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BOUNDS FOR THE RELIABILITY OF MULTISTATE SYSTEMS WITH PARTIALLY ORDERED STATE SPACES AND STOCHASTICALLY MONOTONE MARKOV TRANSITIONS

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Consider a multistate system with partially ordered state space E, which is divided into a set C of working states and a set D of failure states. Let $X(t)$ be the state of the system at time t and suppose $\{X(t)\}$ is a stochastically monotone Markov chain on E. Let T be the failure time, i.e. the hitting time of the set D. We derive upper and lower bounds for the reliability of the system, defined as $P_m(T > t)$ where m is the state of perfect system performance.

Keywords: Multistate system; Markov chain; partial order; stochastic monotonicity; quasi-stationary distribution; exponential distribution

1. Introduction

Consider a system consisting of n components, where the set of possible states for the kth component is a finite set which we denote by E_k $(k = 1, 2, \ldots, n)$. The set of possible states of the system is $E \equiv E_1 \times \cdots \times E_n$. Let the system be monitored from time $t = 0$ and define $X_k(t)$ to be state of component k at time t, defined for $t \geq 0$. The state of the system at time t is then given by $X(t) = (X_1(t), \ldots, X_n(t))$.

For systems of independent components we may construct Markov models for $\{X(t)\}$ by modelling each component process $\{X_k(t)\}$ as a Markov chain. In the case of dependent components, however, Markovian component processes will not in general imply a Markovian system process. Thus, in order to obtain tractable Markov models in the presence of dependence it is convenient to start by the assumption that the system state $\{X(t)\}$ forms a Markov chain on E and then define the possible transitions and transition rates between the states. In the present paper we shall study such Markov chains $\{X(t)\}$ on E.

Suppose $C \subset E$ is the set of states in which the system is working, while $D \equiv E \setminus C$ is the set of failure states. The *failure time* T is the time when the set D is first hit, more precisely $T = \inf\{t \geq 0 : X(t) \in D\}$. We define the *reliability* of the system to be the probability $P_m(T > t)$, with the index m meaning that the Markov chain $\{X(t)\}$ starts at time 0 in state m, which we define to be the perfect functioning state of the system. The purpose of the paper is to derive upper and lower bounds for the reliability function $P_m(T > t)$.

Bounds for $P_m(T > t)$ in the present context have earlier been derived under the assumption that $\{X(t)\}$ is so called associated in time, see Esary and Proschan[1] for the case of binary components (each E_k is the set $\{0, 1\}$) and Funnemark and Natvig[2] for the case of finite totally ordered state spaces E_k. Practice has shown, however, that bounds based on association may often be too wide to be useful. In this paper we use a slightly different approach based on properties of the so-called *quasi-stationary* distribution of the Markov chain $X(t)$ (see Darroch and Seneta[3]). Moreover, we assume that the Markov chain $\{X(t)\}$ is stochastically monotone with respect to a partial order on the state space E. In fact such an assumption is closely reated to association (see Lindqvist[4]), making our approach related to the ones cited above.

To increase generality, we assume that the component state spaces E_k are partially ordered by orders $\preceq_k$, which in turn induce the partial order $\preceq$ on E. The convention is here that larger states with respect to the ordering correspond to better performance of the component. Most literature on multistate systems assume that the state spaces of the components are totally ordered. In practice, however, a totally ordered state space may not be the most natural. For example, the possible states could be "functioning", "failure of type A", "failure of type B", etc. In this case the failure states can not necessarily be ordered in a definite way. Hence we rather have a *partial* ordering, with "functioning" being better than each of the failure states, while no other pairs of states are comparable. More specifically, Caldarola[5] argued that it would be very hard to decide whether or not the state "failed closed" of a curcuit breaker is more failed than the state "failed open". In addition, for a given component the ordering of the states may depend upon the system to which the component belongs. A more artificial reason for introducing partially ordered state spaces of components would be the following. Consider a binary system of binary components, where the component set can be partitioned into mutually independent subsets, each containing possibly stochastically dependent components. Then in order to achieve stochastically independent components, one may define "supercomponents" corresponding to each of the independent sets. A supercomponent involving m binary components would then for example have 2^m partially ordered states. This idea was considered by Caldarola[5].

The precise definition of a stochastically monotone Markov chain on a partially ordered state space is given in the next section. Intuitively, stochastic monotonicity means in the present application that the remaining time to failure decreases stochastically as the system state is getting worse (with respect to the partial order). Stochastically monotone Markov chains on totally ordered state spaces were considered by Keilson[6], while Lindqvist[4] has studied the finite partially ordered case.

As mentioned above, a key ingredient in our approach is the quasi-stationary distribution of the Markov chain $\{X(t)\}$. This is a distribution $\rho = (\rho_i : i \in E)$ with support on C which is defined as the limiting distribution of the state at time $t \to \infty$, conditioned on the event that the process has not yet (at time t) exited from C (see precise definition in Section 2). The basic property of a quasi-stationary distribution ρ that we will use is that

$$P_\rho(T > t) = e^{-t/E_\rho T}$$

which states that the failure time T is exactly exponentially distributed when the initial distribution of the chain is ρ. Quasi-stationary distributions were first considered by Darroch and Seneta[3]. A nice introduction is given by Keilson[6].

The plan of the paper is as follows. In Section 2 we give precise definitions and some basic results concerning partial orders, stochastic monotonicity and quasi-stationary distributions. In Sections 3 and 4 we derive upper and lower bounds for the reliability function. A numerical comparison of the results from the present paper to results of Esary and Proschan[1] is given in Section 5.

2. Precise Definitions and Basic Results

Recall first that a relation $\preceq$ on a set X is a partial order if it is (i) reflexive, i.e. $x \preceq x$ for all x, (ii) antisymmetric, i.e. $x \preceq y$ and $y \preceq x$ imply $x = y$ and (iii) transitive, i.e. $x \preceq y$ and $y \preceq z$ imply $x \preceq z$.

For the systems considered in this paper we assume that the component state spaces are finite partially ordered sets $(E_k, \preceq_k)$. We define the partial order $\preceq$ on E to be the product order defined by

$$(i_1, i_2, \ldots, i_n) \preceq (j_1, j_2, \ldots, j_n) \text{ if and only if } i_k \preceq_k j_k \text{ for } k = 1, 2, \ldots, n$$

We shall assume throughout the paper that each set E_k contains a unique maximal element m_k such that $i_k \preceq m_k$ for all $i_k \in E_k$. The system state $m \equiv (m_1, \ldots, m_n)$ is then a unique maximal element of E. The m_k correspond to perfect functioning of component k while m corresponds to perfect functioning of the system.

Example 1. A simple nontrivial example of a partially ordered state space can be given as follows. Let $E = \{0, 1, 2\}$, where $0 \preceq 1 \preceq 3$, $0 \preceq 2 \preceq 3$ but assume no relation between 1 and 2. Then 3 can be thought of as the perfect state, 0 as the complete failure state while 1 and 2 are intermediate states corresponding to non-perfect conditions which are not necessarily ordered in severity. In fact, the set E will be the state space of a supercomponent (see Section 1) resulting from two binary components. In this case state 3 corresponds to both components working, state 2 corresponds to component 1 working, component 2 failed etc.

A set $A \subseteq E$ is called *increasing (decreasing)* if $i \in A$, $j \succeq i$ ($j \preceq i$) imply $j \in A$. Note that the set A is decreasing if and only if its complement A^c is increasing.

Throughout the present paper we shall assume that the set C of working states is an increasing set. This is a reasonable assumption since it means that if the system is working when in a certain state, then any better state (with respect to the partial order) is also a working state. The set $D = E \setminus C$ is then necessarily a decreasing set.

Let $\lambda = (\lambda_i : i \in E)$ and $\mu = (\mu_i : i \in E)$ be probability distributions on E. We shall say that λ is dominated by μ, written $\lambda \preceq \mu$, if $\lambda(A) \leq \mu(A)$ for all increasing subsets $A \subseteq E$.

Let $\{X(t)\}$ be a time-homogeneous Markov chain on E with intensity matrix $Q = (q_{ij})$. If λ is a distribution on E, then $P_\lambda(\cdot)$ denotes probabilities computed when the distribution of $X(0)$ is λ. In order to stress the dependence on Q, we shall sometimes write $P_{\lambda,Q}(\cdot)$ for $P_\lambda(\cdot)$. For short, we write $P_i(\cdot)$ when λ is the measure concentrated in state i. The Markov chain $\{X(t)\}$ is *stochastically monotone* if for any increasing set $A \subseteq E$ and for any $t > 0$, $P_i(X(t) \in A)$ is a non-increasing function of i (with respect to the order $\preceq$ in E). The following equivalent definition in terms of the intensity matrix Q is given in Lindqvist[4]:

The intensity matrix Q is stochastically monotone if for all $i \preceq j$ in E we have

$$Q_i(A) \leq Q_j(A) \text{ for all increasing } A \subseteq E \text{ with } i, j \notin A \tag{1}$$

$$Q_i(A) \geq Q_j(A) \text{ for all decreasing } A \subseteq E \text{ with } i, j \notin A \tag{2}$$

Here $Q_i(A) = \sum_{j \in A} q_{ij}$.

By a trivial generalization of the proof of Theorem 9.3F in Keilson[6] (to allow *partial* order) it follows that if $\{X(t)\}$ is stochastically monotone, and E contains a unique maximal element m, then $P_m(X(t) \in A)$ is a non-increasing function of t for any increasing set A. In the setting of the present paper this means that when the system state is defined by a stochastically monotone Markov chain, starting in its "best" state, then the system deteriorates (stochastically) with time.

To describe additional properties of stochastically monotone Markov chains we introduce the space of *sample paths* of the chain $\{X(t)\}$. Let $u > 0$ be a fixed time point, and consider the process $\{X(t)\}$ on the closed time interval $[0, u]$. It is well known that the sample paths of $\{X(t)\}$ for $t \in [0, u]$ can be chosen as members of the space E^u of functions from $[0, u]$ to E which are right continuous and have left limits at all $t \in [0, u]$. Moreover, E^u becomes a Polish space when furnished with the Skorohod metric (see e.g. Kamae et al.[7]). Now the $P_\lambda(\cdot)$ can be viewed as measures on the measurable space $(E^u, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-field in E^u. We shall in the following tacitly assume that any considered subset of E^u is a member of $\mathcal{B}$. A natural partial order on E^u is the pointwise order, i.e. for $x(\cdot), y(\cdot) \in E^u$ we have $x \leq y$ iff $x(t) \preceq y(t)$ for all $t \in [0, u]$. This makes it possible to consider increasing (decreasing) subsets of E^u.

Let now R and Q be two intensity matrices for the Markov chains $\{X(t)\}$ and $\{Y(t)\}$, respectively, both defined on E, but not necessarily stochastically monotone. We shall say that R is *dominated* by Q, written $R \leq Q$, if for all $i \preceq j$ in E we have

$$R_i(A) \leq Q_j(A) \text{ for all increasing } A \subseteq E \text{ with } i, j \notin A \tag{3}$$

$$R_i(A) \geq Q_j(A) \text{ for all decreasing } A \subseteq E \text{ with } i, j \notin A \tag{4}$$

If at least one of R and Q is stochastically monotone, then it is enough to consider $i = j$ in the above definition. If $R \leq Q$, then it follows from Theorem 5 of Kamae et al.[7] that for initial distributions $\lambda \preceq \mu$ and any increasing set $C \subseteq E^u$ we have

$$P_{\lambda,R}(C) \leq P_{\mu,Q}(C) \tag{5}$$

Let λ and μ be probability distributions on E such that $\lambda \preceq \mu$, and let $\{X(t)\}$ be stochastically monotone with intensity matrix Q. Then by (5) applied to the case $R = Q$ it follows that for any increasing set $C \subseteq E^u$ we have

$$P_\lambda(C) \leq P_\mu(C) \tag{6}$$

Now, let D be a decreasing subset of E and let T denote the hitting time of the set D, as defined in Section 1. Then $\{T > t\}$ defines an increasing set in E^u when $u > t$. Hence by (6), for stochastically monotone $\{X(t)\}$, we have

$$P_\lambda(T > t) \leq P_\mu(T > t) \text{ whenever } \lambda \preceq \mu \tag{7}$$

The relation (7) is intuitively reasonable in our setting, as it may be interpreted to say that with a better initial state, the probability of no failure before time t increases.

Finally, we introduce the concept of an *ML-matrix.* A square finite matrix $B = (b_{ij})$ is called an ML-matrix if $b_{ij} \geq 0$ for all $i \neq j$. If B is an ML-matrix, then for sufficiently large a, $T = aI + B$ is a nonnegative matrix, where I is the identity matrix. We call B irreducible if T is irreducible, i.e. if for any pair i, j there is a positive integer n with $t_{ij}^{(n)} > 0$. If B is irreducible, then by Chapter 2 in Seneta[8] there exists an eigenvalue τ (which we shall call the Perron-Frobenius eigenvalue of B) such that $\tau > \text{Re}(\nu)$ for any other eigenvalue ν of B. Moreover, to τ correspond unique, up to constant multiples, strictly positive left and right eigenvectors.

Any intensity matrix Q of a Markov chain $\{X(t)\}$ is an ML-matrix. The property that Q is irreducible as an ML-matrix then corresponds to the irreducibility of the Markov chain $\{X(t)\}$ in the ordinary terminology.

The intensity matrix Q is called *up-down* if $q_{ij} > 0$ implies $i \preceq j$ or $i \succeq j$ for all $i, j \in E,\ i \neq j$. This means that any change of state of the system is to a state which is either better or worse with respect to the partial order.

In the paper we will use the notation $Q_C = (q_{ij} : i, j \in C)$. Thus Q_C is the restriction of Q to C. We will also use need the corresponding notation for vectors, e.g. $\rho_C = (\rho_i : i \in C)$.

It is well known that Q irreducible implies the existence of a unique stationary distribution $\pi = (\pi_i : i \in E)$. Moreover, Q_C irreducible implies the theorem below.

Theorem 1 (Darroch and Seneta[3]) *If Q_C is irreducible, then for any initial distribution λ with $\lambda(C) > 0$, the limits*

$$\rho_j = \lim_{t\to\infty} P_\lambda(X(t) = j \mid T > t) \;\; ; \; j \in C \tag{8}$$

exist, with the ρ_j being strictly positive and not depending on λ. The vector ρ_C is the unique normalized (to norm 1) left eigenvector of Q_C for the Perron-Frobenius eigenvalue $\tau \equiv -a$ (< 0). Moreover, define ρ to be the probability distribution on all of E with probability mass ρ_j for $j \in C$ and mass 0 for $j \in D$. Then

$$P_\rho(T > t) = e^{-at} \quad \text{for all } t > 0 \tag{9}$$

i.e. the distribution of T is exactly exponential when ρ is the initial distribution.

Following Darroch and Seneta[3] we shall call the distribution ρ the *quasi-stationary distribution* of $\{X(t)\}$. Note the dependence of ρ on the set C. As indicated by (8), the quasi-stationary distribution can be interpreted as the conditional distribution of the state of the system, $X(t)$, at a large time t, conditioned on the event that the system has not yet failed (i.e. $X(t)$ has not yet left the set C). Moreover, (9) can be interpreted to say that under this condition, the remaining time to failure is exactly exponentially distributed, with expectation $E_\rho T = 1/a$.

Example 2. Suppose E is given as the E_k in Example 1 and let Q be the intensity matrix of a Markov-chain on E. Then by Example 2 in Lindqvist[4] Q is stochastically monotone if the following eight relations all hold:

$$\begin{array}{rccc}
\text{(i)} & q_{10}+q_{12} & \geq & q_{30}+q_{32} \\
\text{(ii)} & q_{20}+q_{21} & \geq & q_{30}+q_{31} \\
\text{(iii)} & q_{12}+q_{13} & \geq & q_{02}+q_{03} \\
\text{(iv)} & q_{21}+q_{23} & \geq & q_{01}+q_{03} \\
\text{(v)} & q_{10} & \geq & q_{30} \\
\text{(vi)} & q_{20} & \geq & q_{30} \\
\text{(vii)} & q_{13} & \geq & q_{03} \\
\text{(viii)} & q_{23} & \geq & q_{03}
\end{array}$$

If Q is also up-down, then $q_{12} = q_{21} = 0$ and it is seen that the inequalities (v)-(viii) are implied by (i)-(iv). If we further assume that direct transitions between the extreme states 0 and 3 are not possible, then we obtain the four relations $q_{10} \geq q_{32},\ q_{20} \geq q_{31},\ q_{13} \geq q_{02},\ q_{23} \geq q_{01}$.

3. Bounds for the Reliability Function

In the present section we are concerned with the computation of simple upper and lower bounds for the reliability function $P_m(T > t)$ in terms of the quasi-stationary distribution ρ and the associated eigenvalue $-a$.

From (7) follows directly that if Q is stochastically monotone and Q_C is irreducible, then

$$0 \leq P_m(T > t) - P_\rho(T > t) \leq (1 - \rho_m) P_m(T > t)$$

From this we obtain

$$e^{-at} \leq P_m(T > t) \leq \rho_m^{-1} e^{-at} \tag{10}$$

The upper bound in (10) appears, however, to be of little value when ρ_m is not close to 1. There is thus a need for improvement of this bound. A possible approach is as follows. First, write

$$P_\rho(T > t) = \sum_{i \in C} \rho_i P_i(T > t) = \rho_m P_m(T > t) + \sum_{i \neq m} \rho_i P_i(T > t)$$

so that

$$P_m(T > t) = \rho_m^{-1} \left[P_\rho(T > t) - \sum_{i \neq m} \rho_i P_i(T > t) \right]$$

From this we get directly:

Theorem 2 *Assume that Q is stochastically monotone and Q_C is irreducible. Suppose lower bounds $b_i(t)$ are available for all $i \in J$, where J is some subset of $C \setminus \{m\}$. Then*

$$e^{-at} \leq P_m(T > t) \leq \rho_m^{-1} \left[e^{-at} - \sum_{i \in J} \rho_i b_i(t) \right] \tag{11}$$

This inequality generalizes the upper bound of (10), as the latter corresponds to $J = \emptyset$ (the empty set). It seems worthwhile, however, to try to compute nonzero $b_i(t)$ for at least the i corresponding to the largest ρ_i, $i \neq m$. The question remains to derive functions $b_i(t)$ that are relatively simple to compute and otherwise do fairly well. Our idea here is to define new Markov chains from the old one and then apply the left inequality in Theorem 2 to the derived chains.

In the following, fix one element in J and call it m'. Define the set E' by

$$E' = \{i \in E : i \preceq m'\}$$

Then E' is a finite partially ordered set, with partial order inherited from E, and with a unique maximal element m'. We shall consider a Markov chain $\{X'(t)\}$ on E' with intensity matrix $Q' = (q'_{ij} : i, j \in E')$, where $q'_{ij} = q_{ij}$ when $i, j \in E',\ i \neq j$, and diagonal elements q'_{ii} defined so that the row sums equal 0. (Note that $Q' \neq Q_{E'}$). Define now $D' = D \cap E'$. Then D' is a decreasing set with respect to the partially ordered set $(E', \preceq)$. Let furthermore T' denote the hitting time of D' for the Markov chain $\{X'(t)\}$. Then we have the following result:

Lemma 1 *Suppose Q is stochastically monotone and up-down. Then*

$$P_{m',Q}(T > t) \geq P_{m',Q'}(T' > t)$$

Proof. We shall define another Markov chain $\{Y(t)\}$ on E, given by the intensity matrix $R = (r_{ij})$ with off-diagonal elements given by

$$r_{ij} = \begin{cases} 0 & \text{if } i \in E',\, j \in E \setminus E' \\ q_{ij} & \text{otherwise} \end{cases}$$

and r_{ii} defined so that the row sums of R equal 0.

As there are no positive transition rates from E' to $E \setminus E'$, it is seen that for an initial state in E', the chain $\{Y(t)\}$ behaves exactly as $\{X'(t)\}$. Thus

$$P_{m',R}(T > t) = P_{m',Q'}(T' > t)$$

By (5) we are therefore done if we can show that $R \leq Q$.

Since Q is stochastically monotone, we must prove that, for any i, $R_i(A) \leq (\geq) Q_i(A)$ for all increasing (decreasing) sets $A \subseteq E$ with $i \notin A$.

Let therefore $i \in E$ be given. First, let A be an increasing set with $i \notin A$. Then $R_i(A) \leq Q_i(A)$ by the definition of R.

Next, let A be a decreasing set with $i \notin A$. We split up into the following two cases:

$i \in E \setminus E'$: Then we have $R_i(A) = Q_i(A)$ by the definition of R.

$i \in E'$: Then $R_i(A) = Q_i(A \cap E')$. We are done if we can prove that $Q_i(A \cap E') = Q_i(A)$. To do this, suppose for contradiction that $q_{ik} > 0$ for some $k \in A \setminus E'$. Then, by the assumed up-down property of Q, either $i \preceq k$ or $i \succeq k$. The former case is impossible, as it would imply $i \in A$ because $k \in A$ and A is decreasing. The latter case is, however, also impossible, as it would imply $k \in E'$ because $i \in E'$ and E' is a decreasing set in E. Thus $q_{ik} = 0$ for all $k \in A \setminus E'$ and we are done.

The lemma suggests that lower bounds $b_{m'}(t)$ may be based on the $P_{m'}(T' > t)$, either by direct computation of these probabilities, or by further bounding them from below. If $\{X'(t)\}$ is stochastically monotone (which is not guaranteed from the monotonicity of $\{X(t)\}$) and $E' \cap C$ is irreducible (with respect to $\{X'(t)\}$), then we can use the lower bound of Theorem 2. If $\{X'(t)\}$ is not monotone, then we may construct a monotone R' with $R' \leq Q'$ and use (5) to compute a bound.

The bounds $b_i(t)$ of the preceding paragraph have the property that $b_i(0) = 1$, $b_i(\infty) = 0$. Using them in (11) thus yields an upper bound $h(t)$ for $P_m(T > t)$ with $h(0) = (1 - \sum_{i \in J} \rho_i)\rho_m^{-1}$ and $h(\infty) = 0$. Thus we can have $h(0)$ as close to 1 as we wish, by increasing J. With $J = C \setminus \{m\}$ we get $h(0) = 1$.

4. Alternative Lower Bound for the Reliability Function

Recall from Theorem 2 that under the given conditions we have $P_m(T > t) \geq e^{-at}$, where $-a$ is the Perron-Frobenius eigenvalue of the restriction Q_C of Q to the set C. In practice it may be hard to compute the exact value of a, so one might be interested in more easily available upper bounds c, say, for a. It is the purpose of the present section to show how such bounds may be derived. Of course, if $a \leq c$, then $P_m(T > t) \geq e^{-ct}$.

We start by a lemma, which is a straightforward consequence of the fact that $-a$ is an eigenvalue of Q_C.

Lemma 2 *If Q_C is irreducible, then*

$$a = \sum_{i \in C} \rho_i Q_i(D) \tag{12}$$

Remark: By definition, a is the hazard rate corresponding to T when ρ is the initial distribution. Also, $Q_i(D)$ is the rate of transitions from state $i \in C$ to D. Thus the right hand side of (12) is indeed the transition rate from C to D when the Markov chain is in the quasi-stationary equilibrium.

Proof: The relation $\rho Q_C = \tau \rho$ is equivalent to $\sum_{i \in C} \rho_i q_{ij} = \tau \rho_j$ for all $j \in C$. Summing over j we get $\sum_{i \in C} \rho_i \sum_{j \in C} q_{ij} = \tau$ from which the lemma follows since $\sum_{j \in C} q_{ij} = -Q_i(D)$ by the fact that Q is an intensity matrix.

The result below is a consequence of a more general result of Lindqvist[9] on the Perron-Frobenius eigenvalue of ML-matrices. The possible advantage of the result is that instead of doing computations related to the quasi-stationary distribution one needs only compute an ordinary stationary distribution which is done by a system of linear equations. We also note that the result does not require stochastic monotonicity of Q.

Let as before, Q_C be the restriction of Q to C. Define now Q_C° so that Q_C and Q_C° coincide outside the main diagonal, and let the diagonal elements of Q_C° be defined so that each row of Q_C° sum up to 0. Then Q_C° is the intensity matrix of a Markov chain on C, and hence the Perron-Frobenius eigenvalue of Q_C° is 0. Note, furthermore, that

$$Q_{C,ii} - Q_{C,ii}^\circ = -Q_i(D)$$

Let π° be the stationary distribution of the Markov chain on C with intensity matrix Q_C°. Then we have:

Theorem 3 (Lindqvist[9]) *If Q_C is irreducible, then*

$$a \le \sum_{i \in C} \pi_i^\circ Q_i(D) \equiv c \tag{13}$$

Example 2. Suppose $E = \{0, 1, 2\}$ with usual ordering $0 \le 1 \le 2$ and transition rates given by $q_{21} = q_{12} = q_{02} = 100$, $q_{10} = 1$ and $q_{20} = q_{01} = 0$. The resulting Markov chain is stochastically monotone and with $C = \{1, 2\}$ we have Q_C irreducible. Now Theorem 3 gives $a \le 0.5$, whereas the exact value of a is 0.4988.

Remark: Time-reversible chains.
Recall that the Markov chain $\{X(t)\}$ by definition is *time-reversible* if there exist positive constants $(z_i : i \in E)$ such that

$$z_i q_{ij} = z_j q_{ji} \tag{14}$$

for all $i, j \in E$, $i \neq j$. The stationary distribution $\pi = (\pi_i : i \in E)$ of the Markov chain $\{X(t)\}$ on E is found by norming the z_i to have sum 1. The relations (14) obviously also carry over to the Markov chain with infinitesimal intensity matrix Q_C° encountered in Theorem 3. It therefore follows that in the time-reversible case we have

$$\pi_i^\circ = \frac{\pi_i}{\pi(C)} \equiv \pi_i^*$$

where π^* is the so-called *ergodic exit distribution* considered by Keilson[6]. In Theorem 6.9C of Keilson[6] is shown that in the case of time-reversibility (not assuming stochastic monotonicity) we have

$$P_{\pi^*}(T > t) \leq P_\rho(T > t) = e^{-at}. \tag{15}$$

The left hand side here can be interpreted as the probability that a presently working system which has been running for a long time, will continue to work for at least a time t. The right hand side, on the other hand, is the corresponding probability for the case when the system is working and has not yet visited the failure states.

The next example shows that(15) does not necessarily hold for non-reversible stochastically monotone chains, however.

Example 2 (continued). The Markov chain of the example is not time-reversible since $q_{20} = 0$, while $q_{02} > 0$. Moreover, we compute $\pi^* = (0, 0.4975, 0.5025)$, so $\pi^* \succ \rho$ and hence $P_{\pi^*}(T > t) > P_\rho(T > t)$.

Remark: Bounding the quasi-stationary distribution.
By Lemma 2 and Theorem 3 we have

$$\sum_{i \in C} \rho_i Q_i(D) \leq \sum_{i \in C} \pi_i^\circ Q_i(D) \tag{16}$$

Since $Q_i(D)$ is a decreasing function for $i \in C$ it might be conjectured that

$$\pi^\circ \preceq \rho_C \tag{17}$$

Indeed, (17) holds in Example 2 since we there have $\pi^\circ = (0.5, 0.5)$. However, (17) does not hold in general as will follow from the next example.

Example 3. Let $(E, \preceq)$ be as in Example 1. Let $q_{01} = 1,\ q_{10} = 1,\ q_{02} = 1,\ q_{20} = 1410,\ q_{13} = 5,\ q_{31} = 2,\ q_{23} = 90,\ q_{32} = 1$. Then Q is stochastically monotone. Let $C = \{1, 2, 3\}$. A computation shows that

$$\begin{aligned} \rho_C &= (0.2838, 0.0005, 0.7157) \\ \pi^\circ &= (0.2835, 0.0079, 0.7087) \end{aligned}$$

so $\pi^\circ \not\preceq \rho_C$ since the former gives the largest mass to the increasing set $\{2, 3\}$.

5. Comparison of results: A binary 2-out-of-three system

Consider a system with three binary components, so that $E_1 = E_2 = E_3 = \{0, 1\}$. The components are independent of each other, and each has a failure rate λ (transition rate from state 1 to state 0) and repair rate μ (transition rate from 0 to 1). Let C be the set of system states for which at least two components work. Thus we assume that the system is a 2-out-of-3 system.

This example was considered by Esary and Proschan[1], who computed lower bounds for $P_m(T > t)$ by using properties of associated stochastic processes. We may therefore use the example for a comparison of our bounds to theirs. Table 1 presents some results in terms of the parameters

$$\alpha = \lambda t \text{ and } \beta = \mu/\lambda$$

In the table, EP corresponds to computations by Esary and Proschan[1].

Table 1. Bounds for the reliability of a 2-out-of-3 system

α	β	Exact	e^{-at}	e^{-ct}	EP	$\rho_m^{-1}e^{-at}$
1	2	.44	.37	.30	.36	.74
1	10	.682	.663	.630	.650	.834
1	100	.9449	.9444	.9434	.9439	.9722
10	20	.0895	.0886	.074	.074	.101
10	100	.565	.565	.558	.559	.581

It is seen that the simple bound e^{-at} using the Perron-Frobenius eigenvalue of Q_C is better than the EP bound. On the other hand, the more easily computed bound e^{-ct} from Theorem 3 is beaten by the EP bound, but otherwise seems to behave satisfactorily at least for highly reliable systems. It should be noted that the EP bounds often lead to rather involved computations.

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