) Half normal plots for main effects, (b) Half-normal plots for two-factor interactions involved with factor G.

Table 4: Estimated variances for the *Reactor example* assuming 1,2 and 3 active factors.

1 active		2 active		3 active		3 active	
Factor	$\hat{\sigma}^2$	Factors	$\hat{\sigma}^2$	Factors	$\hat{\sigma}^2$	Factors	$\hat{\sigma}_c^2$
B	130.08	B, D	45.33	B, D, E	7.25	B, D, E	5.93
E	231.42	B, I	109.50	B, F, H	20.25	B, D, I	35.43
K	235.48	B, K	115.00	B, D, I	22.25	B, C, D	41.51
H	240.45	B, E	116.67	B, D, K	35.00	B, D, H	45.91
D	241.08	B, H	130.33	B, E, K	39.50	B, D, J	48.63

The results from Table 4 clearly confirms that B, D and E are the three most important factors. The close agreement between $\hat{\sigma}^2$ and $\hat{\sigma}_c^2$ also supports the absence of three-factor interactions as verified by the full 2^5 design. We also notice that $\hat{\sigma}_c^2$ gives a different ranking between models than $\hat{\sigma}^2$. The model with the two dummy factors that is ranked second by $\hat{\sigma}^2$ for instance will not be among the upper ten if ranked after $\hat{\sigma}_c^2$. Half-normal plots of main effects and two-factor interactions involved with B, D and E are shown in Figure 5. Factor B is clearly standing out in the main effect plot and the grouping on two parallel lines suggests that two two-factor interactions are present. The factors D and E, however, are put in the group together with the most biased inert factors. The reason why is that relatively large noise is present, and that the largest amount of confounding due to the two two-factor interactions BD and DE in absolute value, closely corresponds to the main effects of D and E. Examining the two-factor interaction contrast plots, one get some hint that B, D and E are the active factors, but the interpretation of these plots are rather difficult. Once the three main effects and the two two-factor interactions are estimated the pattern in the plots is easier to understand. Hence in this case they will best serve as a confirmation of an analysis done by other methods. We note that even if the half-normal plots are not always simple to interpret as in this situation, they may still give valuable information pointing at the most important factors or revealing that interactions are present.

In the examples above our analysis is in agreement with analysis already reported. We have also tried the algorithm above on various simulated examples with up to three active

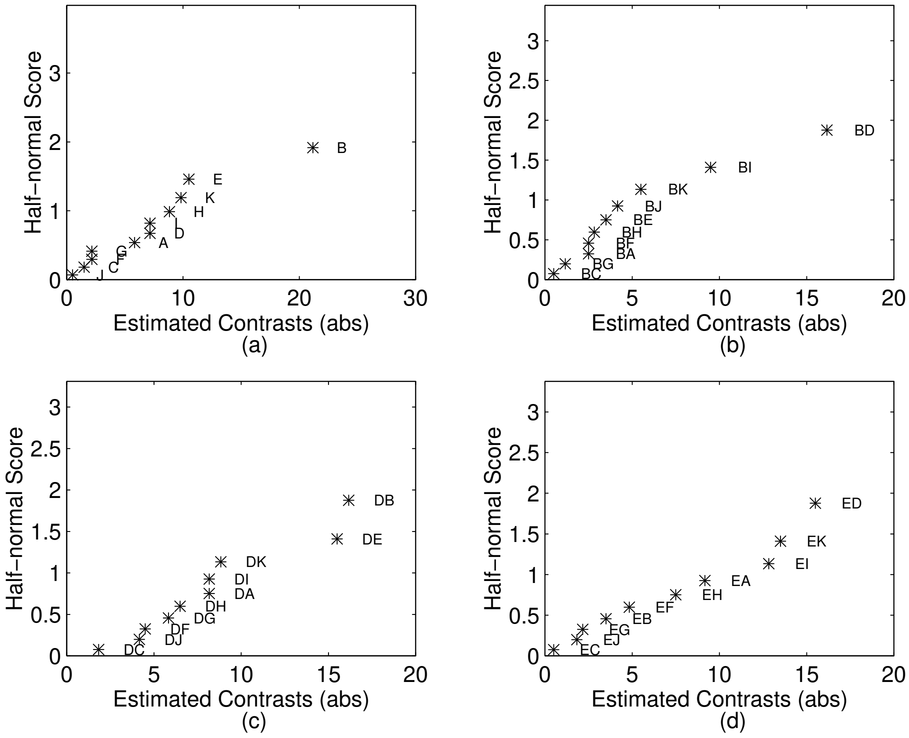

Figure 5: Half-normal plots for main effects and two-factor interactions in the *Reactor example*: (a) Half-normal plot of main effects, (b) - (d) Half-normal plots of two-factor interactions involved with B, D and E.

factors and compared its performance with the Box-Meyer method. In most situation the analysis with these two methods will be in agreement. However, it seems like increasing the error variance will influence the Box-Meyer method to favour a more parsimony model than the correct one. The method above will be less affected. Also if some effects in a model are much larger than the others, the factors with small effects are not identified. This is normally not a problem with the method presented above.

Computing both $\hat{\sigma}^2$ and $\hat{\sigma}_c^2$ gives a very useful check on the presence of three-factor interactions. If these are absent, it is our experience from simulated examples that $\hat{\sigma}^2$ and $\hat{\sigma}_c^2$ are approximately the same for a correctly specified model. If this is no longer so the ranking after $\hat{\sigma}^2$ and $\hat{\sigma}_c^2$ may be completely different and even a clear winner ranked after $\hat{\sigma}^2$ may no longer be suggested as a good candidate model if ranked after $\hat{\sigma}_c^2$. In such cases further analysis should be based on $\hat{\sigma}^2$.

4 Searching for four and more than four active factors

Additional projectivity results concerning the 12 run PB design can be found in Lin and Draper (1992) and Draper and Lin (1995). Especially they have pointed out that the projection into any four dimensions is essential equivalent and contains 7 of the eight level combinations in a 2_{IV}^{4-1} design, that there are three mirror image pairs and that there is one repeated run which is the mirror image of the one missing in the 2_{IV}^{4-1} design. In fact this design is

Table 5: The 12 run PB design and its relation to a 2_{IV}^{4-1} design.

Run PB(12)	Run 2_{IV}^{4-1}	A	B	C	D	AB	AC	AD	BC	BD	CD
9	1	-	-	-	+	+	+	-	+	-	-
6	8	+	+	+	-	+	+	-	+	-	-
10	2	+	-	-	-	-	-	-	+	+	+
7	7	-	+	+	+	-	-	-	+	+	+
11	3	-	+	-	-	-	+	+	-	-	+
4	6	+	-	+	+	-	+	+	-	-	+
2	4	+	+	-	+	+	-	+	-	+	-
5		+	+	-	+	+	-	+	-	+	-
1		+	-	+	-	-	+	-	-	+	-
3		-	+	+	-	-	-	+	+	-	-
8		-	-	+	+	+	-	-	-	-	+
12		-	-	-	-	+	+	+	+	+	+
	5	-	-	+	-	+	-	+	-	+	-

even closer to contain a standard 2_{IV}^{4-1} design for the following reason. Suppose we have a model with no constant term. Taking away the 7 level combinations that are included in a 2_{IV}^{4-1} design, the expected value of the sum of the 5 additional response values will then equal the expected value of the level combination in the 2_{IV}^{4-1} design that is missing, assuming only main effects and two-factor interactions are active. Under these assumptions and depending on which repeated run we use, two 2_{IV}^{4-1} designs with the variance of one observation five times higher than the others are available from each of the 330 projections into four dimension. In practice one may have to subtract $\bar{y}$ from the response values. This projection property into four dimensions is illustrated in Table 5 where we have used the first four columns from Table 1 as an illustration.

Due to the projection property pointed out above, two estimates of main effects are available and these will be unbiased provided we are estimating a correctly specified model with at most four active factors and all three-factor and higher order interactions are zero. Also two-factor interactions may be estimated in strings two and two. The runs 1,3,8 and 12 added together two and two give estimates of strings where main effects are aliased with two two-factor interactions and the sign of one of the two-factor interactions is opposite to the one obtained from the constructed 2_{IV}^{4-1} design. Hence this design has the potential to de-alias main effects from two-factor interactions, see also Wang and Wu (1995). A summary of operations useful for de-aliasing are given below. Here the β-coefficients represent half the main-effects and two-factor interactions.

$$\frac{y_1 + y_3 - (y_8 + y_{12})}{4} = \frac{\hat{\beta}_A + \hat{\beta}_B + \hat{\beta}_C - \hat{\beta}_D}{2} - (\hat{\beta}_{AB} + \hat{\beta}_{CD}) \tag{4.1}$$

$$\frac{y_1 + y_8 - (y_3 + y_{12})}{4} = \frac{\hat{\beta}_A - \hat{\beta}_B + \hat{\beta}_C + \hat{\beta}_D}{2} - (\hat{\beta}_{AD} + \hat{\beta}_{BC}) \tag{4.2}$$

$$\frac{y_3 + y_8 - (y_1 + y_{12})}{4} = \frac{-\hat{\beta}_A + \hat{\beta}_B + \hat{\beta}_C + \hat{\beta}_D}{2} - (\hat{\beta}_{AC} + \hat{\beta}_{BD}) \tag{4.3}$$

We also note that these four runs constitute four orthogonal columns and have a defining relation opposite of the one for the 2^{4-1}_{IV} design. In our case $I = ABCD$. In general these four runs can be used as follow up runs to de-alias two-factor interactions in a 2^{4-1}_{IV} design even if a block effect is assumed.

Now for each of the two repeated runs that are included among the seven runs from an ordinary 2^{4-1}_{IV} design, assume main effects and two-factor interactions (in strings two and two) are estimated from the constructed 2^{4-1}_{IV} designs and thereafter de-aliased using Equations 4.1 to 4.3. Let the fitted response values be denoted by $\hat{y}^1_i$ and $\hat{y}^2_i$. In our example the repeated runs are number 2 and 5. It is obvious then that $\hat{y}^1_i = \hat{y}^2_i = y_i$, $i = 4, 6, 7, 9, 10$ and 11, that $\hat{y}^1_2 = \hat{y}^1_5 = y_2$ and that $\hat{y}^2_5 = \hat{y}^2_2 = y_5$. We will show that $\hat{y}^1_i = \hat{y}^2_i = y_i$, $i = 1, 3, 8$ and 12. Assume $\hat{y}^1_i = y_i + \delta_i$, $i = 1, 3, 8$ and 12. From Equations 4.1 to 4.3 we get the following set of equations.

$$\begin{aligned} \delta_1 + \delta_3 - (\delta_8 + \delta_{12}) &= 0 & (4.4)\\ \delta_1 + \delta_8 - (\delta_3 + \delta_{12}) &= 0 & (4.5)\\ \delta_3 + \delta_8 - (\delta_1 + \delta_{12}) &= 0 & (4.6) \end{aligned}$$

from which it follows that $\delta_1 = \delta_3 = \delta_8 = \delta_{12} = \delta$. Then we must have that

$$\hat{y}^{1*}_5 = y_5 + y_1 + y_3 + y_8 + y_{12} + 4\delta \qquad (4.7)$$

where $\hat{y}^{1*}_5$ is the fifth fitted response and $y_5 + y_1 + y_3 + y_8 + y_{12}$ the fifth observation in the constructed 2^{4-1}_{IV} design written in standard form which implies that $\delta = 0$. Now the same set of arguments applies for $\hat{y}^2_i$ with $\hat{y}^{1*}_5$ interchanged with $\hat{y}^{2*}_2$. Next let $\hat{y}_i = (\hat{y}^1_i + \hat{y}^2_i)/2$. It then follows that $\hat{y}_i = y_i$, $i = 1, 3, 4, 6, 7, 8, 9, 10, 11$ and 12, and that $\hat{y}_2 = \hat{y}_5 = (y_2 + y_5)/2$. But from Equation 3.1 it now follows that the estimated main effects and two-factor interactions are the least squares estimates obtained with a design matrix, $\boldsymbol{X}$, consisting of the ten columns in Table 5. Hence an estimate for half the main effect of factor A, based on the design matrix $\boldsymbol{X}$ is given by

$$\begin{aligned} \hat{\beta}_A &= \Big(\sum_{i=2,4,6,7,9,10,11} \rho^A_i y_i - \sum_{i=1,3,5,8,12} y_i + \sum_{i=4,5,6,7,9,10,11} \rho^A_i y_i - \sum_{i=1,2,3,8,12} y_i\Big)/16 \\ &= \Big(\sum_{i=4,6,7,9,10,11} \rho^A_i y_i - \sum_{i=1,3,8,12} y_i\Big)/8 \qquad (4.8) \end{aligned}$$

and similar for the other main effects and two-factor interactions. Here ρ^A_i is the sign in the i-th run in the column for factor A. We notice that these estimates are a properly scaled sum over two expressions. One is the sum over the signed response values for the six unrepeated runs that are included in the 2^{4-1}_{IV} design. The other is the sum of the four response values with defining relations opposite of the six others multiplied by the sign corresponding to the signs in the row with the mirror image of the repeated runs. We also notice that only the estimates of main effects will be dependent on which of the repeated runs we choose to maintain in the constructed 2^{4-1}_{IV} design. Estimates of two-factor interactions will not be affected. It also follows from the analysis above that estimating the variance using the repeated run is equivalent to assume that the model for the response consists of four main effects and six two-factor interactions.

As an example let us take the model

$$y = 2A + 4C + 2BC - 2CD + \epsilon \qquad (4.9)$$

Table 6: Estimated variances from data given generated by model (4.9) assuming 2 and 3 active factors.

2 active		3 active		3 active	
Factors	$\hat{\sigma}^2$	Factors	$\hat{\sigma}^2$	Factors	$\hat{\sigma}_c^2$
C, D	8.82	A, B, C	4.68	C, D, E	4.09
A, C	12.39	C, D, H	5.03	A, B, C	4.44
C, J	14.59	C, D, I	5.04	C, D, K	5.18
B, C	14.98	C, D, E	5.10	C, I, J	6.27
C, E	15.68	C, E, H	5.86	A, C, D	7.50

from Section 2 and assume that only two and three factors are active. The 5 models with the lowest $\hat{\sigma}^2$ and $\hat{\sigma}_c^2$ are given in the Table 6.

Now let us have a more careful look at the models in Table 6. How can we get a hint that we should search for more than three active factors? First we notice that none of the models is a clear winner, and since the values of $\hat{\sigma}^2$ and $\hat{\sigma}_c^2$ assuming three active factors are substantially lower than the ones we would have got assuming only two active factors, it is an indication that three factors are not enough to explain the variation in the data. If only three factors are active we have for each model four repeated runs. Examining these four runs for each of the models we find large differences between runs that assumingly should give the same expected response values. For the two models with the lowest $\hat{\sigma}_c^2$ for instance this difference is greater than four. This suggests that none of the models above give a thorough explanation of the variation in the data.

Now running the Box-Meyer algorithm on the same set of data allowing main effects and two-factor interactions to be in the model, leaves us with A, C, D and E as the four factors with the highest posterior density (Hynén, 1996), although only factor C stands out as clearly active. But the repeated runs for these four factors have the response values -10.323 and -5.975 (run number 11 and run number 12) compared to -3.205 and -2.481 (run number 2 and run number 5) for factors A, B, C and D which are the four factors most likely to be active from the analysis in Section 2. Moreover considering the constructed 2_{IV}^{4-1} design gives us the estimates of half the main effects and half the two-factor interactions as shown in Table 7. All the observations are corrected for their average.

The difference between the estimates of main effects from the two 2_{IV}^{4-1} design is of course only a reflection of the difference between the repeated runs. As a result we notice that the correspondence between the estimates of main effects based on the 2_{IV}^{4-1} design and the 12 run PB design is convincing when the active factors are assumed to be A, B, C and D but must be considered much less so when A, C, D and E are assumed to be the active factors. Such "quick" calculations can be most valuable in evaluating a model suggested by some method as for instance the main effect and interaction contrast plots given in Section 2 or their corresponding half-normal plots. The most plausible terms from that analysis was exactly the two main effects from factor A, and factor C, and the two two-factor interactions BC and CD. Let us also mention that the adjusted R^2 value for the model with the four terms (A, C, BC and CD) is 99.3. If we assume that A, C, D and E are the active factors and pick the 6 most plausible terms (C, D, AE, CD, CE and DE) the adjusted R^2 value is 89.4 and based on only four terms the best adjusted R^2 is as low as 83.7.

We note that A, B, C and D is not the only choice of four active factors that will give

Table 7: Estimated main-effects and two-factor interactions based on constructed 2_{IV}^{4-1} design.

Factors A, B, C, D	β_A	β_B	β_C	β_D	β_{AB}	β_{AC}	β_{AD}	β_{BC}	β_{BD}	β_{CD}
2_{IV}^{4-1} (y_2 included)	2.15	0.06	4.17	-0.25						
2_{IV}^{4-1} (y_5 included)	2.33	0.24	3.99	-.0.07	-.14	0.16	-.14	2.23	0.16	-1.68
12 run	2.24	0.15	4.08	-.16						
Factors A, C, D, E	β_A	β_C	β_D	β_E	β_{AC}	β_{AD}	β_{AE}	β_{CD}	β_{CE}	β_{DE}
2_{IV}^{4-1} (y_2 included)	1.08	4.17	-0.30	0.96						
2_{IV}^{4-1} (y_5 included)	0.05	3.14	-1.33	-0.07	0.51	-0.59	0.76	-2.35	-0.75	-1.59
12 run	0.56	3.66	-0.82	0.44						

good correspondence between estimates from the 2_{IV}^{4-1} design. There are other repeated runs having smaller distance between response values than run 2 and 5. And even for these two runs any four factors from the set $\{A, B, C, D, F\}$ will have the same convincing agreement in the estimated main effects. This shows that even if a design has good efficiency in estimating the 4 main effects and the 6 two-factor interactions between them as is the case for the 12 run PB design (Wang and Wu, 1995), it does not necessarily have the ability to discriminate between models.

However situations will certainly arise when one will only be confident by searching through all models containing four active factors. Following the idea in Box and Meyer (1993) gives us altogether $\binom{11}{4} = 330$ different models to examine. Each of them will consist of 11 terms when only main effects and two-factor interactions are allowed. Due to over-fitting many different models will normally get a very low $\hat{\sigma}_c^2$ which will complicate the possibility to identify the four correct active factors. A way to reduce terms in the model is needed. Since estimates of main effects is easily achievable the effect of these may be subtracted out of the response. The problem is then reduced to find out for each model how many of the potential 6 two-factor interactions that should be included in addition and hopefully this is a small number. This is a best subsets regression problem.

A more general regression based way of reducing the number of models to examine in searching for k active factors would be to assume that each model contain the k main effects, and then as above examine the models when increasing numbers of two-factor interactions are added. This method does not restrict k to be less or equal to 4, but one should be aware of that for five and more active factors problems with full linear dependence between columns could show up (Wang and Wu, 1995). We have tried this procedure on the data from the example above. The best five set of 4 active factors are given in Table 8, together with their two-factor interaction terms assuming one, two and three two-factor interaction terms respectively. The results clearly underlines that in this case there is rather clear evidence for one single model and that A, B, C and D are the four active factors.

Similar results are obtained if the other method is applied. For four active factors the two suggested methods should be expected to perform similar. Since the estimates of main effects no longer are fixed in the more general regression based approach, a wrong model will have some more flexibility in adapting to the data. Whether this will create any noteworthy difficulties in discriminating between models awaits further research.

Table 8: Estimated variance from data generated from model (4.9) assuming four main effects and one, two and three two-factor interactions.

1 two-factor interaction		2 two-factor interaction		3 two-factor interaction	
Factors	$\hat{\sigma}^2$	Factors	$\hat{\sigma}^2$	Factors	$\hat{\sigma}_c^2$
A, B, C, K, BC	3.03	A, B, C, D, BC, CD	0.18	A, B, C, D, AD, BC, CD	0.18
A, B, C, H, BC	3.39	A, B, C, H, BC, BH	1.90	C, D, I, K, CD, DK, IK	0.18
A, B, C, J, BC	3.39	A, B, C, J, BC, BJ	2.16	C, I, J, K, CI, CJ, IK	0.64
A, B, C, F, BC	3.50	A, B, C, K, AB, AC	2.28	B, C, E, F, BE, CE, EF	1.37
A, B, C, I, BC	3.72	A, B, C, G, AG, BC	2.37	C, E, J, K, CJ, CK, EJ	1.39

The number of models to investigate can be largely reduced if we by other methods have clear evidence that one or more factors are influential. For instance let us once more take a look at the models in Table 6. There will probably not be much of a chance to assume that factor C should be in the model. Considering the half-normal plots of main-effects and two-factor interactions in Figure 3, one would probably have some strong feelings for at least one of the factors A or D too. In general the most important factors will manifest themselves early in an investigation. This may give useful hint at arriving at the most plausible model.

5 Further examples and comments

Though seldom, it may happen that several models will explain the data equally well. Such an example can be found in Hynén (1996). The data are generated with the following model

$$y = A + B + C + 0.5AB + 0.5AC + 0.5BC + \epsilon \tag{5.1}$$

where $\epsilon \sim N(0, 0.25^2)$.

The Box-Meyer method, allowing for two-factor interactions is here slightly in favour of A, B and C being the active factors, but three other models have approximately the same posterior probability as this. These three models include the factors (B, E, K), (A, H, I) and (C, D, F) respectively. Now in order for two models to give the same expected response, we must have

$$\boldsymbol{X}_1\boldsymbol{\beta}_1 = \boldsymbol{X}_2\boldsymbol{\beta}_2 \tag{5.2}$$

Also it is clear that

$$\boldsymbol{\beta}_1 = (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{X}_2\boldsymbol{\beta}_2 \tag{5.3}$$
$$\boldsymbol{\beta}_2 = (\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{X}_1\boldsymbol{\beta}_1 \tag{5.4}$$

since this is the expected bias, wrongly assuming one of the models is correct at the expense of the other. Doing the calculations above, assuming three-factor interactions are not present, one find that the following sets of coefficients (1, 0.5, -0.5, 1,-1, -0.5) for the model with the active factors (B, E, K) and (A, H, I) and (1,-0.5, 0.5, -1, 1, -0.5) for the model with the active factors (C, D, F). In each case it is easily verified that we get the same expected response from all these models.

Table 9: Estimated variance from data generated from model (5.1) assuming 1,2 and 3 active factors.

1 active		2 active		3 active		3 active	
Factor	$\hat{\sigma}^2$	Factors	$\hat{\sigma}^2$	Factors	$\hat{\sigma}^2$	Factors	$\hat{\sigma}_c^2$
C	2.02	C, F	1.46	C, D, F	0.02	B, E, K	0.04
B	2.24	A, B	1.65	A, H, I	0.04	A, H, I	0.04
A	2.32	A, H	1.78	B, E, K	0.05	C, D, F	0.04
G	2.54	A, C	1.80	A, B, C	0.06	A, B, C	0.05
J	2.59	B, I	1.80	C, G, J	0.06	H, J, K	0.96

Running the algorithm in Section 3 allowing for three active factors and using the data in Hynén (1996) gave the following results shown in Table 9. We see that both $\hat{\sigma}^2$ and $\hat{\sigma}_c^2$ both pick the four models above as the ones with the lowest estimate for σ^2, but while the choice is very clear using $\hat{\sigma}_c^2$, this is no longer the case if just $\hat{\sigma}^2$ is used. Then seven models will have an estimate for σ^2 below 0.10. In this situation follow up designs are really needed to find out which of the four plausible models are the right one (Meyer, Steinberg and Box, 1996), since the design leaves no possibility to discriminate between them.

So far all our examples has been on the 12 run PB design. Algorithms based on projection properties may easily be extended to other projectivity $P = 3$ designs, while the main-effect and two-factor interaction contrast plots and hence also the half-normal plot will need some modification in their interpretation. For instance for the 20 run PB design, two-factor interactions are not equally confounded with main effects. The confounding is either 1/3 or 1/5. However, if only one two-factor interaction is present, all except at most one of the non-active factors will have the same expected bias in absolute value, see Lin and Draper (1993). If only two two-factor interactions are present, all except at most two of of the inert factors will fall in two groups. This large reduction of the potential number of active factors may be very useful in analysing the design. We will illustrate with an example. In a 20 run PB design the response is given by

$$y = 1.5A + 1.5B + 2C + 2AB + 1.5AC + \epsilon \tag{5.5}$$

where $\epsilon \sim N(0, 0.25^2)$.

The main effect half-normal contrast plot for this example given in Figure 6a indicates that two two-factor interactions are present, that the active factors will be found among the factors A, B, C, M and Q and that two of them are inert. The plot for the two-factor interactions involved with factor A , Figure 6b, supports this hypothesis. The estimated two-factor interaction contrasts AB, AC, AM and AQ falls of the two lines. Very likely then the two two-factor interactions are involved with factor A, since two columns in the alias matrix will explain the closely parallel lines and the two additional two-factor interactions falling off these lines.

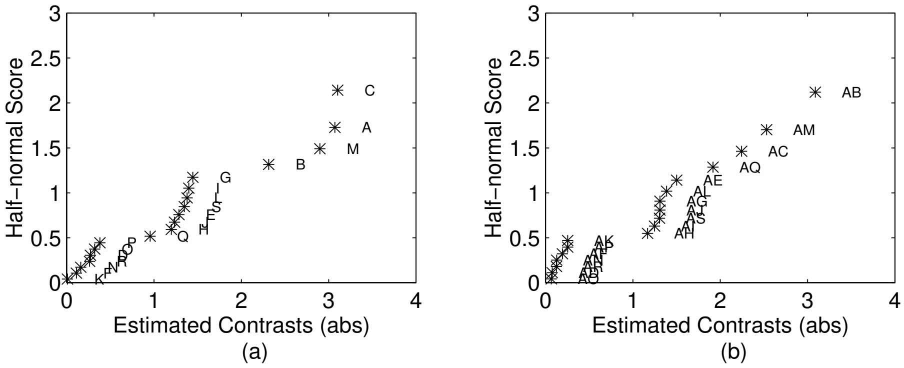

Figure 6: Half-normal plot for main effects and two-factor interactions from the 20 run PB design: (a) Half-normal plot for main effects, (b) Half-normal plots for two-factor interactions involved with factor A.

6 Summary and Conclusions

We have presented methods for analysing designs with complex alias structure, and demonstrated the usefulness of these methods. It is an advantage that these designs are analysed in several ways. With respect to that our methods make a sound alliance taking into consideration both alias structure and projection properties. The first leads to a new interpretation of half-normal plots which may serve as both an explorative and confirmative guide in searching for models. The last ensures more consistent analysis of the data and more efficient implementation of computer algorithms for doing the analysis. The methods are easy to implement and highly competitive as comes to use of computer time. Our analysis has mainly been restricted to the 12 run PB design and we hope to have demonstrated that this design is a very economical screening design, and that efficient methods for analysis are available. With some modification the suggested methods may be extended to other projectivity $P = 3$ designs, among such the additional four 16 run (Box and Tyssedal, 1996) and the 20 run orthogonal arrays are of particular interest. Such work is under progress.

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