Repairable systems with general repair[1]

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ABSTRACT: A repairable system can briefly be characterized as a system which is repaired rather than replaced after a failure. The most commonly used models for the failure process of a repairable system are nonhomogeneous Poisson processes, corresponding to minimal repairs, and renewal processes, corresponding to perfect repairs. The paper reviews models for more general repair actions, often called "imperfect repair" or "better-than-minimal repair" models, using a general framework based on point process theory. A Markov model approach for periodically tested components is also considered.

1 INTRODUCTION

A repairable system is a system which, after failing to perform one or more of its functions satisfactorily, can be restored to fully satisfactory performance by any method, other than replacement of the entire system (Ascher & Feingold 1984).

Much of the literature on repairable systems is motivated by applications to mechanical systems, but important areas of application include software reliability, manufacturing, medicine and other disciplines involving recurrent events. A particular motivation for the work presented here has been studies related to the oil and gas installations at the Norwegian continental shelf, where analyses of operation data are used, e.g., for descriptions and comparisons of equipment, optimization of maintenance procedures, and control and improvement of safety and cost-effectiveness.

The purpose of this paper is to review some basic models and methods used for the analysis of failure data from repairable systems, with emphasis put on various ways of taking into account repair and maintenance. The paper complements recent review papers on imperfect repair models by Phan & Hwang (1996) and Akersten (1998).

[1]To appear in proceedings of European Safety and Reliability Conference, 13-17 September 1999 (ESREL '99), Munich, Germany.

2 NOTATION AND BASIC DEFINITIONS

For brevity, we shall in this paper mainly restrict attention to systems which following a failure are immediately restored to a functioning state. For simplicity of notation we consider systems observed from time $t = 0$, and let $T_1, T_2, \ldots$ denote the successive *failure times*. The assumption of repair time set to 0 is reasonable if repair times are negligible compared to times between failures. It can also be justified provided the time scale is taken to be the *operation time* of the system. In the latter case, a marked point process may keep track of the repair times, and possibly additional information such as production conditions, maintenance and external conditions.

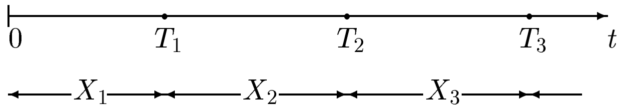

Figure 1: Failure times T_i and interfailure times X_i of a repairable system.

We shall let the *interfailure times*, i.e. times between failures, be denoted $X_1, X_2, \ldots$. Thus we have

$$X_i = T_i - T_{i-1}, \;\; i = 1, 2, \ldots$$

where for convenience we define $T_0 \equiv 0$ (see figure 1). The sequence $T_1, T_2, \ldots$ of failure times and the sequence $X_1, X_2, \ldots$ of interfailure times thus contain exactly the same information about a particular realization of the process. This also holds for the counting process representation

$$N(t) = \text{ number of failures in } (0, t]$$

A plot of $N(t)$ as a function of t (or equivalently, plotting i vs. T_i) is the usual way of depicting data from single repairable systems.

The simplest and most popular stochastic models for repairable systems are the *renewal process*, RP, and the *nonhomogeneous Poisson process*, NHPP. The defining property of an RP is that the interfailure times X_i are i.i.d. with some distribution F. We shall denote this process by RP(F). The NHPP is defined in terms of an intensity function $\lambda(t)$ such that the probability of a failure in a (small) time interval $[t, t+\Delta t)$ is $\lambda(t)\Delta t + o(\Delta t)$, while the number of events in disjoint time intervals are stochastically independent. We denote this process by NHPP($\lambda(\cdot)$).

As we shall see in the next section, the RP and NHPP can be viewed as corresponding to the two extreme repair actions, *perfect repair* and *minimal repair*, respectively. In order to describe models for more general repair strategies, we need some more notation from the theory of point processes. A general reference is Andersen et al. (1992), while a nice reference on point process models in reliability is Thompson (1988).

Let $\mathcal{F}_{t-}$ dentote the *history* of the failure process up to, but not including, t. Usually we shall think of $\mathcal{F}_{t-}$ as $\{N(s) : 0 \leq s < t\}$ which keeps record of all the failures before time t. It could, however, contain more information, e.g. in the case where a marked point process is used as mentioned earlier. The conditional intensity $\lambda(t|\mathcal{F}_{t-})$ is defined by (Andersen et al. 1992)

$$\lambda(t|\mathcal{F}_{t-}) = \lim_{\Delta t \downarrow 0} \frac{Pr(\text{failure in } [t, t+\Delta t)|\mathcal{F}_{t-})}{\Delta t}$$

This is to be interpreted as an ordinary intensity function, but is "adjusted" according to the known history. Thus, $\lambda(t|\mathcal{F}_{t-})\Delta t$ is approximately the probability of failure in the time interval $[t, t+\Delta t)$, conditional on the failure history before time t.

For statistical inference in models defined by the conditional intensity we need to write down the *likelihood function*. Suppose that a single repairable system with failure process $N(t)$ and conditional intensity function $\gamma(t) \equiv \lambda(t|\mathcal{F}_{t-})$ is observed from time 0 to time τ, resulting in failure times $T_1, T_2, \ldots, T_{N(\tau)}$. The *likelihood* function is then given by (Andersen et al. 1992)

$$L_\tau = \{\prod_{i=1}^{N(\tau)} \gamma(T_i)\} \exp(-\int_0^\tau \gamma(u)\mathrm{du}) \tag{1}$$

This likelihood function is valid under the assumption that τ is a so-called *stopping time*, which holds for most censoring schemes occuring in practice. In particular it holds for the two most common observation schemes for repairable systems, *time truncation* in which τ is a prespecified point in time, and *failure truncation* where the process is observed until a given number n of failures have occurred.

For parametric models for $\gamma(t)$ there is a well developed theory for maximum likelihood estimation based on the martingale approach to point processes, see Andersen et al. (1992). A relevant reference for statistical inference in reliability models is Crowder et al. (1991).

3 MINIMAL AND PERFECT REPAIR

A common recipe for analysis of a set of data from a repairable system seems to be as follows. First, apply a test for *trend* in the interfailure times X_i (Ascher & Feingold 1984). If no significant trend is found, then model the X_i by an RP using the well established machinery for analysis of independent and identically distributed (i.i.d.) observations. Otherwise, use an NHPP model which is able to handle various kinds of trends through specification of the intensity function $\lambda(t)$. For example, a *deteriorating* system corresponds to an increasing function $\lambda(t)$, while an *improving* system corresponds to a decreasing $\lambda(t)$. Of course a *homogeneous* Poisson process, HPP(λ), corresponds to a constant intensity $\lambda(t) \equiv \lambda$.

As mentioned in the previous section, the RP and NHPP models correspond to, respectively, perfect and minimal repair. We shall in the present and next sections consider a unifying approach which makes these two models the extremes of a more general model. The unifying approach involves the specification of a *hazard rate* function $z(t)$, which we think of as being the hazard function of the time to first failure of the system. This function will then be used in describing the various models by means of their conditional intensity function $\lambda(t|\mathcal{F}_{t-})$. Recall that if T is a positive, continuous random variable, then the hazard rate is

$$z(t) = \lim_{\Delta t \downarrow 0} Pr(T \leq t + \Delta t | T > t)/\Delta t$$

For a renewal process, $\mathrm{RP}(F)$, it is well known that

$$\lambda(t|\mathcal{F}_{t-}) = z(t - T_{N(t-)})$$

where z is the hazard rate corresponding to F, and $t - T_{N(t-)}$ is the time since the last failure strictly before time t. Thus, after each failure, the rate of failures is returned back to the state at time 0, i.e. the system is as good as new immediately after each failure. This is the reason for calling the RP a *perfect repair* model.

On the other hand, an NHPP has the conditional intensity

$$\lambda(t|\mathcal{F}_{t-}) = z(t)$$

which means that after a failure, the failure rate is exactly as if no failure had ever occurred. Thus we have a reason for calling this a *minimal repair*, since the system is only restored to a state where it is exactly as good (or bad) as it was immediately before the failure. It should also be remarked that for an NHPP the intensity function $\lambda(t)$ equals, for all t, the hazard rate $z(t)$ of the time to first failure.

4 IMPERFECT REPAIR

For many applications it is more reasonable to model the repair action by something in between the two given extremes of perfect and minimal reapir. One such model is suggested by Brown & Proschan (1983), where at the time of each failure a perfect repair occurs with probability p and a minimal repair occurs with probability $1 - p$.

In a recent paper, Dorado et al. (1997) suggest a general repair model which contains a large class of imperfect repair models as special cases. The definition in their paper is given in terms of conditional distributions of the X_i. An equivalent definition in terms of conditional intensities is

$$\lambda(t|\mathcal{F}_{t-}) = \theta(t) z \left(a(t) + \theta(t)(t - T_{N(t-)}) \right)$$

which in addition to the basic hazard rate $z(t)$ that we introduced in Section 3, is defined in terms of two random processes:

$a(t) =$ effective age at last failure before t, i.e. at time $T_{N(t-)}$.

$\theta(t) =$ repair characteristic of last repair before t (called "life supplement"), $0 < \theta(t) \leq 1$.

It is easy to see that an RP corresponds to $\theta(t) \equiv 1$, $a(t) \equiv 0$, while an NHPP has $\theta(t) \equiv 1$, $a(t) = T_{N(t-)}$.

Kijima (1989) suggested two models for imperfect repair, which turn out to be special cases of the model by Dorado et al. (1997). In fact, Kijima's Model I has $\theta(t) \equiv 1$, while the effective age $a(t)$ increases by the amount $D_j X_j$ at the jth failure time T_j, where $D_1, D_2, \ldots$ is a sequence of random variables independently distributed on $[0, 1]$ and independent of other processes.

In Kijima's Model II, we have $\theta(t) \equiv 1$, while the effective age $a(t)$ immediately after the jth failure is set to $\sum_{k=1}^{j} (\prod_{i=k}^{j} D_i) X_k$. Equivalently, the effective age after the jth failure equals D_j multiplied by the effective age just before the jth failure.

When D_j is 1 with probability p and 0 with probability $1 - p$, we obtain the Brown & Proschan (1983) model. Moreover, the model A by Bathe & Franz (1996) is given by letting D_j be, respectively $1, c, 0$ with probabilities $p, q, 1 - p - q$. Here $0 < c < 1$ corresponds to an intermediate repair action.

Dorado et al. (1997) consider statistical inference in their general repair model when the effective ages $a(t)$ and life supplements $\theta(t)$ are observed at each failure, in addition to the failure times T_i. Using the general counting process framework they derive estimates and confidence bands for the distribution of time to first failure, under the condition that this has increasing failure rate.

Statistical inference in the Brown-Proschan model has been studied by Whitaker and Samaniego (1989) and later by Hollander, Presnell and Sethuraman (1992).

Bathe & Franz (1996) considered statistical inference in their model described above, also for the case when nonzero repair times were taken into account.

Block et al. (1985) extended Brown and Proschan's model to age-dependent repair, allowing the parameter p to depend on time. Iyer (1992) in turn extended the age-dependent repair model to the case where repair times are not negligible. He found expressions for the long run fraction of time that the system is down due to minimal repair (perfect repair) as well as for the steady-state availability of the system.

Langseth (1999) investigates the modeling of imperfect maintenance of a system in a competing risk situation, generalizing the Brown and Proschan model to include competing risks, i.e.

situations where more than one failure mechanism "compete" to make the component fail.

Shaked & Shantikumar (1986) analyse a multi-component imperfect repair model. They consider models of systems whose components have dependent lifelengths and are minimally repaired upon failure until they are scrapped

5 LOGLINEAR AND GENERALIZED LINEAR MODELS

Berman and Turner (1992) consider estimation in the fairly general parametric model

$$\lambda(t|\mathcal{F}_{t-}) = g\left\{\sum_{k=1}^{p} \theta_k Q_k(t|\mathcal{F}_{t-})\right\} \tag{2}$$

where g is a known monotonic continuous function and the $Q_k(t|\mathcal{F}_{t-})$ are known functions of t and the history $\mathcal{F}_{t-}$. The data needed for estimation are just the failure times T_i, and they show how to use software for generalized linear models.

The special case when $g(y) = e^y$ is applied to repairable sytems by Lawless & Thiagarajah (1996). In particular they consider the model

$$\lambda(t|\mathcal{F}_{t-}) = e^{\alpha+\beta g_1(t)+\gamma g_2(t-T_{N(t-)})} \tag{3}$$

where α, β, γ are unknown parameters and g_1 and g_2 are known functions. This is, in fact, an example of what Cox (1972) called a *modulated renewal process* (see next section).

Observe that the intensity in (3) depends both on the age (t) of the system and the time $(t - T_{N(t-)})$ since the last failure.

6 MODULATED RENEWAL PROCESSES

Cox (1972) introduced a class of point processes in which the conditional hazard function of a renewal process or of a Poisson process is perturbed by a factor depending on quantities thought to influence the probability of occurrence, each quantity being combined with an unknown regression coefficient. More preciesely, the hazard function has the form

$$\lambda(t|\mathcal{F}_{t-}) = e^{\sum_{i=1}^{p}\beta_i z_i(t)} z(t - T_{N(t-)})$$

This can be viewed as a (general) renewal process modulated by the vector $(z_i(t))$. Cox suggested that inference in this model is made via a conditional likelihood in a way similar to the well-known Cox-regression model in survival analysis.

7 THE TREND-RENEWAL PROCESS

An interesting feature of the modulated renewal process, as well as the model (3) is that they contain both the NHPP and the RP as special cases. This makes it possible to choose between a minimal and perfect repair model by use of statistical (likelihood ratio) tests for the parameters. We shall present an alternative model, which at the outset may seem very similar to (3), but which is not even covered by the general formula (2).

The idea behind the process, which was termed the *trend-renewal process* by Lindqvist (1993), is to generalize the following well known property of the NHPP. First, define the *cumulative* intensity function corresponding to an intensity $\lambda(\cdot)$ by $\Lambda(t) = \int_0^t \lambda(u)\mathrm{d}u$.

The result is that if $T_1, T_2, \ldots$ is an NHPP$(\lambda(\cdot))$, then the time-transformed stochastic process
$\Lambda(T_1), \Lambda(T_2), \ldots$ is HPP(1).

We define the *trend-renewal process* (TRP) simply by allowing the above HPP(1) to be *any* renewal process RP(F). Thus, in addition to the intensity function $\lambda(t)$, for a TRP we need to specify a distribution function F of the interarrival times of this renewal process. Formally we can define the process TRP$(F, \lambda(\cdot))$ as follows:

Let $\lambda(t)$ be a nonnegative function defined for $t \geq 0$, and let $\Lambda(t) = \int_0^t \lambda(u)\mathrm{d}u$. The process $T_1, T_2, \ldots$ is called TRP$(F, \lambda(\cdot))$ if the transformed process $\Lambda(T_1), \Lambda(T_2), \ldots$ is RP(F), i.e. if the $\Lambda(T_i) - \Lambda(T_{i-1})$; $i = 1, 2, \ldots$ are i.i.d. with distribution function F.

Figure 2 illustrates the definition. Note that for an NHPP$(\lambda(\cdot))$, the RP(F) is replaced by HPP(1). Thus TRP$(1 - e^{-x}, \lambda(\cdot))$ = NHPP$(\lambda(\cdot))$. Also, TRP$(F, 1)$ = RP(F), which shows that the TRP generalizes both the RP and the NHPP.

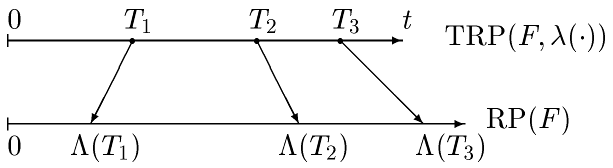

Figure 2: The defining property of the trend-renewal process.

It can be shown that the conditional intensity function for the TRP$(F, \lambda(\cdot))$ is

$$\lambda(t|\mathcal{F}_{t-}) = z(\Lambda(t) - \Lambda(T_{N(t-)}))\lambda(t) \tag{4}$$

This is a product of one factor, $\lambda(t)$, which depends on the age t of the system and one factor which depends on a transformed time from the last previous failure. In fact, time since last failure is measured on a scale depending on the current intensity of failures, in contrast to the model (3) of Lawless and Thiagarajah (1996).

8 BERMAN'S INHOMOGENEOUS GAMMA PROCESS

The *inhomogeneous gamma process* of Berman (1981) is a special case of the TRP, given by $\mathrm{TRP}(G, \lambda(\cdot))$ with G corresponding to a gamma-distribution with scale parameter 1 (see previous section for notation). Berman studied in particular what he calls a *modulated gamma process*, with intensity function given by

$$\lambda(t) = \rho \exp\{\sum_{i=1}^{p} \beta_i z_i(t)\} \qquad (5)$$

where ρ and the β_i are unknown parameters and the $z_i(\cdot)$ are given ("explanatory") functions. Of particular interest is the model for "cyclic trend of fixed frequency", for which

$$\lambda(t) = \rho \exp\{\beta_1 \cos(\omega t) + \beta_2 \sin(\omega t)\}$$

with ω fixed. Testing the null hypothesis $\beta_1 = \beta_2 = 0$ here leads to a test for the null hypothesis of an RP with gamma-distributed interfailure times vs. a *non*-monotonic (cyclic) trend.

9 PERIODICALLY TESTED SYSTEMS

Failures are often classified into severity classes, for example critical or degraded failures. Simple Markov models for the failure mechanism of a component, suitable for making statistical inferences based on reliability data bases, were suggested and studied in Hokstad & Frøvig (1996). Their study involved dormant (hidden) failures of periodically tested standby components.

Lindqvist & Amundrustad (1998) extended their model, embedding it in a quite general and flexible Markov chain approach. The main idea of their approach is to use a continuous time Markov model for the component or system state when time runs *between* testing epochs, and then in addition model repair actions by separate transition matrices.

As an illustration we consider the most general failure model of Hokstad & Frøvig (1996), namely their Failure Mechanism III. Consider a component which starts the operation at time $t = 0$ and is tested at time epochs $\tau, 2\tau, 3\tau, \ldots$, When time is running between testing epochs, the state $X(t)$ of the component at time t is modeled as a Markov chain with state space

$$S = \{O, D, K^I, K^{II}\}, \qquad (6)$$

where $O =$ the component is as good as new, $D =$ the component has a failure classified as *degraded* (noncritical), $K^I =$ the component has a failure classified as *critical*, caused by a *sudden shock*, $K^{II} =$ the component has a failure classified as *critical*, caused by the *degradation process.*

The transition intensities of this Markov chain are, moreover, defined by the state diagram of Figure 3. This means that the two critical failure states are absorbing, i.e. no repairs are done in the time intervals between PM epochs.

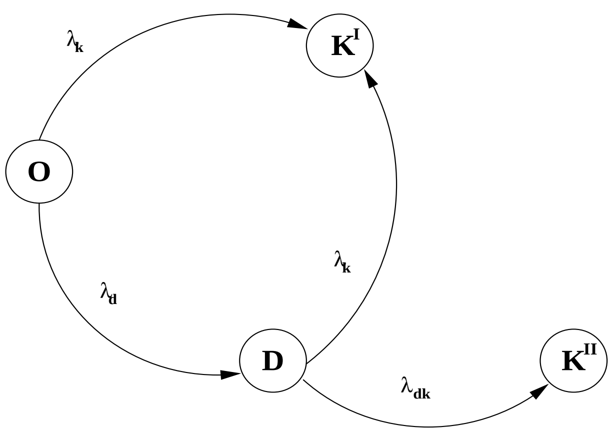

Figure 3: State diagram for the failure mechanism of HF.

Having defined the probabilistic behaviour of the component state between testing, we have to add to the model a specification of the repair policy. In Lindqvist & Amundrustad (1998) this is modeled in the form of a transition matrix R defined on the state space of the component. For example a rather general imperfect repair model could be given by

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-r & r & 0 & 0 \\ 1-r_{k1} & 0 & r_{k1} & 0 \\ 1-r_{k2} & 0 & 0 & r_{k2} \end{bmatrix}$$

where the ordering of the states follows from (6). Here r is the probability that a degraded failure remains unrepaired at testing epochs, while $1 - r_{k1}$ is the probability of successful repair of a K^I failure and $1 - r_{k2}$ is the similar for K^{II}.

10 CONCLUSIONS

The paper reviews probabilistic and statistical models for repairable systems which are repaired at failures or, possibly, at certain predetermined maintenance time epochs. Some models, especially those intended for statistical inference, can be considered as "black box" models where a good fit to data may be the main goal. Others are based on system knowledge and an attempt to model the underlying failure mechanism.

When studying the literature it becomes clear that almost all work in the field has been done for single failure modes, although some exceptions have been presented in this survey. It is believed, however, that future work in the field will be more directed towards more complicated maintenance and failure schemes. An interesting attempt in this direction is Langseth (1999).

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