Geostatistical approach to event migration of seismic reflection times

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Abstract

Zero offset traveltimes from reflection seismic, are integrated in a Bayesian framework to localize a geological subsurface. The first arrival is fitted into the framework of piecewise affine operators. A partially analytical expression for the posterior distribution is used to develop an algorithm to sample from the posterior distribution. An efficient approximate algorithm to sample from the posterior distribution is also proposed. A synthetic example illustrate the results.

1 Introduction

Inverse problems arising in geophysics are frequently solved by using geostatistical methodology. Working with reflection time inversion, Delprat-Jannaud and Lailly (1993), recognize the benefit of defining a continuous model independent of discretization. They pose the solution as being the argument that minimize an objective function. The objective function is a trade off between the residual sum of squares and a regularization term. The uncertainty is assessed by evaluating the Hessian of the residual sum of squares for a linearized problem (Delprat-Jannaud and Lailly 1992).

When the regularization term is a norm in a reproducing kernel Hilbert space, there exists a dual probability measure for random functions, (Tarantola 1987; Wahba 1990). Using this measure as a prior in a Bayesian model, the maximum posterior estimate, is identical to the solution found by regularization. The maximum posteriori estimate is frequently used for Bayesian models (O'Sullivan 1986), although the classical Bayes estimate is the conditional expectation (Robert and Casella 1999). In general the posterior distribution may be regarded as the Bayesian answer to an inverse problem. The Bayesian methodology hence provide a stable estimate and assess the associated uncertainty.

The objective is to localize a geological subsurface, by using zero offset non-migrated reflection times. The subsurface is the borderline between two layers, it is assumed that the velocity is constant in the top layer. If the exact traveltimes are known in a region, there is an one to one relation between the traveltimes and the subsurface, see for example Kleyn (1977). The traveltimes are however discretely sampled with errors. A standard solution is to fit a spline function through the traveltimes, see for example Gjøystdal and Ursin (1981). Event migration as used in this article differs from Kirchhoff migration (Bleistein, 1987). For a background model having constant velocity, Kirchhoff migration takes events in data space and back projects them along elliptic curves. Event migration localize the recorded event along the same elliptic curve. In the current work the problem is solved by a Bayesian methodology in a Gaussian framework. The traveltime observations are fit into the framework of piecewise affine operators (Kolbjørnsen and Omre 2002). This formulation gives an expression for the posterior distribution that is partially analytical tractable. Using this expression, an algorithm to sample the posterior distribution is proposed, as well as an efficient sampling algorithm that samples an approximation to the posterior distribution. The approximate approach still honor the nonlinearities in the observations. The method is illustrated in a synthetic example.

The constant velocity model is not sufficiently complex to describe a realistic earth model. The problem is however related to reflection tomography (Farra and Madariaga 1988; Delprat-Jannaud and Lailly 1993). In reflection tomography both the subsurface and the velocity field above the subsurface is unknown. A linearized analysis of reflection tomography show that large components of the velocity field and reflector position remains undetermined by traveltime observations (Delprat-Jannaud and Lailly 1992; Bube and Meadows 1999). A Bayesian solution to the inverse problem in reflection tomography, would give a contribution by its ability represent uncertainty in such nonlinear problems.

2 Model

Consider a two layer model, with constant seismic velocity, v, in the top layer. The twice continuously differentiable Gaussian random function $\{Z(x)\,,\, x \in \mathcal{R}\}$, represents the geological subsurface being the depth to the boundary between the two layers.

The Gaussian random function is defined by its mean and its covariance,

$$\begin{aligned}
\mu_Z(x) &= \mathrm{E}\{Z(x)\}\ ,\ x \in \mathcal{R} \\
\mathrm{C}_Z(x_1, x_2) &= \mathrm{Cov}\{Z(x_1), Z(x_2)\}\ ,\ x_1, x_2 \in \mathcal{R}.
\end{aligned}$$

For the Gaussian random function to be twice differentiable with probability one, the expectation is required to be twice differentiable, and regularity conditions must be enforced to the covariance function, see Stein (1999) for details. There is however no stationarity assumption, hence a model formulation as in Bayesian kriging (Omre and Halvorsen 1989), is admissible. The space of twice continuous differentiable functions on $\mathcal{R}$ is denoted $C^2(\mathcal{R})$. To keep notation short, Z will be used to denote the random function it self, $Z(x)$ will denote the random function evaluated at the location x. Lower case letters will denote deterministic functions such as specific sample paths. Bold letters are used to denote vectors and vector valued functions.

3 Observations

The current section describes the experimental set up that produces the traveltime observations, and phrases the problem in the setting of a piecewise affine inverse problem.

Geometrical aspects of the data collection, is illustrated in Figure 1. A pulse is generated on the surface in the shot location, x_s. The pulse propagates into the ground and is reflected by the subsurface. The part of the pulse that is reflected in the location x propagates back to the surface and is the first to arrive in the receiver location x_r. In the point x, where this reflection occurs, the elliptic curve with foci at x_s and x_r is tangent to the subsurface, see Figure 1. The time from the pulse is generated in x_s until it is received at x_r is denoted the traveltime. The offset is half of the horizontal distance between the shot and receiver location. In the current presentation only zero offset traveltimes are considered hence $x_s = x_r$, and the ellipse degenerate to a circle.

For a given shot/receiver location the two way reflection time is a nonlinear multi valued functional of the geological subsurface $\{z(x)\,,\ x \in \mathcal{R}\}$. In the geological model described above there is at least one reflection for each shot. This reflection occur at minimum distance from the shot location to the subsurface. Only this first arrival is considered in this article, but a similar approach can be used to deal with multiple arrivals. For a subsurface z and a shot/receiver location x_s the traveltime of the first arrival can be expressed as:

$$t(z, x_s) = \min_{x \in \mathcal{R}} \frac{2}{v} \sqrt{z(x)^2 + (x - x_s)^2},$$

with v being the velocity in the top layer, and the value of x that obtains the minimum is the reflection location. Observations of first arrival will be fitted into the framework of piecewise affine operators as defined in Kolbjørnsen and Omre (2002).

Definition 1 (Piecewise affine operator) *An operator $\boldsymbol{K} : \mathcal{Z} \to \mathbf{R}^n$, is said to be piecewise affine, if it can be represented as:*

$$\boldsymbol{K}(z) = \boldsymbol{K}_x z + \boldsymbol{k}_x \quad \text{for} \quad z \in \mathcal{A}_x \,; \quad x \in \mathcal{X}$$

with $\mathcal{X}$ being an index set, $\{\mathcal{A}_x\}_{x\in\mathcal{X}}$ being a partition of $\mathcal{Z}$; $\boldsymbol{K}_x : \mathcal{Z} \to \mathbf{R}^n$ being bounded linear operators on $\mathcal{Z}$ and $\boldsymbol{k}_x$ being n dimensional vectors. The indexed set of triplets $\{\mathcal{A}_x, \boldsymbol{K}_x, \boldsymbol{k}_x\}_{x\in\mathcal{X}}$ are the parameters of the piecewise affine operator.

Kolbjørnsen and Omre (2002) considers both finite and continuous index sets, $\mathcal{X}$. The problem of event migration is in the current presentation adapted to the framework of piecewise affine operators having a continuous index set. For the case of a continuous index set, the partition is assumed to have a special form.

Definition 2 (Restricted linear partition) *A partition* $\{\mathcal{A}_x\}_{x\in\mathcal{X}}$ *with* $\mathcal{X} \subset \mathbf{R}^d$ *of* $\mathcal{Z}$ *is a restricted linear partition if,*

$$\mathcal{A}_x = \mathcal{C}_x \cap \{\boldsymbol{R}_x(z) = \mathbf{0}\},$$

with $\mathcal{C}_x$ *being any subset of* $\mathcal{Z}$*;* $\boldsymbol{R}_x(z) = \boldsymbol{R}_x z + \boldsymbol{r}_x$*; with* $\boldsymbol{R}_x : \mathcal{Z} \to \mathbf{R}^d$ *being a bounded linear operator; and* $\boldsymbol{r}_x$ *being a d-dimensional vector function. The set of triplets* $\{\mathcal{C}_x, \boldsymbol{R}_x, \boldsymbol{r}_x\}_{x\in\mathcal{X}}$ *are the parameters of the restricted linear partition.*

For zero offset traveltimes, measured for the shot location, x_s, define the indexed functionals,

$$K_x(z) = \frac{2}{v}\sqrt{z(x)^2 + (x - x_s)^2}\ , \tag{1}$$

$$R_x(z) = z(x) \cdot z'(x) + x - x_s\ , \tag{2}$$

with v being the velocity in the top layer; z being the subsurface; z' being the derivative of the subsurface; and x being the spatial reference. In addition define the function sets

$$\mathcal{C}_x = \left\{z \in C^2(\mathcal{R}) : z(x)^2 + (x - x_s)^2 < z(u)^2 + (u - x_s)^2;\ u \in \mathbf{R}^1 \setminus \{x\}\right\} \tag{3}$$

The functionals K_x, R_x and the function sets $\mathcal{C}_x$ correspond to the parameters of the piecewise affine operator, and the restricted linear partition. Any reflection from the subsurface, z, back to the shot location, x_s, will occur in a location x with $R_x(z) = 0$, that is the place where the subsurface is tangent to the circle with center x_s. A reflection from this point will have the traveltime $K_x(z)$. In addition only the first arrival is considered, hence the reflection from x must have the minimum time, that is $z \in \mathcal{C}_x$. The parameters of the traveltime operator are hence identified by the expressions above, but does not automatically conform to the framework of the piecewise affine operators since K_x and R_x are nonlinear functionals. However since $Z(x)$ and $Z'(x)$ can be solved exactly when $K_x(Z)$ and $R_x(Z)$ are known only a minor adjustments is needed. The exact details for this is in Appendix A.

Having observed the minimum traveltime at n shot locations, denoted $\boldsymbol{x_s} = [\,x_{s\,1},\ ...,\ x_{s\,n}\,]$, an indexed operator is made by stacking the operators corresponding to each shot location into a vector. For each shot location a new dimension is added to the index, $\boldsymbol{x} = [\,x_1,\ ...,\ x_n\,]$. This gives

$$\boldsymbol{K}(z) = \boldsymbol{K}_{\boldsymbol{x}}(z) \ \text{ for }\ z \in \mathcal{A}_{\boldsymbol{x}}\ ;\quad \boldsymbol{x} \in \mathcal{X}\ , \tag{4}$$

with

$$\boldsymbol{K}_{\boldsymbol{x}}(z) = [\, K_{x_1}(z),\, \dots, K_{x_n}(z)\,]^T \quad ,$$
$$\boldsymbol{R}_{\boldsymbol{x}}(z) = [\, R_{x_1}(z),\, \dots, R_{x_n}(z)\,]^T \quad ,$$
$$\mathcal{A}_{\boldsymbol{x}} = \bigcap_{i=1}^{n} \mathcal{C}_{x_i} \cap \{\boldsymbol{R}_{\boldsymbol{x}}(z) = \boldsymbol{0}\} \; .$$

The index, $\boldsymbol{x}$, of the resulting piecewise affine operator have an intuitive interpretation. When $z \in \mathcal{A}_{\boldsymbol{x}}$, z have the minimum distance to the shot locations $\boldsymbol{x}_s$ in the reflection locations $\boldsymbol{x}$. Since only the minimum distance is considered there will be a monotone relation between the shot locations and the reflection locations.

The exact relation between the traveltimes and the subsurface is described above, but the traveltimes are observed with errors. Let the observed random variable, $\boldsymbol{Y}$, be defined by its conditional distribution,

$$p(\boldsymbol{y}|Z = z) = N_n(\boldsymbol{K}(z), \boldsymbol{\Sigma}_\varepsilon)$$

with $\boldsymbol{K}(z)$ being defined by Expression (4); and $\boldsymbol{\Sigma}_\varepsilon$ being the the covariance matrix for the observation error. The marginal distribution of $\boldsymbol{Y}$ is not Gaussian due to the nonlinearity of $\boldsymbol{K}$.

4 Sampling the posterior

The main result in this section is an algorithm for sampling the posterior when conditioning to traveltimes. This algorithm is based on a decomposition of the posterior obtained in Kolbjørnsen and Omre (2002). The Jacobian of the restrictions in the partition, $\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{R}_{\boldsymbol{x}}(z)$, is needed to obtain the decomposition. It is further convenient to introduce the extended operators

$$\tilde{\boldsymbol{K}}_{\boldsymbol{x}}(z) = \tilde{\boldsymbol{K}}_{\boldsymbol{x}}(z) \quad \text{for} \qquad \boldsymbol{x} \in \mathcal{X},$$

with

$$\tilde{\boldsymbol{K}}_{\boldsymbol{x}} z = \begin{bmatrix} \boldsymbol{K}_{\boldsymbol{x}}(z) \\ \boldsymbol{R}_{\boldsymbol{x}}(z) \\ \boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{R}_{\boldsymbol{x}}(z) \end{bmatrix} .$$

Note that this is a collection of operators indexed by the same index as the original problem. Define the random variables $\tilde{\boldsymbol{Y}}_{\boldsymbol{x}}$ corresponding to observations of each of the operators $\tilde{\boldsymbol{K}}_{\boldsymbol{x}}$,

$$p(\tilde{\boldsymbol{y}}_{\boldsymbol{x}}|Z = z) = N_{n+d+d^2}(\tilde{\boldsymbol{K}}_{\boldsymbol{x}}(z), \tilde{\boldsymbol{\Sigma}}_\varepsilon) \; ,$$

with $\tilde{\boldsymbol{K}}_{\boldsymbol{x}}$ being the extended operator for index $\boldsymbol{x}$; and $\tilde{\boldsymbol{\Sigma}}_\varepsilon$ being an extension of $\boldsymbol{\Sigma}_\varepsilon$. The values of $\tilde{\boldsymbol{\Sigma}}_\varepsilon$ are like $\boldsymbol{\Sigma}_\varepsilon$ in the upper left $n \times n$ corner and zero otherwise.

The objective is to compute the posterior distribution of Z given $\boldsymbol{Y} = \boldsymbol{y}$. In Kolbjørnsen and Omre (2002) this distribution is derived as a mixing of truncated Gaussian distributions, under given regularity conditions. The result for the traveltime observations reads,

$$
\begin{aligned}
&p\{Z = z, Z \in \mathcal{C}_{\boldsymbol{x}}, \boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{R}_{\boldsymbol{x}}(Z) = \boldsymbol{r} | \boldsymbol{Y} = \boldsymbol{y}\} \\
&\qquad = p\{Z = z |\, Z \in \mathcal{C}_{\boldsymbol{x}}, \tilde{\boldsymbol{Y}}_{\boldsymbol{x}} = (\boldsymbol{y}, \boldsymbol{0}, \boldsymbol{r})\,\} \\
&\qquad\quad \times\ \mathrm{P}\{Z \in \mathcal{C}_{\boldsymbol{x}} |\, \tilde{\boldsymbol{Y}}_{\boldsymbol{x}} = (\boldsymbol{y}, \boldsymbol{0}, \boldsymbol{r})\,\} \times \frac{|\det(\boldsymbol{r})| p\{\,\tilde{\boldsymbol{Y}}_{\boldsymbol{x}} = (\boldsymbol{y}, \boldsymbol{0}, \boldsymbol{r})\,\}}{p\{\boldsymbol{Y} = \boldsymbol{y}\}}
\end{aligned} \tag{5}
$$

The distribution on the left hand side is the distribution of Z when Z is in $\mathcal{C}_{\boldsymbol{x}}$ and $\boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{R}_{\boldsymbol{x}}(Z)$ have the value $\boldsymbol{r}$. The posterior distribution of Z is obtained by integrating out $\boldsymbol{x}$ and $\boldsymbol{r}$. In Appendix A the expressions above are given in terms of $Z(x)$, $Z'(x)$ and $Z''(x)$, for a traveltime observation with reflection in the location x. The first term on the right hand side of Expression (5) is a truncated Gaussian distribution when the observations are without error, the case with errors being treated in Appendix A. The second term is a probability and the third term is a measure on $\mathcal{X} \times \mathbf{R}^{d^2}$. The product of the second and third term is the joint posterior density for Z being in $\mathcal{C}_{\boldsymbol{x}}$ and $\boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{R}_{\boldsymbol{x}}(Z)$ having the value $\boldsymbol{r}$. Note that $\boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{R}_{\boldsymbol{x}}(z)$ is diagonal and that the elements on the diagonal are positive for $z \in \mathcal{C}_{\boldsymbol{x}}$ due to the second order criterion for minima.

The mixing distribution of Expression (5), provides a sampling algorithm for the posterior.

Algorithm 1 *CTGM-algorithm (Continuous Truncated Gaussian Mixing)*

1. *Sample* $\boldsymbol{x}^*, \boldsymbol{r}^* \sim q(\boldsymbol{x}, \boldsymbol{r})$ *with* $q(\boldsymbol{x}, \boldsymbol{r}) \propto |\det(\boldsymbol{r})| p\{\,\tilde{\boldsymbol{Y}}_{\boldsymbol{x}} = (\boldsymbol{y}, \boldsymbol{0}, \boldsymbol{r})\,\}$
2. *Sample* $Z^* \sim p\{Z | \tilde{\boldsymbol{Y}}_{\boldsymbol{x}^*} = (\boldsymbol{y}, \boldsymbol{0}, \boldsymbol{r}^*)\}$
3. *If* $Z^* \in \mathcal{C}_{\boldsymbol{x}^*}$ *stop.*

The algorithm is a variant of rejection sampling. The acceptance rate of the algorithm is given by

$$
p_{accept} = \frac{p\{\boldsymbol{Y} = \boldsymbol{y}\}}{\int_{\mathcal{X}} \int_{\mathbf{R}^{d^2}} |\det(\boldsymbol{r})| p\{\,\tilde{\boldsymbol{Y}}_{\boldsymbol{x}} = (\boldsymbol{y}, \boldsymbol{0}, \boldsymbol{r})\,\}\, d\boldsymbol{r}\, d\boldsymbol{x}}
$$

The algorithm yields samples from the posterior when conditioning to a piecewise affine operator. The posterior density for Z being in $\mathcal{C}_{\boldsymbol{x}}$ with $\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{R}_{\boldsymbol{x}}(Z)$ having the value $\boldsymbol{r}$ is sampled correctly since the proposal distribution in Step 1 is proportional to the third factor of Expression(5) and the acceptance part in Step 3 is exactly the second factor of Expression (5). When a pair $(\boldsymbol{x}^*, \boldsymbol{r}^*)$ is accepted, the sampled value z^* is a valid sample. Note that the reflection locations, $\boldsymbol{x}$, are monotone as a function of the shot locations, $\boldsymbol{x}_s$ and that the elements of $\boldsymbol{r}$ are positive. These inequality constraints should be imposed when sampling $q(\boldsymbol{x}, \boldsymbol{r})$.

The challenging part of the algorithm is to sample the proposal distribution,

$$q(\boldsymbol{x}, \boldsymbol{r}) \propto |\det(\boldsymbol{r})| p\{\, \tilde{\boldsymbol{Y}}_{\boldsymbol{x}} = (\boldsymbol{y}, \boldsymbol{0}, \boldsymbol{r})\,\}, \tag{6}$$

which is known apart from the normalizing constant, see Appendix A for details. There are several ways to sample this distribution. One approach is to sample the distribution using a McMC algorithm. As an alternative, an algorithm that samples an approximation to $q(\boldsymbol{x}, \boldsymbol{r})$ is proposed. The approximate approach still honor the nonlinear structure of the problem since only Step 1 of the CTGM-algorithm is approximated.

The distribution $q(\boldsymbol{x}, \boldsymbol{r})$ is approximated by a sequence of truncated Gaussian distributions. Firstly $q(\boldsymbol{x}, \boldsymbol{r})$ is approximated by a Gaussian distribution, $q_G(\boldsymbol{x}, \boldsymbol{r})$. The distribution $q_G(\boldsymbol{x}, \boldsymbol{r})$ is sampled sequentially, but for each variable in the sequence the relevant constraint is imposed on the conditional distribution. The resulting approximation is neither a Gaussian nor a truncated Gaussian distribution, but the absolute magnitude of the proposal density can be assessed directly in the sampling approach. Since the exact distribution is known to a multiplicative factor, importance weights for the sampled values can be calculated. These weights can be used to remove bias in the sample. This is not done in the current study, however.

5 Example

The methodology is tested in a synthetic example. Uncertainty in the shot/receiver location is included in addition to the observation error. To correctly account for these effects, shot locations, $\boldsymbol{x_s}$, and observation errors, $\boldsymbol{\varepsilon}$, should be sampled simultaneously with $\boldsymbol{x}$ and $\boldsymbol{r}$, see Appendix A.

The prior model for the reflector is defined to be a stationary Gaussian random field,

$$Z(x) = Z_0 + \widetilde{Z}(x) \quad ; \quad x \in [0,\, 10];$$

with $p\{Z_0\} = N(2, 1)$; and $\widetilde{Z}(x)$ being a zero mean Gaussian random field with covariance $\mathrm{Cov}\{\widetilde{Z}(x), \widetilde{Z}(x+h)\} = 0.15^2 \exp\{-3\cdot(h/2)^2\}$. Several samples from the prior distribution are shown in Figure 2. Note the extreme uncertainty in the level of the samples, that make several of the samples lie partially or completely outside the scales of the figure. The realization used in this example is plotted with a thick line.

Figure 3 shows the lines that connect the shot/receiver locations with the reflecting point at the subsurface to be recovered. The uncertainty in shot location is assigned standard deviation $\sigma_{x_s} = 10$ m. The observation error of the traveltime is assigned standard deviation $\sigma_t = 0.05$ sec. The interval velocity is assumed to be $v = 1$ km/sec. In Figure 4 the observed traveltimes and assumed shot locations are plotted together with the true traveltimes, which of course can be exactly computed in this synthetic example. The CTGM-algorithm is implemented in two simulation studies. In the first study only the five traveltimes that is marked in Figure 4 is used, in the second all 81 traveltime observations are used.

Figure 5 display the sample values from the study when only five traveltime observations are used. The proposal distribution $q(\boldsymbol{x}, \boldsymbol{r})$ is sampled by a McMC-algorithm. The acceptance rate in the final step of the CTGM-algorithm, is 98%, leaving the sampling of q as the most time consuming part. Compared with the samples from the prior distribution, see Figure 2, there is a dramatic reduction in uncertainty. The level is well determined and, the true subsurface is within the ensemble of samples from the posterior distribution.

Figure 6 display the sample values from the study when all 81 traveltime observations are used. The approximate sampling approach is used. The observed acceptance rate is 4.95% in the final step of the CTGM-algorithm. The relatively low acceptance rate is partially due to numerical instability in this particular problem since extreme smoothness is imposed by the second order exponential correlation function. In this case the uncertainty is very low within the region containing observations. At both ends there is larger uncertainty. Some of the samples have extreme values at the end of the interval. This is an artifact that is caused by the interaction of the approximate sampling algorithm and the extreme smoothness of the second order exponential correlation function. Apart from the valley between 6 and 7 km the true subsurface is well within the ensemble of samples from the posterior distribution. The deviation in this valley can partially be explained by the actual observation errors in this region, see Figure 4.

The ensemble of samples can be combined to a single estimate. To reduce the influence of the outliers, see Figure 6, the pointwise median is used as an estimator. For comparison the standard estimate based on inverting the smoothed traveltimes is computed. In Figure 7 (a)-(c) the estimates are compared with each other and to the true subsurface. The standard estimate have higher, sharper tops and more shallow, broader valleys, than both the true subsurface and the proposed estimate. The standard estimate miss out both the valley around 4 km and the valley between 6 and 7 km. In the flat regions such as the interval between 1 and 3 km the standard estimate is as good as the proposed. The cause of the difference in the estimates is that the smoothing of the traveltimes in the standard approach, implicitly impose a spatial assumption of stationarity for the traveltime as a function of the shot location. In the proposed approach the stationarity is imposed directly on the subsurface.

6 Conclusions

Zero offset traveltimes from reflection seismic are used to localize a geological subsurface. A Bayesian approach is developed by first defining the prior and likelihood and next condition to the observations. The posterior distribution is explored by sampling. Further an approximate sampling algorithm that honor the nonlinearities of the problem is proposed.

The methodology yields satisfactory results when evaluated in an example. The uncertainty can be represented when few observations are present, and the function is well recovered within the shot section when a realistic amount of data is used. When compared to a standard estimate, the proposed approach is computationally more expensive, but gives a better estimate in curved sections.

Observations being linear operators of the random field, such as well observations can be included, by using the conditional distribution as input to the algorithm, or by extending the affine operator. Seismic observations with offset, can be treated in the same frame work and the method extend to traveltimes in 3-D. A small inhomogeneous deviation from the constant velocity can also be accounted for by using a perturbation argument, this will produce a colored error term that is dependent on the reflection point, and is a first step in the direction of surface reflection tomography.

Acknowledgment

I am grateful to my advisor Henning Omre for valuable comments. Attending the mathematical geophysical summer school at Stanford university, august 1999, and several discussions with professor Paul Switzer at Stanford university have been beneficial. Funding for the research is provided by a PhD grant from the the Research Council of Norway.

References

Bleistein, N. (1987) "On the imaging of reflectors in the earth." Geophysics, Vol. 52, pp 931-942.

Bube K.P. and Meadows M.A. (1999) "The null space of a generally anisotropic medium in linearized surface reflection tomography", Geophys.J.Int., Vol. 139, pp 9-50.

Delprat-Jannaud, F. & Lailly, P. (1992) "What information on the Earth Model Do the Reflection traveltimes Provide?" Journal of Geophysical research, Vol. 97, no. B13, pp 19827- 19844.

Delprat-Jannaud, F. & Lailly, P. (1993) "Ill-Posed and well posed Formulations of the Reflection traveltime Tomography Problem." Journal of Geophysical research, Vol. 98, no. B4, pp 6589 - 6605.

Farra, V. and Madariaga, R. (1988) "Non-linear reflection tomography." Geophysical Journal Vol. 95, pp 135-147.

Gjøystdal, G. and Ursin, B. "Inversion of reflection times in three dimensions" Geophysics, Vol. 46, no. 7, pp 972-983.

Kleyn, A. H. (1977) On the migration of reflection time contour maps: Geophys.Prosp, Vol. 25, pp 125-140.

Kolbjørnsen, O. & Omre, H. (2002) "Bayesian inversion of piecewise affine operators in a Gaussian framework."

Omre, H. & Halvorsen, K.B. (1989) "The Bayesian bridge between Simple and Universal Kriging." Mathematical Geology, Vol.21, no. 7, pp 767- 786.

O'Sullivan, F. (1986), "A statistical perspective on ill-posed inverse problems," *Statistical Science*, Vol 1,no. 4, pp 502-518.

Robert C.P. and Casella G. (1999) "Monte Carlo statistical methods", Springer.

Tarantola A. (1987) "Inverse problem theory : methods for data fitting and model parameter estimation",Elsevier.

Stein, M.L. (1999) "Interpolation of spatial data : some theory for kriging." Springer, New York.

Wahba, G. (1990) "Spline models for observational data." Society for Industrial and Applied Mathematics.

A Adapting traveltime information to the CTGM algorithm

This appendix contain detailed calculations to fit traveltime observations from reflection seismic into the frame work piecewise affine operators. The calculations below are for the case of one observation, but the extension to several observations is obvious, since the transforms apply locally to each set of variables corresponding to each single traveltime.

Firstly assume that the observed traveltimes do not have any observation error. The mixing distribution to be sampled in the CTGM-algorithm is then,

$$q(x,r) = \text{const} \times |r| p\{K_x Z = t, R_x(Z) = 0, \nabla_x R_x(Z) = r\},$$

with K_xZ, $R_x(Z)$ and $\nabla_x R_x(Z)$ referring to Expressions (1) and (2). For a given x rename the random variables $(K_xZ,\ R_x(Z),\ \nabla_x R_x(Z))$ by (T, R_0, R_1) and $(\,Z(x),\ Z'(x),\ Z''(x)\,)$ by (Z_0, Z_1, Z_2). The relation between (T, R_0, R_1) and (Z_0, Z_1, Z_2) is given by:

$$\begin{aligned} T &= \frac{2}{v}\sqrt{Z_0^2 + (x - x_s)^2}\,, \\ R_0 &= Z_0 \cdot Z_1 + x - x_s\,, \\ R_1 &= Z_0 \cdot Z_2 + Z_1^2 + 1\,. \end{aligned} \tag{7}$$

This relation can be inverted to find :

$$\begin{aligned} Z_0 &= \sqrt{\left(\frac{Tv}{2}\right)^2 - (x - x_s)^2}\,, \\ Z_1 &= \frac{x_s - x - R_0}{Z_0}\,, \\ Z_2 &= \frac{R_1 - Z_1^2 - 1}{Z_0}\,. \end{aligned} \tag{8}$$

The variables (Z_0, Z_1, Z_2), have a known Gaussian distribution, hence the distribution of (T, R_0, R_1) may be calculated by the usual transformation rule. Assuming $(T, R_0, R_1) = (t, 0, r)$, the outcome of (Z_0, Z_1, Z_2) is given by the following expressions:

$$\begin{aligned} z_0(t, x, x_s) &= \sqrt{\left(\frac{tv}{2}\right)^2 - (x - x_s)^2}\,, \\ z_1(t, x, x_s) &= \frac{x_s - x}{z_0(t, x, x_s)}\,, \\ z_2(t, x, x_s, r) &= \frac{r - z_1(t, x, x_s)^2 - 1}{z_0(t, x, x_s)}\,. \end{aligned}$$

Using the transformation rule yields the result

$$\begin{aligned} q(x, r; x_s, t) = \text{const} \times &\left|\frac{r\,\frac{tv}{2}}{z_0(t, x, x_s)^3}\right| \\ &\times p\{Z(x) = z_0(t, x, x_s), Z'(x) = z_1(t, x, x_s), Z''(x) = z_2(t, x, x_s, r)\}, \end{aligned}$$

with x_s and t regarded as given parameters; and the joint probability distribution of $(Z(x),\ Z'(x),\ Z''(x))$ being known from the prior.

When observation error is included in traveltime and shot location, the mixing distribution must be extended such that it include the true traveltime and the true shot location as well. Hence by denoting the observed traveltime y, the task is to sample the distribution

$$\tilde{q}(x, r, t, x_s; y) = q(x, r; x_s, t)p\{\varepsilon = y - t\}p\{x_s\},$$

with; $p\{\varepsilon = y - t\}$ being the likelihood of the observation; and $p\{x_s\}$ being a prior distribution for the shot location.

In general two conditional probabilities are not identical even if the events behind the conditioning bar match. Conditional statements are relative to the $\sigma-$algebra generating the events. The equivalence between sampling the distribution $p\{Z|Z(x), Z'(x), Z''(x)\}$ in place of $p\{Z|K_x(Z), R_x(Z), \nabla_x R_x(Z)\}$, is due to the one to one relation between $Z(x)$, $Z'(x)$ and $Z''(x)$; and $K_x Z$, $R_x(Z)$ and $\nabla_x R_x(Z))$, that is given by Expressions (7) and (8).

B Figures

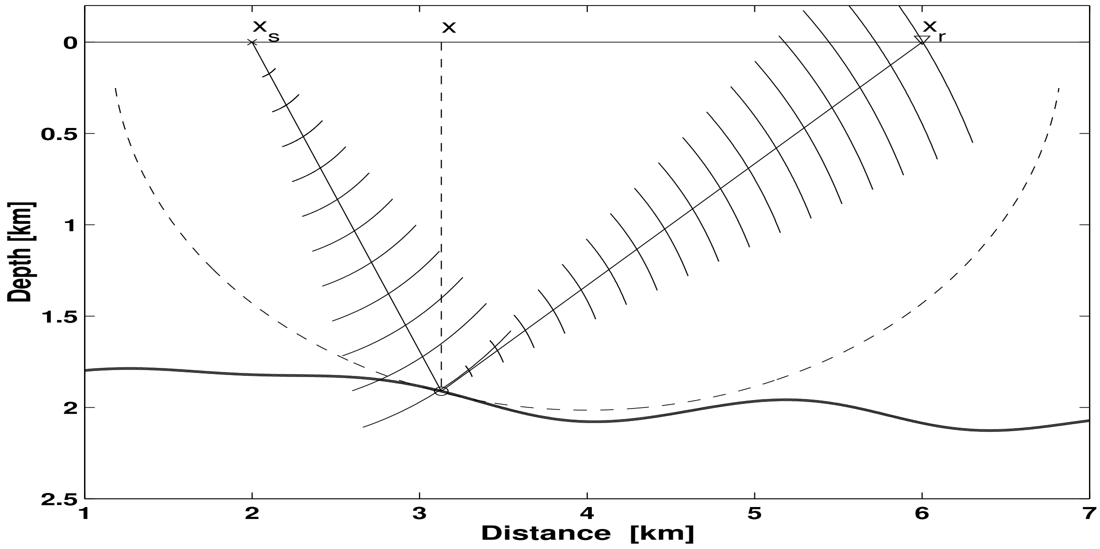


Figure 1: Traveltime geometry. The pulse generated in x_s is reflected in the point $(\,x,\, z(x)\,)$ at the subsurface and is detected by the receiver in x_r. In the point, x, where the reflection occur the ellipse with foci in x_s and x_r, (dashed line) is tangent to the surface.

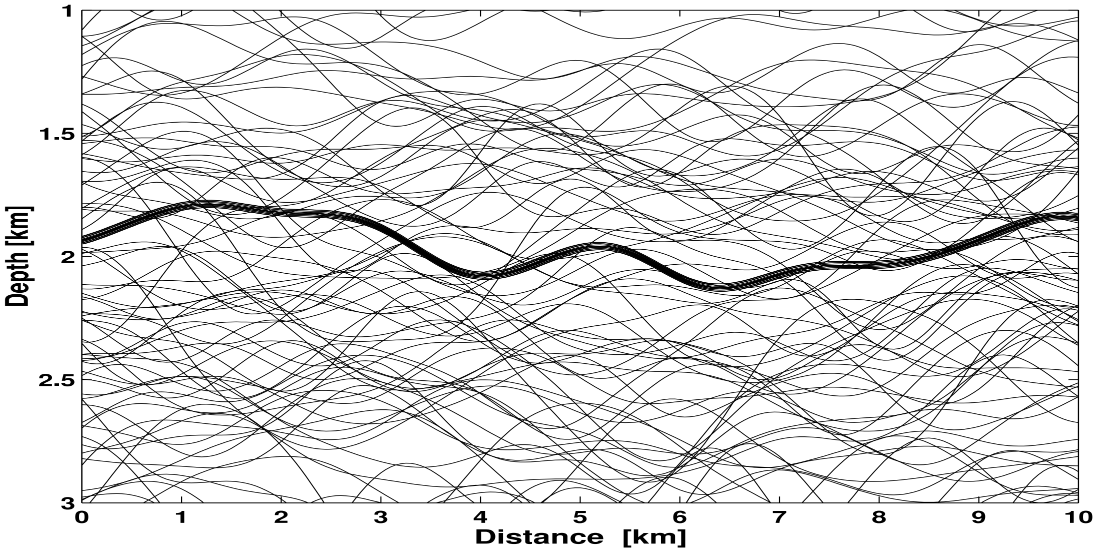


Figure 2: Traveltime migration; the prior. The actual curve is shown with a thick black line together with 100 samples from the prior model for the geological horizon, several of the samples are partially or completely outside the scales of the figure.

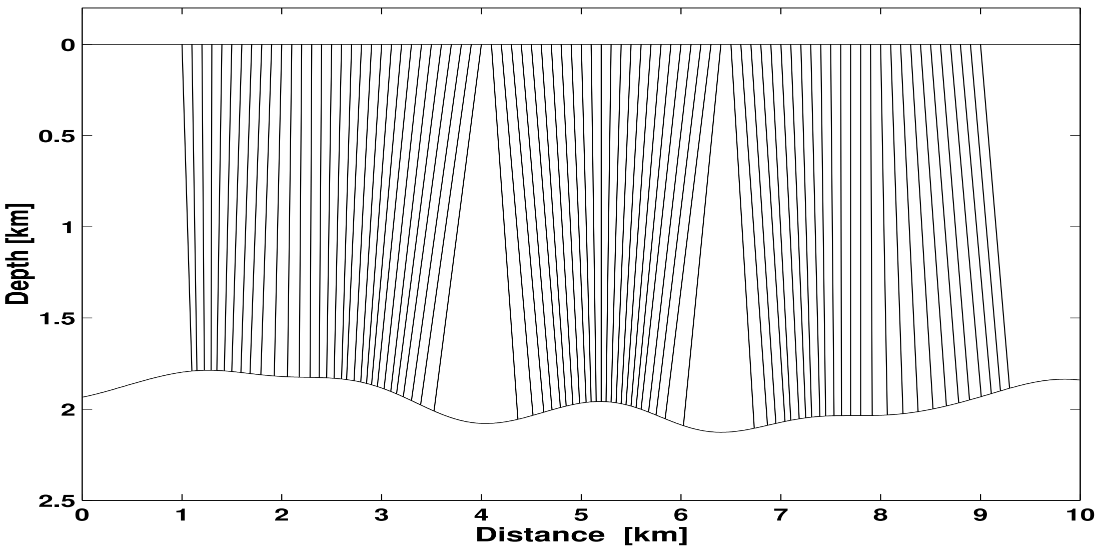


Figure 3: Traveltime geometry. The upper straight horizontal line is the surface, the geological horizon to be recovered is the lower curved horizontal line. The vertical lines are the ray paths connecting the shot location at the surface to to the nearest point at the geological horizon.

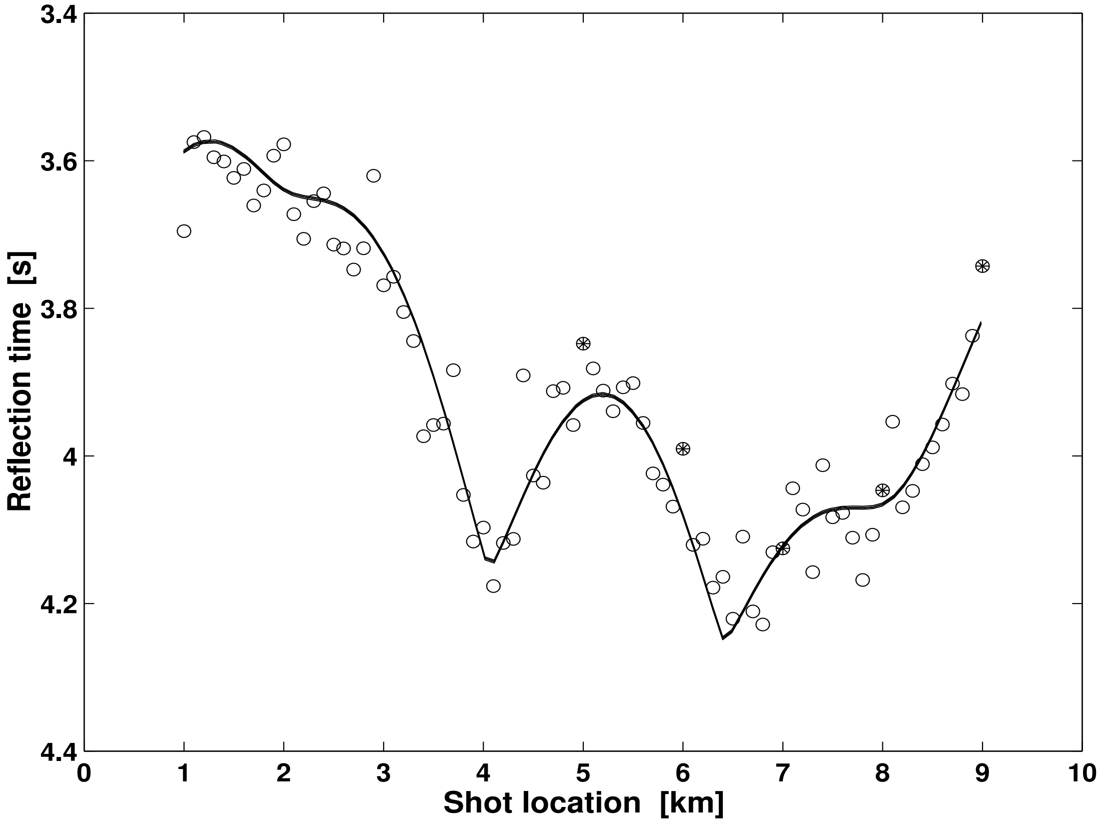


Figure 4: Traveltime migration; the observations. The observed traveltimes are plotted as circles together with the true traveltimes as a solid line. The observations have errors in both x_s and t. The filled circles is the subset of observations used when only five observations are considered.

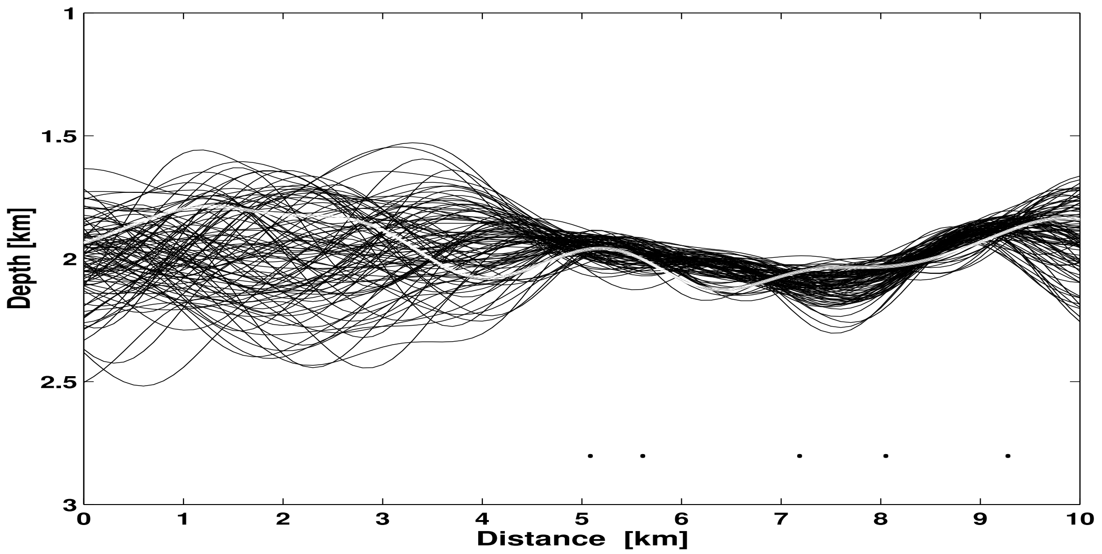


Figure 5: Traveltime migration; the posterior. The actual horizon is displayed in white together with 99 samples from the posterior, using only the five observations marked in Figure 4. The dots in the bottom of the figure indicates the actual reflection locations for the 5 observations.

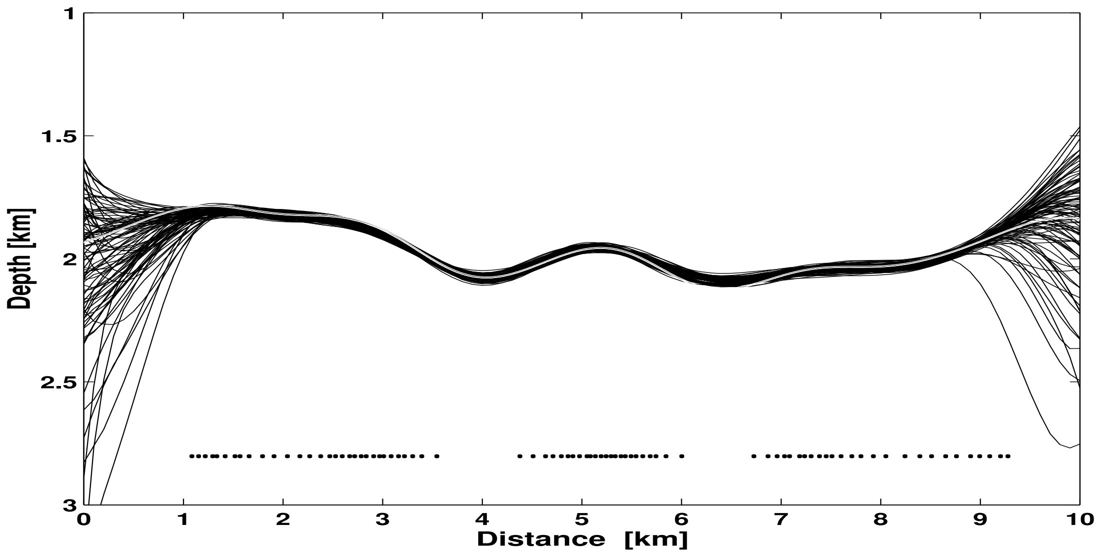
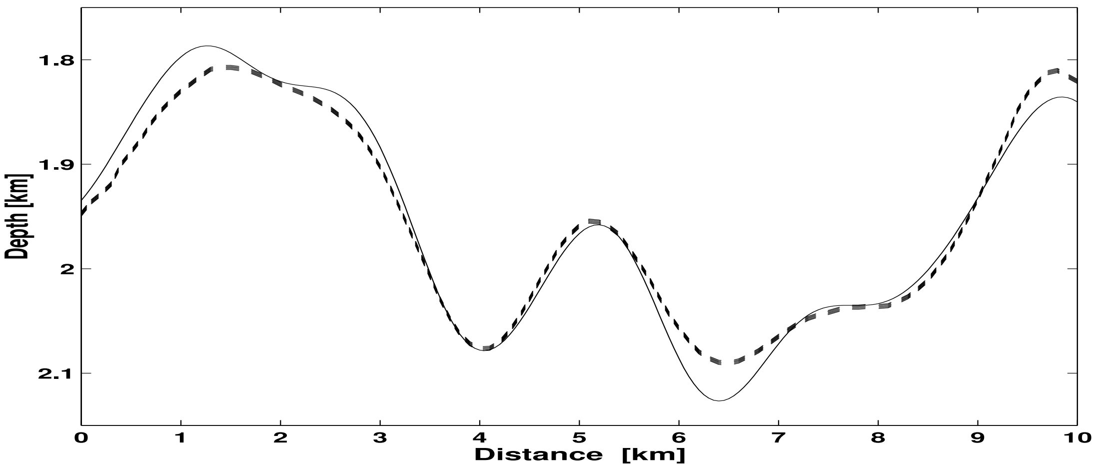
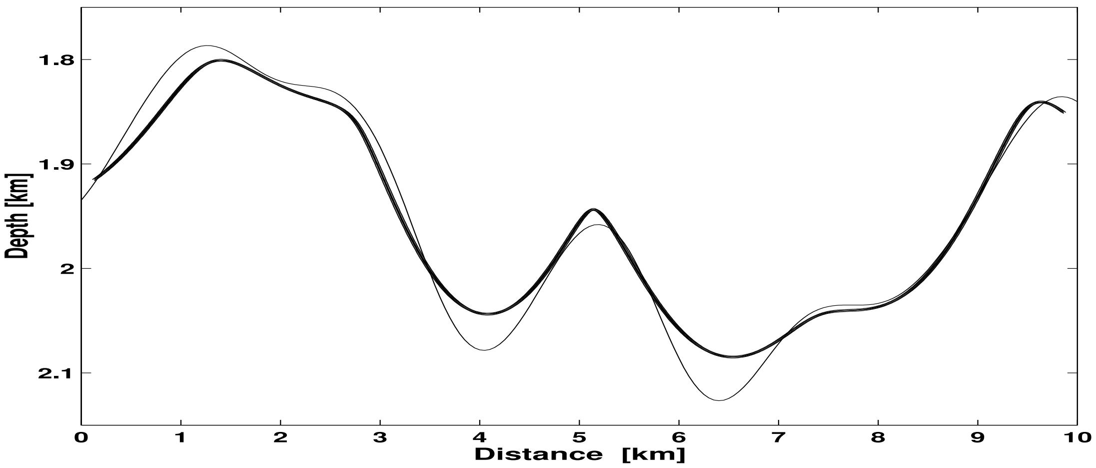


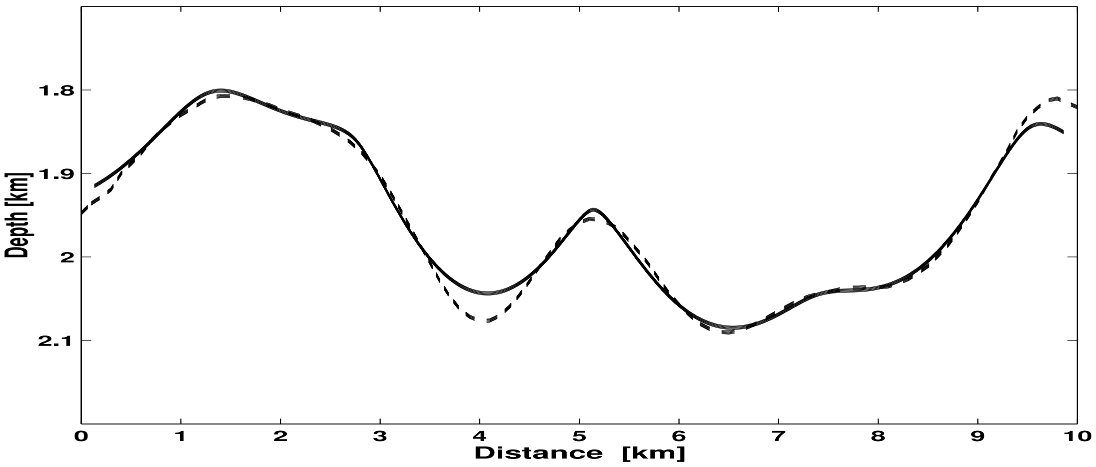
Figure 6: Traveltime migration; the posterior. The actual horizon is displayed in white together with 99 samples from the posterior, using all 81 observations in Figure 4. The dots in the bottom of the figure indicates the actual reflection locations for all 81 observations.



(a)



(b)



(c)

Figure 7: Comparison of estimates and true subsurface; The truth is plotted with a thin full line; the standard estimate with thick full line; and the proposed estimate with a thick dashed line. Comparison of truth and proposed estimate (a); truth and standard estimate (b); proposed and standard estimate (c).