NORGES TEKNISK-NATURVITENSKAPELIGE UNIVERSITET

# Representation of the Response Process of Quadratic Filters Subjected to Gaussian Noise

by

A. Naess

PREPRINT
STATISTICS NO. 13/2001
ISSN 0804-9173

NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
TRONDHEIM, NORWAY

This report has URL
http://www.math.ntnu.no/preprint/statistics/2001/S13-2001.ps
Arvid Naess has homepage: http://www.math.ntnu.no/~arvidn
E-mail: arvidn@math.ntnu.no
Address: Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491 Trondheim, Norway.

# REPRESENTATION OF THE RESPONSE PROCESS OF QUADRATIC FILTERS SUBJECTED TO GAUSSIAN NOISE

A. Naess
Department of Mathematical Sciences
Norwegian University of Science and Technology
NO-7491 Trondheim, Norway

## 1 Introduction

It appears that the first major contribution to the statistical analysis of quadratic filters driven by stationary Gaussian processes came with the seminal paper by Kac & Siegert (1947). Their work related to the theory of noise in radio receivers with square law detectors as the title of their paper clearly states. Typically, the noise process that was considered could be represented as a filtered white noise process.

Later, this theory found interesting applications in the statistical analysis of the response of moored, floating offshore structures (Neal 1974), and the engineering literature has seen a large number of contributions focusing on the development of statistical techniques to deal with this kind of applications. The extension of these techniques to the case of multi-directional waves was achieved by Naess (1990). This required a generalization of the theory developed by Kac & Siegert (1947), which could only be used for the case of uni-directional waves. In these approaches, the ocean wave elevation was invariably modeled as a Gaussian process.

Based on the principal ideas proposed by Naess (1990), the main purpose of the present paper is to derive a generalization of the theory developed by Kac & Siegert (1947). To achieve this generalization, the theory of random measures turns out to provide the crucial tool. In fact, a slightly generalized version of the theory of multiple Wiener-Itô integrals (Major 1980) will be employed. It will be shown how this will lead to representation theorems for the response process of an integrated class of parametrized quadratic filters subjected to a stationary Gaussian process.

## 2 Representation of the Input Process

To achieve our goal in an expedient manner, the input process, which is stationary and Gaussian, will be constructed in a particular way. We start by constructing an orthogonal Gaussian measure $B$ defined on the $\sigma$-algebra of the Borel sets of the real plane, that is, $\mathbf{R}^2$.

Let $(\Omega, \mathcal{S}, P)$ be a probability space. Let $\mathcal{R}^+$ denote the family of (finite) rectangular subsets of $\mathbf{R}_+ \times \mathbf{R}$, where $\mathbf{R}_+$ denotes the non-negative real numbers. For each subset $\Delta \in \mathcal{R}^+$, define the complex Gaussian variable $B(\Delta) = R(\Delta) + iS(\Delta)$ $(i = \sqrt{-1})$ by

$$E[B(\Delta)] = 0 \tag{1}$$

and

$$E[B(\Delta_1)B(\Delta_2)^*] = m(\Delta_1 \cap \Delta_2) \tag{2}$$

where $E[\cdot]$ denotes mathematical expectation and $m$ denotes Lebesgue measure on $\mathbf{R}^2$. $^*$ denotes complex conjugation. Equation (2) is complemented by the requirements that

$$E[R(\Delta_1)R(\Delta_2)] = E[S(\Delta_1)S(\Delta_2)] \tag{3}$$

and

$$E[R(\Delta_1)S(\Delta_2)] = 0 \tag{4}$$

Equations (1)-(4) completely specify the Gaussian variables $B(\Delta)$ for every $\Delta \in \mathcal{R}^+$. This definition is transferred to the family $\mathcal{R}^-$ of all (finite) rectangular subsets of $\mathbf{R}_- \times \mathbf{R}$ by defining

$$B(\Delta) = B(\Delta^-)^* \qquad \forall\ \Delta \in \mathcal{R}_- \tag{5}$$

where $\Delta^- = (-b, -a) \times (c, d)$ if $\Delta = (a, b) \times (c, d)$, and with analogous definitions for the other types of pertinent rectangular subsets. A further extension to the Boolean algebra $\mathcal{A}$ generated by $\mathcal{R}^+ \cup \mathcal{R}^-$ is now made by adding the following condition.

For every $k = 2, 3, \ldots$

$$B(\Delta_1 \cup \ldots \cup \Delta_k) = B(\Delta_1) + \ldots + B(\Delta_k) \qquad \text{(a.s.)} \tag{6}$$

if $\Delta_i \in \mathcal{R}^+ \cup \mathcal{R}^-$, $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j = 1, \ldots, k$, where $\emptyset$ denotes the empty set, and (a.s.) denotes almost surely wrt $P$. It is realized that equation (6) implies the relation $B(A_1 \cup A_2) = B(A_1) + B(A_2)$ (a.s.) for any pair $A_1, A_2 \in \mathcal{A}$ if $A_1 \cap A_2 = \emptyset$.

This makes $B$ a well-defined elementary orthogonal Gaussian measure on $\mathcal{A}$ (Gihman & Skorohod 1974). Since $m$ is a complete measure, $B$ has a unique extension in quadratic mean (q.m.) to an orthogonal Gaussian measure, which is still denoted by $B$, on the $\sigma$-algebra $\mathcal{B}$ of Borel sets in $\mathbf{R}^2$ (Gihman & Skorohod 1974). $B$ satisfies the property that $E[B(A_1)B(A_2)^*] = m(A_1 \cap A_2)$ for every pair of sets $A_1, A_2 \in \mathcal{B}$.

The following incremental form of equation (2) is used extensively in the subsequent derivations. Letting $B_{ab}(da\,db) = B((a, a+da) \times (b, b+db))$, it is obtained that

$$E[B_{x_1 x_2}(dx_1 dx_2) B_{y_1 y_2}(dy_1 dy_2)^*] = \delta_{x_1 y_1}\, \delta_{x_2 y_2}\, dx_1\, dx_2 \tag{7}$$

where $\delta_{xy} = 1$ if $x = y$ and $\delta_{xy} = 0$ if $x \neq y$. Usually the indices on $B$ will not be written if the values of these indices are clear from the context or are unimportant. By construction, the relation $B(d(-x)\, dy) = -\, B(dx\, dy)^*$ also obtains.

Let $L^p = L^p(\mathbf{R}^2)$ $(p > 0)$ denote the space of $p$-integrable functions on $\mathbf{R}^2$, that is, $f \in L^p$ if $\|f\|_p = \{\int_{\mathbf{R}^2} |f|^p\}^{1/p} < \infty$. In applications, a central concept in the modeling of stationary stochastic processes is the spectral density or the variance spectrum. In keeping with this, let $\Psi \in L^1$, $\Psi \geq 0$ be given. Then the following equation defines a stochastic process $X(t)$, $t \in \mathbf{R}$, (Gihman & Skorohod 1974)

$$X(t) = \int_{\mathbf{R}^2} \sqrt{\Psi(\omega, \beta)}\, e^{i\omega t}\, B(d\omega\, d\beta) \tag{8}$$

where the stochastic integral on the rhs of equation (8) is defined in a q.m. sense. We now have the following theorem

**Theorem 1** *If the function $\Psi \in L^1$, $\Psi \geq 0$, satisfies the symmetry relation $\Psi(-\omega, \beta) = \Psi(\omega, \beta)$, then the stochastic process $X(t)$ defined by equation (8) is a real, q.m. continuous, zero-mean, stationary Gaussian process with a spectral density $S_X(\omega)$ given by*

$$S_X(\omega) = \int_{\mathbf{R}} \Psi(\omega, \beta)\, d\beta \tag{9}$$

*Proof*: $X(t)$ is a zero-mean Gaussian process by the contruction of $B$. $X(t)$ is also real since

$$\begin{aligned} X(t)^* &= \int_{\mathbf{R}^2} \sqrt{\Psi(\omega, \beta)}\, e^{-i\omega t}\, B(d\omega\, d\beta)^* \\ &= \int_{\mathbf{R}^2} \sqrt{\Psi(-\,\omega, \beta)}\, e^{i\omega t}\, B(d\omega\, d\beta) = X(t) \end{aligned} \tag{10}$$

where the relation $B(d(-\omega)\, d\beta) = -\, B(d\omega\, d\beta)^*$ and the symmetry property of $\Psi$ have been used. Also, from equation (8) it is obtained, by invoking the properties of $B$, that

$$E[X(t)\, X(t+\tau)] = \int_{\mathbf{R}^2} \Psi(\omega, \beta)\, e^{i\omega\tau}\, d\omega\, d\beta \tag{11}$$

The Bochner and spectral representation theorems (Cramer & Leadbetter 1967) then show that $X(t)$ is stationary and that the spectral density of $X(t)$ is given by equation (9). It is clear that the stochastic process $\zeta(\omega)$ defined by $\zeta(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\omega} \sqrt{\Psi(\sigma, \beta)}\, B(d\sigma\, d\beta)$, provides us with a spectral process of orthogonal increments associated with $X(t)$. $\zeta(\omega)$ satisfies the relation $E[\,|\, d\zeta(\omega)\,|^2] = S_X(\omega)\, d\omega$. $\square$

Note that $d_\beta X(t)$ defined by the equation

$$d_\beta X(t) = \int_{\mathbf{R}} \sqrt{\Psi(\omega, \beta)}\, e^{i\omega t}\, B(d\omega\, d\beta) \tag{12}$$

where the integration is carried out wrt $\omega$, can be considered as the incremental components wrt the parameter $\beta$ adding up to give $X(t)$. In the context of modeling a multi-directional random wave field, the $d_\beta X(t)$ expresses the incremental directional components adding up to produce the multi-directional sea. In this case, the parameter $\beta$ would signify the direction of propagation of the incremental component interpreted as a unidirectional wave field, and $\beta$ would typically be a number in the interval $[0, 2\pi)$. In the same manner, $\omega$ would then represent the wave frequency (Naess 1990). It is obvious that $d_\beta X(t)$ and $d_{\beta'} X(t)$ are independent for $\beta \neq \beta'$.

## 3 Representation of the Response Process

In the context of an offshore structure subjected to a random multi-directional stationary wave field, a second order, non-linear structural response mode, $Z(t)$ say, can be expressed as

$$Z(t) = \int_{\mathbf{R}^4} h(\tau, \tau'; \beta, \beta')\, d_\beta X(t-\tau)\, d_{\beta'} X(t-\tau')\, d\tau\, d\tau' \tag{13}$$

where the real (finite) function $h(\tau, \tau'; \beta, \beta')$ denotes what is known as a bidirectional quadratic impulse response function for the considered response mode. Obviously, there is no restriction imposed by assuming that $h$ is symmetrical, that is, $h(\tau, \tau'; \beta, \beta') = h(\tau', \tau; \beta', \beta)$. Note that for a physically realizable system, $h(\tau, \tau'; \beta, \beta') = 0$ if $\tau < 0$ or $\tau' < 0$. A representation of the response process as given by equation (13) (strictly speaking without the dependence on $\beta$) is sometimes referred to as the second order term of a stochastic Volterra series, cf. Schetzen (1980).

Which conditions must be satisfied by the function $h$ to ensure the existence of the integral on the rhs of equation (13) as a q.m. integral? This is answered by the next theorem, where the following function is introduced

$$r(t, \beta) = \int_{\mathbf{R}} \Psi(\omega, \beta)\, e^{i\omega t}\, d\omega \tag{14}$$

**Theorem 2** *Assume that $X(t)$ is given as in Theorem 1. Then the integral on the rhs of equation (13) is well defined as a q.m. integral if the following two integrals are well defined as Riemann integrals*

$$I_1 = \int_{\mathbf{R}^4} h(t, t'; \beta, \beta)\, h(s, s'; \beta', \beta')\, r(t-t'; \beta)\, r(s-s'; \beta')\, dt\, dt'\, ds\, ds'\, d\beta\, d\beta' \tag{15}$$

$$I_2 = \int_{\mathbf{R}^4} h(t, t'; \beta, \beta')\, h(s, s'; \beta, \beta')\, r(t-s; \beta)\, r(t'-s'; \beta')\, dt\, dt'\, ds\, ds'\, d\beta\, d\beta' \tag{16}$$

*Proof*: The proof proceeds in a manner entirely analogous to that given by Cramer & Leadbetter (1967) for existence of q.m. integrals. From that proof it is realized that the existence is ensured if $E[|Z(t)|^2]$ can be expressed in terms of well defined deterministic Riemann integrals. Carrying through the calculations of $E[|Z(t)|^2]$, it is obtained that $E[|Z(t)|^2] = I_1 + 2I_2$. $\square$

**Corollary 1** *Let $\Psi$ be as in Theorem 1. If there exists $f \in L^1(\mathbf{R})$ such that $\Psi(\omega, \beta) \leq f(\omega)$ for a.e. $(\omega, \beta) \in \mathbf{R}^2$, then the integral on the rhs of equation (13) is well defined as a q.m. integral if the function $h$ is continuous and satisfies the following three conditions: 1) The function $(s, t, \beta) \to h(s, t; \beta, \beta) \in L^1(\mathbf{R}^3)$; 2) For a.e. $(\beta, \beta') \in \mathbf{R}^2$, the function $(s, t) \to h(s, t; \beta, \beta') \in L^1(\mathbf{R}^2)$; and 3) The function $(\beta, \beta') \to \int_{\mathbf{R}^2} h(s, t; \beta, \beta')\, dsdt \in L^2(\mathbf{R}^2)$.*

*Proof*: Clearly $|r(t, \beta)| \leq \|f\|_1$, so $r$ is bounded. Continuity of $r$ follows by dominated convergence. The claimed result now follows from standard results for Riemann integration (Apostol 1957; Rudin 1987). □

A (time-invariant) quadratic filter is usually not specified by its quadratic impulse response function, but rather by the associated Fourier transform. For the situation at hand, this Fourier transform $\hat{H}(\omega, \omega'; \beta, \beta')$ is defined as follows, assuming that $h$ satisfies condition 1) of Corollary 1.

$$\hat{H}(\omega, \omega'; \beta, \beta') = \int_{\mathbf{R}^2} h(\tau, \tau'; \beta, \beta') e^{-i(\omega\tau + \omega'\tau')}\, d\tau\, d\tau' \tag{17}$$

$\hat{H}$ is called a bidirectional quadratic transfer function (BQTF) and it depends on two frequencies and two directions. This is typically the quantity provided by numerical calculations of the hydrodynamics related to the structure (Naess 1990). It should be noted that the BQTF is a more general specification of a quadratic filter than the bidirectional quadratic impulse response function, in the sense that not every valid BQTF can be obtained as the Fourier transform of a time domain function as in equation (17).

It is now assumed that a real, continuous function $h$, which is symmetrical in the same manner as above, and a bounded function $\hat{H} : \mathbf{R^4} \to \mathbf{C}$ are given. It is further assumed that $h$ and $\hat{H}$ constitute a Fourier transform pair wrt the first two variables, where $\hat{H}$ is given by equation (17). It can easily be proved that $\hat{H}$ has the following properties: $\hat{H}(\omega, \omega'; \beta, \beta') = \hat{H}(\omega', \omega; \beta', \beta)$ (symmetry) and $\hat{H}(-\omega, -\omega'; \beta, \beta') = \hat{H}(\omega, \omega; \beta, \beta')^*$.

The two functions $h$ and $\hat{H}$ determine a quadratic filter. The response of this filter to the stochastic process $X(t)$ can be expressed as in equation (13). By straight-forward calculation it can be shown that the response can be written in the alternative way

$$Z(t) = \int_{\mathbf{R}^4} \Gamma(\omega,\, \beta; \omega', \beta')\, e^{i(\omega - \omega')t}\, B(d\omega\, d\beta)\, B(d\omega'\, d\beta')^* \tag{18}$$

where we have introduced the kernel

$$\Gamma(\omega,\,\beta;\omega',\beta') = \hat{H}(\omega,-\,\omega';\beta,\beta')\,\sqrt{\Psi(\omega,\beta)\,\Psi(\omega',\beta')} \tag{19}$$

From its definition, it follows that $|\Gamma(\omega,\omega';\beta,\beta')|^2 \leq \text{const.}\,\Psi(\omega,\beta)\,\Psi(\omega',\beta')$, since $\hat{H}$ is bounded. This implies that $\Gamma \in L^2(\mathbf{R}^4)$, and that the (real) function $(\omega,\beta) \to \Gamma(\omega,\beta;\omega,\beta) \in L^1(\mathbf{R}^2)$. Hence by Corollary 2 below, the stochastic integral in equation (18) is well defined as a q.m. integral. Invoking the properties of $\hat{H}$, it follows that $\hat{H}(\omega,-\,\omega';\beta,\beta') = \hat{H}(\omega',-\,\omega;\beta',\beta)^*$. Hence the function $\Gamma$ satisfies the relation $\Gamma(\omega,\beta;\omega',\beta') = \Gamma(\omega',\beta';\omega,\beta)^*$, that is, $\Gamma$ is a Hermitian kernel. With this kernel we can associate a self-adjoint operator $\bar{\Gamma} : L^2(\mathbf{R}^2) \to L^2(\mathbf{R}^2)$, defined by the relation $\bar{\Gamma}\psi(\omega,\beta) = \int_{\mathbf{R}^2} \Gamma(\omega,\beta;\omega',\beta')\,\psi(\omega',\beta')\,d\omega'\,d\beta'$. The following important result will be needed in the later developments

**Theorem 3** *The kernel $\Gamma$ has the expansion*

$$\Gamma(\omega,\beta;\omega',\beta') = \sum_{j=1}^{\kappa} \lambda_j\,\psi_j(\omega,\beta)\,\psi_j(\omega',\beta')^* \qquad \textit{(q.m.)} \tag{20}$$

*where $\kappa \leq \infty$, $\{\lambda_j\}_{j=1}^{\kappa}$ is the (at most denumerable) set of eigenvalues ($\neq 0$), which are all real, of the operator $\bar{\Gamma}$. $\{\psi_j\}_{j=1}^{\kappa}$, $\psi_j \in L^2(\mathbf{R}^2)$, is a set of associated orthonormal eigenfunctions. That is, for each $j \in \{1,2,\ldots\,\kappa\}$,*

$$\int_{\mathbf{R}^2} \Gamma(\omega,\beta;\omega',\beta')\,\psi_j(\omega',\beta')\,d\omega'\,d\beta' = \lambda_j\psi_j(\omega,\beta) \tag{21}$$

*Proof*: $\bar{\Gamma}$ is a self-adjoint Hilbert-Schmidt operator, and the properties of such operators have been extensively discussed in the literature. Our theorem is a standard result from this theory, cf. e.g. Helmberg (1969) or Halmos & Sunder (1978). □

For the remaining part of this paper we shall assume that $\kappa = \infty$, as this is the most general and difficult case. The situation with finite $\kappa$ is basically straight-forward and simple.

The following property of the eigenfunctions is also required

**Theorem 4** *The set of orthonormal eigenfunctions $\{\psi_j\}_{j=1}^{\infty}$ can be assumed to satisfy the property that*

$$\psi_j(-\,\omega,\beta) = \psi_j(\omega,\beta)^* \qquad \textit{(a.s.)} \tag{22}$$

*Proof*: Define the function $\tilde{\psi}$ by the relation $\tilde{\psi}(\omega, \beta) = \psi(-\omega, \beta)^*$. Invoking the property $\hat{H}(-\omega, -\omega'; \beta, \beta') = \hat{H}(\omega, \omega; \beta, \beta')^*$, it is seen that $\Gamma(-\omega, \beta; \omega', \beta') = \Gamma(\omega, \beta; -\omega', \beta')^*$. Hence it follows that

$$\int_{\mathbf{R}^2} \Gamma(\omega, \beta; \omega', \beta')\, \tilde{\psi}_j(\omega', \beta')\, d\omega'\, d\beta' = \int_{\mathbf{R}^2} \Gamma(\omega, \beta; -\omega', \beta')\, \psi_j(\omega', \beta')^*\, d\omega'\, d\beta'$$

$$= \int_{\mathbf{R}^2} \Gamma(-\omega, \beta; \omega', \beta')^*\, \psi_j(\omega', \beta')^*\, d\omega'\, d\beta' = \lambda_j\, \psi_j(-\omega, \beta)^* = \lambda_j\, \tilde{\psi}_j(\omega, \beta) \quad (23)$$

showing that $\tilde{\psi}$ is an eigenfunction for the eigenvalue $\lambda$ whenever $\psi$ is an eigenfunction for the same eigenvalue $\lambda$.

Consider a specific eigenvalue $\lambda_j$ of multiplicity $m$, which is finite since the selfadjoint Hilbert-Schmidt kernel $\Gamma$ defines a compact integral operator having zero as the only accumulation point of the eigenvalues. If $\psi_j + \tilde{\psi}_j = 0$ a.s., then $\psi_j$ is replaced by $\phi_j = i\,\psi_j$, and it is seen that $\phi_j(\omega, \beta) = \phi_j(-\omega, \beta)^*$ a.s. If $\psi_j + \tilde{\psi}_j \neq 0$ a.s., then we define the function $\phi_j = \psi_j + \tilde{\psi}_j$, which is an eigenfunction clearly satisfying the required relation. If $m > 1$, then there exists an eigenfunction for the same eigenvalue $\lambda_j$, $\varphi$ say, that is orthogonal to $\phi_j$. Since $\phi_j$ satisfies equation (22), $\tilde{\varphi}$ is also orthogonal to $\phi_j$. Repeating the construction used for $\phi_j$, it is realized that the constructed set of orthogonal eigenfunctions for the eigenvalue $\lambda_j$, which satisfy equation (22), can be extended by one, and consequently by the number required to get $m$ such eigenfunctions. This can be done for every eigenvalue, and since we have a selfadjoint operator with real eigenvalues, eigenfunctions belonging to different eigenvalues are clearly orthogonal. By simple normalization we therefore obtain an orthonormal set of eigenfunctions satisfying equation (22). □

The next theorem provides us with the 'eigenprocesses' needed for the series expansion of the response process

**Theorem 5** *For every $j = 1, 2, \ldots$, the equation*

$$W_j(t) = \int_{\mathbf{R}^2} \psi_j(\omega, \beta)\, e^{i\omega t}\, B(d\omega\, d\beta) \quad (24)$$

*defines a real, q.m. continuous, stationary Gaussian process of zero mean and unit variance, and with a spectral density $S_j(\omega)$ given by the expression*

$$S_j(\omega) = \int_{\mathbf{R}} |\psi_j(\omega, \beta)|^2\, d\beta \quad (25)$$

*Proof*: Since $\psi_j \in L^2(\mathbf{R}^2)$, the stochastic integral is well defined as a q.m. integral, and clearly provides us with a zero-mean Gaussian process. It is also obtained that

$$\begin{aligned} W_j(t)^* &= \int_{\mathbf{R}^2} \psi_j(\omega,\beta)^*\, e^{-i\omega t}\, B(d\omega\, d\beta)^* \\ &= \int_{\mathbf{R}^2} \psi_j(-\omega,\beta)\, e^{-i\omega t}\, B(d(-\omega)\, d\beta) = W_j(t) \end{aligned} \tag{26}$$

showing that each $W_j(t)$ is a real process. Calculating the covariance function of $W_j(t)$, it is found that

$$E[\, W_j(t)\, W_j(t+\tau)\,] = \int_{\mathbf{R}^2} |\psi_j(\omega,\beta)|^2\, e^{i\omega\tau}\, d\omega\, d\beta = \int_{\mathbf{R}^2} |\psi_j(\omega,\beta)|^2\, d\beta\, e^{i\omega\tau}\, d\omega \tag{27}$$

This shows that each $W_j(t)$ has a spectral density given by equation (25), which immediately leads to $Var[W_j(t)] = 1$. $\square$

The main theorem is now the following

**Theorem 6** *For a self-adjoint Hilbert-Schmidt kernel $\Gamma$ as defined above, the stochastic response $Z(t)$ of the associated quadratic filter subjected to a zero-mean stationary Gaussian input, can be represented as follows*

$$Z(t) - E[Z(t)] = \sum_{j=1}^{\infty} \lambda_j\, \left\{W_j(t)^2 - 1\right\} \qquad \textit{(q.m.)} \tag{28}$$

*and uniformly in $t$, where $\{\lambda_j\}_{j=1}^{\infty}$ denote the eigenvalues of $\Gamma$, and $W_j(t)$, $j \in \mathbf{N}$, are defined by equation (24).*

*Proof*: It is convenient to divide the proof into two lemmas.

**Lemma 1** *Assume that $\gamma \in L^2(\mathbf{R}^4)$, $\gamma(\omega,\beta;\omega',\beta') = \gamma(\omega',\beta';\omega,\beta)^*$, and that the function $(\omega,\beta) \to \gamma(\omega,\beta;\omega,\beta) \in L^1(\mathbf{R}^2)$. Then*

$$\begin{aligned} &E\left[\left|\int_{\mathbf{R}^4} \gamma(\omega,\,\beta;\omega',\beta')\, e^{i(\omega-\omega')t}\, B(d\omega\, d\beta)\, B(d\omega'\, d\beta')^*\right|^2\right] \\ &= \left(\int_{\mathbf{R}^2} \gamma(\omega,\beta;\omega,\beta)\, d\omega\, d\beta\right)^2 + 2\int_{\mathbf{R}^4} |\gamma(\omega,\beta;\omega'\beta')|^2\, d\omega\, d\omega'\, d\beta\, d\beta' \end{aligned} \tag{29}$$

*Proof*: By writing out the lhs of equation (29), and invoking the relation between higher order and lower order statistical moments of Gaussian variables, specifically $E[X_1X_2X_3X_4] = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4]E[X_2X_3]$ if $X_j$, $j = 1, ..., 4$, are Gaussian variables, the result of the lemma follows by using the orthogonality property of the increments of the Gaussian measure $B$, cf. equation (7). Note that $\gamma(\omega, \beta; \omega, \beta)$ is real.

**Corollary 2** *Let the function $\gamma$ be defined as in Lemma 1. Then the stochastic integral $\int_{\mathbf{R}^4} \gamma(\omega,\, \beta; \omega', \beta')\;\; e^{i(\omega-\omega')t}\, B(d\omega\, d\beta)\, B(d\omega'\, d\beta')^*$ exists as a q.m. integral if and only if the two multiple integrals on the rhs of equation (29) exist and are finite.*

**Lemma 2** *Let $\gamma_n$ denote the function defined by the equation $\gamma_n(\omega, \beta; \omega', \beta') = \Gamma(\omega, \beta; \omega', \beta') - \sum_{j=1}^{n} \lambda_j\, \psi_j(\omega, \beta)\, \psi_j(\omega', \beta')^*$. Define the quantity $\epsilon_n$ by the equation*

$$\epsilon_n = E\left[\left|Z(t) - E[Z(t)] - \sum_{j=1}^{n} \lambda_j\; \left\{W_j(t)^2 - 1\right\}\right|^2\right] \tag{30}$$

*Then $\epsilon_n = 2\, ||\gamma_n||_2^2$.*

*Proof*: It is first observed that $\gamma_n$ satisfies the same conditions as the function $\gamma$ in Lemma 1, so that Lemma 1 applies to $\gamma_n$. It is seen that $\epsilon_n = E[\,|Z(t) - \sum_{j=1}^{n} \lambda_j W_j(t)^2 - D_n|^2\,]$ where $D_n = E[Z(t)] - \sum_{j=1}^{n} \lambda_j = E[\,Z(t) - \sum_{j=1}^{n} \lambda_j W_j(t)^2\,]$. Hence, $\epsilon_n = Var[Z(t) - \sum_{j=1}^{n} \lambda_j W_j(t)^2] = E[\,|Z(t) - \sum_{j=1}^{n} \lambda_j W_j(t)^2|^2\,] - D_n^2$. By introducing the representations of $Z(t)$ and $W_j(t)$ given by equations (18) and (24), respectively, and invoking Lemma 1, it is obtained that $\epsilon_n = 2\int_{\mathbf{R}^4} |\gamma_n(\omega, \beta; \omega'\beta')|^2\, d\omega\, d\omega'\, d\beta\, d\beta' = 2\, ||\gamma_n||_2^2$, since clearly, $D_n = \int_{\mathbf{R}^2} \gamma_n(\omega, \beta; \omega, \beta)\, d\omega\, d\beta$.

The proof of Theorem 6 now follows from the results of Theorem 3. □

Clearly,

$$E[\,Z(t)\,] = \int_{\mathbf{R}^2} \Gamma(\omega, \beta; \omega, \beta)\, d\omega\, d\beta \tag{31}$$

The next theorem sums up some basic properties of the response process $Z(t)$ given by equations (18) and (19).

**Theorem 7** *The response process $Z(t)$ given by equation (18) is a q.m. continuous, (weakly) stationary process with mean value given by equation (31) and a spectral density*

$$S_Z(\omega) = 2 \int_{\mathbf{R}^3} |\Gamma(\omega', \beta; \omega' - \omega, \beta')|^2 \, d\omega' \, d\beta \, d\beta' \tag{32}$$

*The variance of $Z(t)$ is given as follows*

$$Var[\, Z(t)\,] = \int_{\mathbf{R}} S_Z(\omega) \, d\omega = 2\|\Gamma\|_2^2 = 2 \sum_{j=1}^{\infty} \lambda_j^2 \tag{33}$$

*Proof*: Calculating $E[\, Z(t)\, Z(t+\tau)\,]$ it is found that

$$\begin{aligned}
& E[\, Z(t)\, Z(t+\tau)\,] \\
= \;& 2 \int_{\mathbf{R}^4} |\Gamma(\omega, \beta; \omega', \beta')|^2 \, e^{i\,(\omega - \omega')\tau} d\omega \, d\omega' \, d\beta \, d\beta' + \left( \int_{\mathbf{R}^2} \Gamma(\omega, \beta; \omega, \beta) \, d\omega \, d\beta \right)^2 \\
= \;& \int_{\mathbf{R}} \left( 2 \int_{\mathbf{R}^3} |\Gamma(\omega, \beta; \omega - \varpi, \beta')|^2 \, d\omega \, d\beta \, d\beta' \right) e^{i\,\varpi\,\tau} \, d\varpi \\
+ \;& \left( \int_{\mathbf{R}^2} \Gamma(\omega, \beta; \omega, \beta) \, d\omega \, d\beta \right)^2
\end{aligned} \tag{34}$$

This proves that $Z(t)$ is q.m. continuous and weakly stationary with a spectral density as specified.

The two first equalities of equation (33) are obvious. To prove the last equality, we define $\Gamma_n(\omega, \beta; \omega', \beta') = \sum_{j=1}^{n} \lambda_j \, \psi_j(\omega, \beta) \, \psi_j(\omega', \beta')^*$. Then $\Gamma_n \to \Gamma$ in $L_2(\mathbf{R}^4)$ by equation (20), which implies that $\|\Gamma_n\|_2 \to \|\Gamma\|_2$ as $n \to \infty$. It is verified that $\|\Gamma_n\|_2^2 = \sum_{j=1}^{n} \lambda_j^2$. Hence it follows that $\sum_{j=1}^{\infty} \lambda_j^2$ converges, and $\|\Gamma\|_2^2 = \sum_{j=1}^{\infty} \lambda_j^2$. □

A similar result to the last equality of equation (33) cannot be proved for the mean value without additional information. The following theorem provides a partial answer.

**Theorem 8** *If $\sum_{j=1}^{\infty} |\lambda_j|$ converges and $\Gamma(\omega, \beta; \omega, \beta) = \sum_{j=1}^{\infty} \lambda_j \, |\psi_j(\omega, \beta)|^2$ a.s., then*

$$E[\, Z(t)\,] = \sum_{j=1}^{\infty} \lambda_j \tag{35}$$

*Proof*: Let $g_n(\omega,\beta) = \sum_{j=1}^n \lambda_j \, |\psi_j(\omega,\beta)|^2$ and $f_n(\omega,\beta) = \sum_{j=1}^n |\lambda_j| \, |\psi_j(\omega,\beta)|^2$. Then $f = \lim_{n\to\infty} f_n$ is a well defined, measurable function, and by monotone convergence, $\|f_n\|_1 = \sum_{j=1}^n |\lambda_j| \to \|f\|_1 = \sum_{j=1}^\infty |\lambda_j|$ as $n \to \infty$. Now, $|g_n| \le |f_n| \le |f|$. Hence, by dominated convergence, $\int_{\mathbf{R}^2} g_n(\omega,\beta)\, d\omega\, d\beta = \sum_{j=1}^n \lambda_j \to \int_{\mathbf{R}^2} \Gamma(\omega,\beta;\omega,\beta)\, d\omega\, d\beta$ as $n \to \infty$. $\square$

## 4 Slow-Drift Approximation

In the application of quadratic filters to the investigation of the response statistics of large compliant offshore structures, it is of particular interest to impose the so-called slow-drift condition. This amounts to introducing the approximation that $\hat{H}(\omega,\omega';\beta,\beta') = 0$ whenever $\omega \cdot \omega' > 0$. Invoking equation (19), it follows that this is equivalent to the condition $\Gamma(\omega,\beta;\omega',\beta') = 0$ whenever $\omega \cdot \omega' < 0$. In this case, Theorem 6 can be modified as follows

**Theorem 9** *Let the kernel $\Gamma$ be as specified in Theorem 6, and assume that it satisfies the slow-drift condition. Then each eigenvalue of equation (21) is of multiplicity at least two. The stochastic response $Z^{(-)}(t)$ of the associated quadratic filter subjected to a zero-mean stationary Gaussian input as given by equation (8), can be represented as follows*

$$Z^{(-)}(t) - E[Z^{(-)}(t)] = \sum_{j=1}^{\infty} \mu_j \, \left\{ W_{2j-1}(t)^2 + W_{2j}(t)^2 - 2 \right\} \qquad \textit{(q.m.)} \tag{36}$$

*and uniformly in $t$, where the eigenvalues $\lambda_j$ of equation (21) have been ordered so that $\mu_j = \lambda_{2j-1} = \lambda_{2j}$ for $j \in \mathbf{N}$. Each $W_j(t)$ is given by equation (24), where $\psi_{2j}(\omega,\beta) = \chi(\omega)\, \psi_{2j-1}(\omega,\beta)$ and $\chi(\omega) = -\,i$ for $\omega > 0$, $= 0$ for $\omega = 0$, and $= i$ for $\omega < 0$*

*Proof*: Consider the reduced eigenvalue problem

$$\int_{-\infty}^{\infty} \int_0^{\infty} \Gamma(\omega,\beta;\omega',\beta')\, \phi(\omega',\beta')\, d\omega'\, d\beta' = \mu\phi(\omega,\beta) \qquad \big((\omega,\beta) \in \mathbf{R}_+ \times \mathbf{R}\big) \tag{37}$$

Let $\mu_j$ and $\phi_j$, $j = 1, 2, \ldots$, be the eigenvalues ($\neq 0$) and an associated orthonormal set of eigenfunctions of this integral equation, cf. Theorem 3, which is also valid with the required changes for this reduced problem. We now make the following definitions: For $j = 1, 2, \ldots,$

$$\lambda_{2j-1} = \lambda_{2j} = \mu_j \tag{38}$$

$$\psi_{2j-1}(\omega,\beta) = \begin{cases} \frac{1}{\sqrt{2}}\,\phi_j(\omega,\beta) & : \quad \omega \geq 0, \beta \in \mathbf{R} \\ \frac{1}{\sqrt{2}}\,\phi_j(-\,\omega,\beta)^* & : \quad \omega < 0, \beta \in \mathbf{R} \end{cases} \tag{39}$$

$$\psi_{2j}(\omega,\beta) = \begin{cases} \frac{-i}{\sqrt{2}}\,\phi_j(\omega,\beta) & : \quad \omega \geq 0, \beta \in \mathbf{R} \\ 0 & : \quad \omega = 0, \beta \in \mathbf{R} \\ \frac{i}{\sqrt{2}}\,\phi_j(-\,\omega,\beta)^* & : \quad \omega < 0, \beta \in \mathbf{R} \end{cases} \tag{40}$$

Clearly, $\psi_{2j}(\omega,\beta) = \chi(\omega)\,\psi_{2j-1}(\omega,\beta)$ by construction. Invoking the properties of $\Gamma$, it is now a straight-forward task to check that $\{\psi_j\}_{j=1}^{\infty}$ is an orthonormal set of eigenfunctions associated with the set of eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ satisfying equation (21). Since Theorem 3 also applies to the reduced eigenvalue problem, it now follows that the constructed set of eigenvalues and eigenfunctions gives us the results of Theorem 3, specifically equation (20), for the full problem. It is also seen that by construction, the results of Theorem 4 are valid. $\Box$

As is well known, the function $\chi(\cdot)$ is the linear transfer function of the Hilbert transform, cf. Cramer & Leadbetter (1967). This implies, by linearity of the transform, that $W_{2j}(t)$ is the Hilbert transform of $W_{2j-1}(t)$. Consequently, $Z_j(t) = W_{2j-1}(t)^2 + W_{2j}(t)^2$ becomes the square of the Hilbert envelope of $W_{2j-1}(t)$ (or $W_{2j}(t)$). This fact points to the physical implication of the slow-drift approximation. Specifically, the spectral density of the slow-drift response can be non-zero at most at the frequencies that are obtained as the difference of the positive frequencies occurring in the spectral density $S_X(\cdot)$ of the input process, that is, for positive frequencies where $S_X(\cdot)$ is non-zero. Let Supp$(f)$ denote the support of the real-valued function $f$. That is, if $f$ is defined in some set $D$, then Supp$(f) = \{x \in D : f(x) \neq 0\}$. From the integral on the rhs of the first equality of the proof of theorem 7, it is seen that the slow-drift condition implies that Supp$(S_Z) \subseteq D_X$, where $D_X = \{\omega - \omega' : \omega, \omega' \in \text{Supp}(S_X^+)\}$. Here, $S_X^+(\omega) = 2\,S_X(\omega)$ for $\omega \geq 0$ and

$S^+(\omega) = 0$ for $\omega < 0$, that is, $S_X^+$ denotes the one-sided, or physical, spectral density of $X(t)$. Similarly, it follows that $\mathrm{Supp}(S_{Z_j}) \subseteq D_X$, where $S_{Z_j}$ denotes the spectral density of $Z_j(t)$. By the same token, $\mathrm{Supp}(S_{W_j}) \subseteq \mathrm{Supp}(S_X)$

# 5 Sum-Frequency Approximation

Similarly as with the slow-drift approximation, the sum-frequency approximation is also motivated by a practical observation. It has turned out that certain motion responses of a moored offshore structure like a tension leg platform, is only sensitive to frequencies in the force spectrum that are larger than those found in the wave spectrum, that is in $\mathrm{Supp}(S_X^+)$. Typically, such a motion response is also modeled as a second order, non-linear response mode as described in section 3.

To achieve an expedient representation of such a response process, the sum-frequency approximation is implemented by imposing the condition that $\hat{H}(\omega, \omega'; \beta, \beta') = 0$ whenever $\omega \cdot \omega' < 0$. From equation (19), it follows that this is equivalent to the condition $\Gamma(\omega, \beta; \omega', \beta') = 0$ whenever $\omega \cdot \omega' > 0$. Note that this condition immediately implies that $E[\, Z(t)\, ] = 0$ by equation (31).

By a discussion similar to that used for the slow-drift approximation, it follows from the proof of Theorem 7 that the spectral density of the response process obtained by introducing the sum-frequency approximation only contains the sum of frequencies occurring in the wave spectrum, that is, in $\mathrm{Supp}(S_X^+)$. This can in fact also be seen from the response representation given by equation (18) in view of the sum-frequency condition imposed on the kernel $\Gamma$. For the present case, Theorem 6 can be expressed in the following form

**Theorem 10** *Let the kernel $\Gamma$ be as specified in Theorem 6, and assume that it satisfies the sum-frequency condition. Then to each eigenvalue of equation (21) there corresponds an eigenvalue of the same magnitude, but of opposite sign. The stochastic response $Z^{(+)}(t)$ of the associated quadratic filter subjected to a zero-mean stationary Gaussian input as given by equation (8), can be represented as follows*

$$Z^{(+)}(t) = \sum_{j=1}^{\infty} \nu_j \, \left\{ W_{2j-1}(t)^2 - W_{2j}(t)^2 \right\} \qquad (q.m.) \tag{41}$$

*and uniformly in $t$, where the eigenvalues $\lambda_j$ of equation (21) have been ordered so that $\nu_j = \lambda_{2j-1} = -\lambda_{2j}$ for $j \in \mathbf{N}$. Each $W_j(t)$ is given by equation (24), where $\psi_{2j}(\omega,\beta) = \chi(\omega)\,\psi_{2j-1}(\omega,\beta)$ and $\chi(\omega) = -i$ for $\omega > 0$, $= 0$ for $\omega = 0$, and $= i$ for $\omega < 0$*

*Proof*: With the proof of Theorem 6 in mind, we only need to verify that $-\lambda$ is an eigenvalue of the operator $\bar{\Gamma}$ if $\lambda$ is, and that the corresponding eigenfunctions can be related as stated. Hence, assume that $\lambda$ is an eigenvalue of $\bar{\Gamma}$ with an associated eigenfunction $\psi$. Define the function $\hat{\psi}$ by the relation $\hat{\psi}(\omega,\beta) = \chi(\omega)\,\psi(\omega,\beta)$ for $(\omega,\beta) \in \mathbf{R}^2$.

For $\omega < 0$:

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(\omega,\beta;\omega',\beta')\,\hat{\psi}(\omega',\beta')\,d\omega'\,d\beta' \\
= &\int_{-\infty}^{\infty}\int_{0}^{\infty} \Gamma(\omega,\beta;\omega',\beta')\,\hat{\psi}(\omega',\beta')\,d\omega'\,d\beta' \\
= &-i\int_{-\infty}^{\infty}\int_{0}^{\infty} \Gamma(\omega,\beta;\omega',\beta')\,\psi(\omega',\beta')\,d\omega'\,d\beta' \\
= &-i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(\omega,\beta;\omega',\beta')\,\psi(\omega',\beta')\,d\omega'\,d\beta' \\
= &-i\,\lambda\,\psi(\omega,\beta) = -\lambda\,\hat{\psi}(\omega,\beta)
\end{aligned} \tag{42}$$

An entirely similar argument shows that the far lhs and the far rhs of equation (42) are equal also for $\omega > 0$. For $\omega = 0$, it is obviously satisfied. In view of Theorem 6, and the fact that $E[\,Z^{(+)}(t)\,] = 0$, we can clearly obtain the stated representation of the response process $Z^{(+)}(t)$. $\square$

The representations we have obtained in the previous sections for the response process of a quadratic filter subjected to stationary Gaussian noise, are ideally suited for practical calculations of the response statistics of such filters. Such questions will be pursued in a subsequent paper. It may also be noted that the theory presented may easily be extended from the case of one scalar parameter $\beta$ to cover the situation where this parameter is replaced by a vector of scalar parameters $\beta = (\beta_1,\dots,\beta_n)$, $n \in \mathbf{N}$. We have chosen to focus here on the case of one scalar parameter since this has a direct physical motivation.

# References


Apostol, T. M. (1957). *Mathematical Analysis.* Reading, Mass.: Addison-Wesley Publ. Comp.

Cramer, H. & M. R. Leadbetter (1967). *Stationary and Related Stochastic Processes.* New York: John Wiley & Sons, Inc.

Gihman, I. I. & A. V. Skorohod (1974). *The Theory of Stochastic Processes I.* New York: Springer-Verlag.

Halmos, P. R. & V. S. Sunder (1978). *Bounded Integral Operators on $L^2$-Spaces.* Berlin: Springer-Verlag.

Helmberg, G. (1969). *Introduction to Spectral Theory in Hilbert Space.* Amsterdam: North-Holland Publ. Comp.

Kac, M. & A. J. F. Siegert (1947). On the theory of noise in radio receivers with square law detectors. *Journal of Applied Physics 18*, 383–397.

Major, P. (1980). *Multiple Wiener-Itô Integrals.* Lecture Notes in Mathematics No. 849. Berlin: Springer-Verlag.

Naess, A. (1990). Statistical Analysis of Nonlinear Second-Order Forces and Motions of Offshore Structures in Short-Crested Random Seas. *Probabilistic Engineering Mechanics 5*(4), 192–204.

Neal, E. (1974). Second order hydrodynamic forces due to stochastic excitation. In *Proceedings 10th ONR Symposium*, Cambridge, Mass.

Rudin, W. (1987). *Real and Complex Analysis* (third edition ). New York: McGraw-Hill.

Schetzen, M. (1980). *The Volterra and Wiener Theories of Non-Linear Systems.* New York: John Wiley and Sons, Inc.