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PREPRINT
STATISTICS NO. 1/2009

NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
TRONDHEIM, NORWAY

This report has URL http://www.math.ntnu.no/preprint/statistics/2009/S1-2009.ps
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# A method of parametric solution of convolution equations

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**Abstract**

A variant of the method of moments is developed for parametric solution of convolution equations of the first kind. Two models — the gamma model and the shifted gamma model — are studied in details.

*Keywords*: Convolution equation; Distribution of delay; Dynamical systems with delay; Gamma distribution; Shifted gamma distribution

## 1 Introduction

A convolution equation of the first kind is an integral equation of the form

$$\int_{-\infty}^{\infty} f(t-s)g(s)ds = h(t), \tag{1}$$

where $g(t)$ and $h(t)$, the kernel and the right hand side, respectively are known functions, and $f(t)$ is the unknown function to be found. Practically, the functions $g(t)$ and $h(t)$ are observed on some discrete grids with errors (usually random). Equation (1) arises in many scientific and engineering disciplines, and there is a broad literature devoted to solution of these equations, see for example Tikhonov and Arsenin (1977).

The convolution equation of the first kind is an ill-posed problem. This makes its solution in the general case difficult, and requires some additional information about measurement errors, which is often not available. The situation essentially improves if a parametric form of $f(t)$ is known. In this case the problem is reduced to estimation of a few parameters. The traditional way of parametric solution, the least squares method, is however often not applicable in practice, because it leads to cumbersome, strongly nonlinear equations which are sensitive to measurement errors. This is the case, for example, when $f(t)$ is a gamma density

$$f(t) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t}, \quad t > 0, \alpha > 0, \beta > 0, \tag{2}$$

or a beta density

$$f(t) = \frac{\Gamma(\alpha, \beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1}, \quad 0 < t < 1, \alpha > 0, \beta > 0,$$

where one needs to differentiate under the integral sign a combination of gamma functions. At the same time, these families are among the most frequently arising in many applications.

In this work, we develop a method, based on moments of the functions $g(t)$ and $h(t)$. It usually gives an easily computable solution and is stable with respect to measurement errors. A special attention is paid to the case when the unknown function is a gamma or a shifted gamma density, since these functions are good approximations in many dynamical systems.

We will assume that the functions $f(t)$, $g(t)$, $h(t)$ are integrable and integrate to one. This assumption is, however, usually not necessary and is made only for mathematical convenience. The method can in fact be used in much more general cases because, even when the functions are not integrable, the problem can often be reduced to the case when the functions are integrable and integrate to one, see Ushakova (2008).

The rest of the paper is organized as follows. In Section 2 we prove the main theorem on which the method is based, and give a description of the method. In Section 3 the method is used for $f(t)$ of two parametric forms: gamma densities and shifted gamma densities. The mean squared error of the estimators are studied asymptotically using the delta method.

## 2 Method

Denote as usual

$$\left( \begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}$$

**Theorem 1.** *Let $f(x)$ and $g(x)$ be two integrable functions such that*

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} g(x)dx = 1,$$

*and let $h(x)$ be their convolution,*

$$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

*Define for $k = 0, 1, 2, \ldots$*

$$\mu_f^{(k)} = \int_{-\infty}^{\infty} x^k f(x)dx, \quad \mu_g^{(k)} = \int_{-\infty}^{\infty} x^k g(x)dx,$$

$$\mu_h^{(k)} = \int_{-\infty}^{\infty} x^k h(x)dx,$$

$$\gamma_f^{(k)} = \int_{-\infty}^{\infty} (x-\mu_f^{(1)})^k f(x)dx, \quad \gamma_g^{(k)} = \int_{-\infty}^{\infty} (x-\mu_g^{(1)})^k g(x)dx,$$

$$\gamma_h^{(k)} = \int_{-\infty}^{\infty} (x-\mu_h^{(1)})^k h(x)dx.$$

*provided that the corresponding integrals exist. Then*

$$\mu_h^{(n)} = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \mu_f^{(k)} \mu_g^{(n-k)}$$

*and*

$$\gamma_h^{(n)} = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \gamma_f^{(k)} \gamma_g^{(n-k)}$$

**Proof.**

$$\begin{aligned}
\mu_h^{(n)} &= \int_{-\infty}^{\infty} x^n \left( \int_{-\infty}^{\infty} f(x-y)g(y)dy \right) dx = \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y+y)^n f(x-y)g(y)dxdy = \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (x-y)^k y^{n-k} f(x-y)g(y)dxdy = \\
&= \sum_{k=0}^{n} \left[ \binom{n}{k} \int_{-\infty}^{\infty} y^{n-k} \left( \int_{-\infty}^{\infty} (x-y)^k f(x-y)dx \right) g(y)dy \right] = \\
&= \sum_{k=0}^{n} \left[ \binom{n}{k} \mu_f^{(k)} \int_{-\infty}^{\infty} y^{n-k} g(y)dy \right] = \sum_{k=0}^{n} \binom{n}{k} \mu_f^{(k)} \mu_g^{(n-k)}.
\end{aligned}$$

Now, using $\mu_h^{(1)} = \mu_f^{(1)} + \mu_g^{(1)}$,

$$\begin{aligned}
\gamma_h^{(n)} &= \int_{-\infty}^{\infty} (x-\mu_h^{(1)})^n \left( \int_{-\infty}^{\infty} f(x-y)g(y)dy \right) dx = \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y-\mu_f^{(1)}+y-\mu_g^{(1)})^n f(x-y)g(y)dxdy = \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (x-y-\mu_f^{(1)})^k (y-\mu_g^{(1)})^{n-k} f(x-y)g(y)dxdy = \\
&= \sum_{k=0}^{n} \left[ \binom{n}{k} \int_{-\infty}^{\infty} (y-\mu_g^{(1)})^{n-k} \times \right. \\
&\quad \left. \times \left( \int_{-\infty}^{\infty} (x-y-\mu_f^{(1)})^k f(x-y)dx \right) g(y)dy \right] = \\
&= \sum_{k=0}^{n} \left[ \binom{n}{k} \gamma_f^{(k)} \int_{-\infty}^{\infty} (y-\mu_g^{1})^{n-k} g(y)dy \right] = \sum_{k=0}^{n} \binom{n}{k} \gamma_f^{(k)} \gamma_g^{(n-k)}.
\end{aligned}$$

Taking into account that $\gamma_f^{(1)} = \gamma_g^{(1)} = \gamma_h^{(1)} = 0$, we derive the following

**Corollary.** *Under conditions of the theorem,*

$$\gamma_h^{(2)} = \gamma_f^{(2)} + \gamma_g^{(2)},$$

$$\gamma_h^{(3)} = \gamma_f^{(3)} + \gamma_g^{(3)}.$$

The method for parametric solution of the integral equation (1) is as follows. Let $f(t)$ be known up to $m$ unknown parameters $f(t) = f(t;\theta_1,\ldots,\theta_m)$. Express the first $m$ moments $\mu_f^{(1)},\ldots,\mu_f^{(m)}$ or/and central moments $\gamma_f^{(2)},\ldots,\gamma_f^{(m+1)}$ of the function $f(t)$ in terms of the parameters $\theta_1,\ldots,\theta_m$

$$\mu_f^{(i)} = \mu_f^{(i)}(\theta_1,\ldots,\theta_m), \quad i = 1,\ldots,m; \tag{3}$$

$$\gamma_f^{(j)} = \gamma_f^{(j)}(\theta_1,\ldots,\theta_m), \quad i = 2,\ldots,m+1. \tag{4}$$

Choose $m$ of the $2m$ equations (3), (4) (sometimes it is more convenient to use a moment, sometimes an absolute moment) and solve them with respect to $\theta_1,\ldots,\theta_m$:

$$\theta_i = \theta_i(\mu_f^{(1)},\ldots,\mu_f^{(m)},\gamma_f^{(2)},\ldots,\gamma_f^{(m+1)}) \quad i = 1,\ldots,m. \tag{5}$$

Moments of the functions $g(t)$ and $h(t)$ can be assumed known. Practically, they are found using some method of numerical integration. Since $\mu_g^{(1)},\ldots,\mu_g^{(m)}$, $\mu_h^{(1)},\ldots,\mu_h^{(m)}$, $\gamma_g^{(2)},\ldots,\gamma_g^{(m+1)}$, $\gamma_h^{(2)},\ldots,\gamma_h^{(m+1)}$ are known, we can find $\mu_f^{(1)},\ldots,\mu_f^{(m)}$, $\gamma_f^{(2)},\ldots,\gamma_f^{(m+1)}$ using the following recursive formulas, which follow from Theorem 1

$$\mu_f^{(k)} = \mu_h^{(k)} - \sum_{i=0}^{k-1} \binom{k}{i} \mu_f^{(i)} \mu_g^{(k-i)}, \quad k = 1,\ldots,m; \tag{6}$$

$$\gamma_f^{(k)} = \gamma_h^{(k)} - \sum_{i=0}^{k-1} \binom{k}{i} \gamma_f^{(i)} \gamma_g^{(k-i)}, \quad k = 2,\ldots,m+1. \tag{7}$$

The moments $\mu_f^{(1)},\ldots,\mu_f^{(m)}$, $\gamma_f^{(2)},\ldots,\gamma_f^{(m+1)}$, found from (6) and (7), are substituted in (5), and this gives estimates of parameters $\theta_1,\ldots,\theta_m$.

# 3 Gamma and shifted gamma models

In this section, we apply the suggested method to two important parametric families of functions, gamma and shifted gamma densities. Thus the unknown function $f(t)$ is either (2) or

$$f(t) = f(t;\alpha,\beta,\tau) = \frac{\beta^\alpha}{\Gamma(\alpha)}(t-\tau)^{(\alpha-1)}e^{-\beta(t-\tau)}, \quad t > \tau; \tag{8}$$

$$\alpha > 0, \quad \beta > 0, \quad \tau > 0.$$

Of course, (2) is a special case of (8), but we consider these two families separately because the estimators of $\alpha$ and $\beta$ are different for the two models, even when $\tau$ is estimated by zero.

## 3.1 Gamma model

Approximation by a gamma distribution is very popular and useful in many areas, in particular in dynamical systems. In such systems, the unknown function $f(t)$ is the distribution (density) of a delay, and this distribution is often well approximated by a gamma distribution. It is so, in particular, for many biological systems, see Mittler et al. (1998)

So, suppose that the unknown function has the form (2), where the parameters $\alpha$ and $\beta$ are unknown and need to be estimated. Since $\alpha = (\mu_f^{(1)})^2/\gamma_f^{(2)}$, $\beta = \mu_f^{(1)}/\gamma_f^{(2)}$, and since due to Theorem 1 and its Corollary, $\mu_f^{(1)} = \mu_h^{(1)} - \mu_g^{(1)}$ and $\gamma_f^{(2)} = \gamma_h^{(2)} - \gamma_g^{(2)}$, we have

$$\alpha = \frac{(\mu_h^{(1)} - \mu_g^{(1)})^2}{\gamma_h^{(2)} - \gamma_g^{(2)}}, \quad \beta = \frac{\mu_h^{(1)} - \mu_g^{(1)}}{\gamma_h^{(2)} - \gamma_g^{(2)}}.$$

Let $\hat{\mu}_h^{(1)}$, $\hat{\mu}_g^{(1)}$, $\hat{\gamma}_h^{(2)}$, $\hat{\gamma}_g^{(2)}$ be estimators of the moments $\mu_h^{(1)}$, $\mu_g^{(1)}$, $\gamma_h^{(2)}$, $\gamma_g^{(2)}$, respectively. Then the suggested estimators of the parameters $\alpha$ and $\beta$ are

$$\hat{\alpha} = \frac{(\hat{\mu}_h^{(1)} - \hat{\mu}_g^{(1)})^2}{\hat{\gamma}_h^{(2)} - \hat{\gamma}_g^{(2)}}, \quad \hat{\beta} = \frac{\hat{\mu}_h^{(1)} - \hat{\mu}_g^{(1)}}{\hat{\gamma}_h^{(2)} - \hat{\gamma}_g^{(2)}}. \tag{9}$$

Calculation of the integrals $\mu_h^{(1)}$, $\mu_g^{(1)}$, $\gamma_h^{(2)}$, $\gamma_g^{(2)}$, i.e. construction of estimates $\hat{\mu}_h^{(1)}$, $\hat{\mu}_g^{(1)}$, $\hat{\gamma}_h^{(2)}$, $\hat{\gamma}_g^{(2)}$, is a routine problem of numerical integration, which can be solved using standard techniques, see for example Cheney and Kincaid (2000). Methods for estimating errors of these estimates are also well known. Therefore, we do not consider this problem here and assume that distributions of these errors are given. We express the distributions of the estimators $\hat{\alpha}$ and $\hat{\beta}$ for the parameters of interest in terms of distributions of $\hat{\mu}_h^{(1)}$, $\hat{\mu}_g^{(1)}$, $\hat{\gamma}_h^{(2)}$, $\hat{\gamma}_g^{(2)}$. Denote

$$\mathbf{X}_n = (\hat{\mu}_h^{(1)}, \hat{\mu}_g^{(1)}, \hat{\gamma}_h^{(2)}, \hat{\gamma}_g^{(2)})^{\mathrm{T}}$$

and

$$\mathbf{c} = (\mu_h^{(1)}, \mu_g^{(1)}, \gamma_h^{(2)}, \gamma_g^{(2)})^{\mathrm{T}}.$$

Suppose that $\mathbf{X}_n$ is asymptotically normal

$$\sqrt{n}(\mathbf{X}_n - \mathbf{c}) \xrightarrow{D} \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}). \tag{10}$$

Denote (vectors of derivatives)

$$\mathbf{a} = \left( \frac{2(\mu_h^{(1)} - \mu_g^{(1)})}{\gamma_h^{(2)} - \gamma_g^{(2)}}, -\frac{2(\mu_h^{(1)} - \mu_g^{(1)})}{\gamma_h^{(2)} - \gamma_g^{(2)}}, -\frac{(\mu_h^{(1)} - \mu_g^{(1)})^2}{(\gamma_h^{(2)} - \gamma_g^{(2)})^2}, \frac{(\mu_h^{(1)} - \mu_g^{(1)})^2}{(\gamma_h^{(2)} - \gamma_g^{(2)})^2} \right)^{\mathrm{T}},$$

$$\mathbf{b} = \left( \frac{1}{\gamma_h^{(2)} - \gamma_g^{(2)}}, -\frac{1}{\gamma_h^{(2)} - \gamma_g^{(2)}}, -\frac{\mu_h^{(1)} - \mu_g^{(1)}}{(\gamma_h^{(2)} - \gamma_g^{(2)})^2}, \frac{\mu_h^{(1)} - \mu_g^{(1)}}{(\gamma_h^{(2)} - \gamma_g^{(2)})^2} \right)^{\mathrm{T}}.$$

Using the multivariate delta method, we obtain the following

**Theorem 2.** *Let (10) hold. Then*

$$\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{D} \mathcal{N}(\mathbf{0}, \mathbf{a}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{a}),$$

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{D} \mathcal{N}(\mathbf{0}, \mathbf{b}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{b}).$$

## 3.2 Shifted gamma model

When the distribution of a delay in a dynamical system is estimated, the gamma approximation has the following disadvantage. A delay, described by a gamma distribution, starts right after zero. For many dynamical systems, this is unrealistic: the event must “mature”. For example, in some biological models of HIV infection, such as Mittler et al. (1998), Nelson and Perelson (2002), describing the in-host dynamics of the infection, the model allows a delay from the time of infection of a cell by the virus until the production of new viral particles. In both of the aforementioned papers, a gamma delay density is assumed. However, the viral replication cycle is a multistage process which includes: entry to the cell, reverse transcription, integration into host DNA, transcription, translation, assembly of the viral particle and release of virions from the cell. All these processes definitely take some time to be accomplished. Hence, an extension of the gamma family to shifted gamma distributions seems to be a reasonable solution of the problem.

In this subsection, we consider this model, i.e. assume that the function $f(t)$ in (1) has the form (8). Since

$$\mu_f^{(1)} = \frac{\alpha}{\beta} + \tau, \quad \gamma_f^{(2)} = \frac{\alpha}{\beta^2}, \quad \gamma_f^{(3)} = \frac{2\alpha}{\beta^3},$$

we have the following expressions for $\alpha$, $\beta$, and $\tau$ in terms of $\mu_f^1$, $\gamma_f^2$, and $\gamma_z^3$:

$$\alpha = \frac{4(\gamma_f^{(2)})^3}{(\gamma_f^{(3)})^2}, \quad \beta = \frac{2\gamma_f^{(2)}}{\gamma_f^{(3)}}, \quad \tau = \mu_f^{(1)} - \frac{2(\gamma_f^{(2)})^2}{\gamma_f^{(3)}}.$$

Due to Theorem 1 and its Corollary, this gives

$$\alpha = \frac{4(\gamma_h^{(2)} - \gamma_g^{(2)})^3}{(\gamma_h^{(3)} - \gamma_g^{(3)})^2}, \quad \beta = \frac{2(\gamma_h^{(2)} - \gamma_g^{(2)})}{\gamma_h^{(3)} - \gamma_g^{(3)}},$$

$$\tau = \mu_h^{(1)} - \mu_g^{(1)} - \frac{2(\gamma_h^{(2)} - \gamma_g^{(2)})^2}{\gamma_h^{(3)} - \gamma_g^{(3)}}$$

Thus, in the considered case, the suggested estimators of $\alpha$, $\beta$, and $\tau$ are

$$\hat\alpha = \frac{4(\hat\gamma_h^{(2)} - \hat\gamma_g^{(2)})^3}{(\hat\gamma_h^{(3)} - \hat\gamma_g^{(3)})^2}, \quad \hat\beta = \frac{2(\hat\gamma_h^{(2)} - \hat\gamma_g^{(2)})}{\hat\gamma_h^{(3)} - \hat\gamma_g^{(3)}}, \tag{11}$$

$$\hat\tau = \hat\mu_h^{(1)} - \hat\mu_g^{(1)} - \frac{2(\hat\gamma_h^{(2)} - \hat\gamma_g^{(2)})^2}{\hat\gamma_h^{(3)} - \hat\gamma_g^{(3)}}. \tag{12}$$

Denote

$$\mathbf{X}_n = (\hat\mu_h^{(1)}, \hat\mu_g^{(1)}, \hat\gamma_h^{(2)}, \hat\gamma_g^{(2)}, \hat\gamma_h^{(3)}, \hat\gamma_g^{(3)})^{\mathrm{T}}$$

and

$$\mathbf{c} = (\mu_h^{(1)}, \mu_g^{(1)}, \gamma_h^{(2)}, \gamma_g^{(2)}, \gamma_h^{(3)}, \gamma_g^{(3)})^{\mathrm{T}}.$$

As in the previous section, suppose that $\mathbf{X}_n$ is asymptotically normal

$$\sqrt{n}(\mathbf{X}_n - \mathbf{c}) \xrightarrow{D} \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}). \tag{13}$$

Using the multivariate delta method, we obtain the following asymptotics for $\hat\alpha$, $\hat\beta$, $\hat\tau$ in terms of the limit distribution of the vector $\mathbf{X}_n$. Denote

$$\mathbf{a} = \left(0, 0, \frac{12(\gamma_h^{(2)} - \gamma_g^{(2)})^2}{(\gamma_h^{(3)} - \gamma_g^{(3)})^2}, -\frac{12(\gamma_h^{(2)} - \gamma_g^{(2)})^2}{(\gamma_h^{(3)} - \gamma_g^{(3)})^2}, -\frac{8(\gamma_h^{(2)} - \gamma_g^{(2)})^3}{(\gamma_h^{(3)} - \gamma_g^{(3)})^3}, \frac{8(\gamma_h^{(2)} - \gamma_g^{(2)})^3}{(\gamma_h^{(3)} - \gamma_g^{(3)})^3}\right),$$

$$\mathbf{b} = \left(0, 0, \frac{2}{\gamma_h^{(3)} - \gamma_g^{(3)}}, -\frac{2}{\gamma_h^{(3)} - \gamma_g^{(3)}}, -\frac{2(\gamma_h^{(2)} - \gamma_g^{(2)})}{(\gamma_h^{(3)} - \gamma_g^{(3)})^2}, \frac{2(\gamma_h^{(2)} - \gamma_g^{(2)})}{(\gamma_h^{(3)} - \gamma_g^{(3)})^2}\right),$$

$$\mathbf{d} = \left(1, -1, -\frac{4(\gamma_h^{(2)} - \gamma_g^{(2)})}{\gamma_h^{(3)} - \gamma_g^{(3)}}, \frac{4(\gamma_h^{(2)} - \gamma_g^{(2)})}{\gamma_h^{(3)} - \gamma_g^{(3)}}, \frac{2(\gamma_h^{(2)} - \gamma_g^{(2)})^2}{(\gamma_h^{(3)} - \gamma_g^{(3)})^2}, -\frac{2(\gamma_h^{(2)} - \gamma_g^{(2)})^2}{(\gamma_h^{(3)} - \gamma_g^{(3)})^2}\right).$$

**Theorem 3.** *Let (13) hold. Then*

$$\sqrt{n}(\hat\alpha - \alpha) \xrightarrow{D} \mathcal{N}(\mathbf{0}, \mathbf{a}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{a}),$$

$$\sqrt{n}(\hat\beta - \beta) \xrightarrow{D} \mathcal{N}(\mathbf{0}, \mathbf{b}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{b}),$$

$$\sqrt{n}(\hat\tau - \tau) \xrightarrow{D} \mathcal{N}(\mathbf{0}, \mathbf{d}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{d}).$$

Note that the estimators (9) of $\alpha$ and $\beta$ in the ordinary gamma model differ from the estimators (11) in the shifted gamma model when $\tau = 0$. At the same time, negative values of $\tau$ are “nonphysical”. Therefore, if the estimate $\hat\tau$ given by (12) is negative, we suggest to let $\tau$ be estimated by zero instead of (12), and to let $\alpha, \beta$ be estimated using the ordinary gamma model, i.e. by (9).

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