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Split-plot Designs with Mirror Image Pairs as Sub-plots

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In this article we investigate two-level split-plot designs where the sub-plots consist of only two mirror image trials. Assuming third and higher order interactions negligible, we show that these designs divide the estimated effects into two orthogonal sub-spaces, separating sub-plot main effects and sub-plot by whole-plot interactions from the rest. Further we show how to construct split-plot designs of projectivity $P \geq 3$. We also introduce a new class of split-plot designs with mirror image pairs constructed from non-geometric Plackett-Burman designs. The design properties of such designs are very appealing with effects of major interest free from full aliasing assuming that 3rd and higher order interactions are negligible.

KEY WORDS: Alias structure; Plackett-Burman designs; Projective properties; Restriction on randomization; Screening designs; Two-level designs.

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1. INTRODUCTION

Split-plot experiments are common in industry and typically are used when some factors are harder to change, such that a complete randomization of the experimental runs is difficult or impossible to perform. Split-plotting may also occur when two or more process steps are involved, in robust product design experimentation and when it is of interest to estimate some factors with higher precision than others where the latter are then handled as sub-plot factors.

Split-plot experiments conceptually consist of two experimental designs where the “sub-plot” design is embedded within another called the whole-plot design. In the original agricultural application, split-plotting refers to a situation where larger experimental units, called whole-plots – typically larger pieces of land, are split into smaller subunits called sub-plots. Whole-plot factor combinations from a whole-plot design are then randomly applied to the larger pieces of land and for each whole-plot, sub-plot factor combinations from the sub-plot design are randomly applied to the sub-plots.

When there are many factors, using two-level experimental plans helps keeping the number of experimental runs at an acceptable size. This is especially important for factor screening experiments in the early stages of an exploratory experimentation. Further reductions can be achieved by using fractional designs. Examples of such experimental plans for split-plot experiments can be found in Addelman (1964), Huang et al. (1998), Bingham and Sitter (1999, 2001, 2003), Bingham et al. (2004, 2005) and Bisgaard (2000). Another avenue for savings was pursued by Kulahci and Bisgaard (2005) using two-level Plackett-Burman (PB) designs (Plackett and Burman, 1946) and in some cases their half-fractions. Kulahci (2001) further suggested using two mirror image pairs at the sub-plot level for each level combination of the whole-plot factors (henceforth abbreviated as SPMIP designs). An advantage with these designs, besides their economical run sizes, is that their analysis can be done easily using ordinary least squares (OLS) in two subsequent and independent steps (Tyssedal and Kulahci, 2005). This simplifies the analysis of these designs and is particularly helpful when performing early stage factor screening where the goal is to identify the active factor space and not necessarily to estimate a given parametric model. The active factor space is typically expected to be of much lower dimension than the number of factors in the screening experiment. The identification then typically consists of a search procedure evaluating the ability of a number of subsets of factors to explain the variation in the response. The use of OLS will ease the interpretation and save time. Being able to perform the search in two independent steps instead of one large search, enables us to handle more factors and to identify active factor spaces of higher dimension. Moreover fitting specific parametric split plot models seems dubious at this early stage when it is not clear if the response may need a transformation; see Box and Cox (1964). For more explanation see Tyssedal and Kulahci, (2005).

Lists of two-level split-plot designs generated from regular two-level factorial designs have been provided by several authors. Huang et al. (1998) list minimum aberration designs for $N = 16, 32$ and 64, and Bingham and Sitter (1999) provide lists for $N = 8$ and 16. Designs with two subplots for each whole-plot level combination are also SPMIP designs. However, there are many design alternatives that are not provided by these lists. Bingham et al. (2004, 2005) also provide lists of minimum aberration SPMIP designs for $N = 16$ and 32, but only for a small number of whole-plot factors. All the designs on these lists are computer generated. In this paper we focus on the construction of SPMIP designs with desirable projective properties. Our designs are generated from both regular and non-regular orthogonal arrays. For reasons to be explained below we do not focus on minimum aberration designs. Instead we provide practitioners with information about which projectivity $P \geq 3$, see Section 4, SPMIP designs that are possible to construct and show a flexible way to construct them. We expect these designs to be particularly useful in screening situations when a complete randomization is difficult or impossible. However, they may also prove useful for robust design experimentation when a small subset of the factors investigated is of importance.

This article is organized as follows. In Section 2, we introduce the industrial experiment that originally inspired this research. We then proceed to develop the alias structure for SPMIP designs in Section 3. Thereafter, in Section 4, we explain projectivity of two-level designs as a useful and relevant criterion both for geometric (regular) and for non-geometric (non-regular) screening designs. In Section 5 we show how to construct SPMIP designs of projectivity $P \geq 3$ from geometric designs. A discussion example is included to illustrate how various experimental situations affect the choice of design. Section 6 deals with the construction of two-level designs from non-geometric designs. The article concludes with a general discussion in Section 7.

2. AN INDUSTRIAL EXAMPLE

The motivation for this research stems from consulting with a specialty paper company that needed to develop a new laminated paper product. The product was made in two-stages, the first involving making the paper and the second consisting of laminating the paper with another material. For a schematic illustration of the process, see Figure 1. For an early screening experiment the product development team wanted to experiment with 6 factors *A, B, C, D, E* and *F* for the first stage of the process and 5 factors *P, Q, R, S* and *T* for the second laminating stage. Each experimental trial in the first stage necessitated, as a minimum, the production of a whole roll of material; a changeover between factor settings was relatively expensive. The team felt that to produce reliable results the first stage experiment should as a minimum include 16 runs.

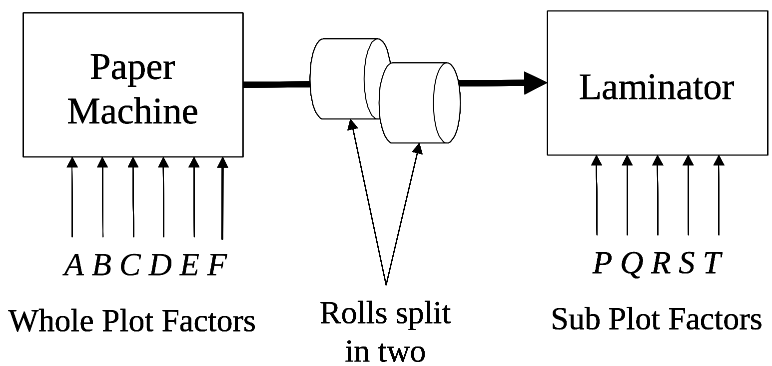

FIGURE 1. A two-stage manufacturing process for a laminated paper product.

For the second lamination stage, changeovers were somewhat easier and less expensive. Moreover, the two process steps were not connected. After products were made and laminated, samples were tested in a rather lengthy and expensive process that produced the final response. Because the final tests were expensive it was important to keep the total number of final tests small. For evaluating all these factors economically, it was suggested that rolls from the first stage should be cut in twin halves and fed through the lamination process doubling the number of units going through the second process stage. That implied a split-plot structure since two halves from the same roll would be more alike than two halves from different rolls.

To construct this experiment a standard 16 run 2^{6-2} design with generators $E = ABC$ and $F = BCD$ was used for the first stage. Each first stage unit was then divided in two and randomly assigned to second stage treatment combinations.

To design the second stage experiment, we exploited the property that any two-level factorial experiment blocked in blocks of two produces runs that are mirror images; see e.g. Box, Hunter and Hunter (1978, p. 340). Thus a regular $2^5 = 32$ run full factorial was blocked in 16 blocks of two trials each. It is clear that this design matrix can be written as

$$\mathbf{X} = \begin{pmatrix} \mathbf{W} & \mathbf{S} \\ \mathbf{W} & -\mathbf{S} \end{pmatrix}. \quad (1)$$

where $\mathbf{W}$ is the 2^{6-2} design and $\begin{bmatrix} \mathbf{S} \\ -\mathbf{S} \end{bmatrix}$ contains the 16 blocks.

Due to unfortunate circumstances, the sample identification numbers were messed up and so results from the experiment are not useful. However, the example nevertheless shows that split-plot experiments in this form represent a desirable way of planning experiments for an important class of practical situations. The blocking described in the second stage can be done in several ways leading to designs with different properties. Before discussing this issue we discuss the alias structure of SPMIP designs.

3. ALIAS STRUCTURE OF SPMIP DESIGNS

Let us consider split plot designs with pairs of sub plots constructed by fold-over. We can write the first order model matrix as

$$\mathbf{X}_1 = \begin{bmatrix} \mathbf{i} & \mathbf{W} & \mathbf{S} \\ \mathbf{i} & \mathbf{W} & -\mathbf{S} \end{bmatrix} \tag{2}$$

where $\mathbf{i}$ is a column of plus 1's. The matrix of two-factor interaction columns is

$$\mathbf{X}_2 = \begin{bmatrix} \mathbf{W}\times\mathbf{W} & \mathbf{W}\times\mathbf{S} & \mathbf{S}\times\mathbf{S} \\ \mathbf{W}\times\mathbf{W} & -\mathbf{W}\times\mathbf{S} & \mathbf{S}\times\mathbf{S} \end{bmatrix} \tag{3}$$

where $\mathbf{W}\times\mathbf{W}$ is the block matrix of the two-factor interaction columns between whole-plot factors, $\mathbf{W}\times\mathbf{S}$ is the block matrix of the two-factor interaction columns between a whole-plot and a sub-plot factor and $\mathbf{S}\times\mathbf{S}$ is the block matrix of the two-factor interaction columns between sub-plot factors. We will assume that interactions of order higher than 2 are negligible.

The alias structure of any design can according to Box and Wilson (1951) be found as follows. Suppose we fit the regression model $\mathrm{E}(\mathbf{Y}) = \mathbf{X}_1\boldsymbol{\beta}_1$, but in fact the true expectation is given by $\mathrm{E}(\mathbf{Y}) = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2$. The expected value of the least squares estimator $\hat{\boldsymbol{\beta}}_1 = \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{Y}$ is then $E\left(\hat{\boldsymbol{\beta}}_1\right) = \boldsymbol{\beta}_1 + \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{X}_2\boldsymbol{\beta}_2$. The matrix $\mathbf{A} = \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{X}_2$ is the *alias matrix* showing to what extent assumed model parameters will potentially be biased by additional active effects represented in $\mathbf{X}_2$.

If the error term $\boldsymbol{\varepsilon}$ is correlated and its variance-covariance matrix is $\boldsymbol{\Sigma}$ then the generalized alias matrix $\mathbf{A}^{GLS} = \left(\mathbf{X}_1'\boldsymbol{\Sigma}^{-1}\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2$ is more appropriate (Kulahci and Bisgaard (2006)). We will in this article only consider two-level designs with orthogonal first order model matrices and where the normal assumptions about the split-plot error covariance structure applies. Kulahci and Bisgaard (2006), then showed that for split-plot designs with a design matrix as given in (2), the alias matrix is still given by $\mathbf{A} = \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{X}_2$.

For $\mathbf{X}_1$ an orthogonal matrix we have $\left(\mathbf{X}_1'\mathbf{X}_1\right) = N\mathbf{I}$ and

$$(\mathbf{X}_1'\mathbf{X}_2) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2\mathbf{W}'(\mathbf{W}\times\mathbf{W}) & \mathbf{0} & 2\mathbf{W}'(\mathbf{S}\times\mathbf{S}) \\ \mathbf{0} & 2\mathbf{S}'(\mathbf{W}\times\mathbf{S}) & \mathbf{0} \end{bmatrix} \qquad (4)$$

Thus

$$\mathbf{A} = \frac{2}{N}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{W}'(\mathbf{W}\times\mathbf{W}) & \mathbf{0} & \mathbf{W}'(\mathbf{S}\times\mathbf{S}) \\ \mathbf{0} & \mathbf{S}'(\mathbf{W}\times\mathbf{S}) & \mathbf{0} \end{bmatrix} \qquad (5)$$

In the following we denote whole-plot main effects by *w* and sub-plot main effects main effects by *s*. Similarly two-factor interactions will be denoted by *ww*, *ws* and *ss* depending on whether they are between whole-plot factors, a whole-plot and a sub-plot factor or between sub-plot factors respectively.

The alias matrix (5) shows that *w* main effects are aliased with *ww* and *ss* interactions while *s* main effects are aliased with *ws* interactions only. From (3) we see that *ww* and *ss* interactions are orthogonal to *ws* interactions. We note that these properties are a consequence of the fold-over structure for the sub-plot columns and do not require the main effect columns to be orthogonal. Further, notice that the aliasing will be the same as that of the design $[\mathbf{W} \quad \mathbf{S}]$ and that the results obtained in this section are valid whether or not the design is run in split-plot mode. Restrictions on randomization, however, do influence the variance of the estimated effects. In the following we take a closer look at properties of split-plot designs with the above structure.

4. THE CONCEPT OF PROJECTIVITY FOR TWO-LEVEL DESIGNS.

For screening experimentation where only a few factors out of many are assumed to be active, the projective properties of the designs are important. The projection rationale makes the identification of the active factors less dependent on model assumptions and on effect hierarchy and effect heredity principles that are not necessarily valid in all situations, see Box and Tyssedal (2001). Indeed projectivity is a more relevant design criterion for constructing screening designs than any of the traditional alphabetic criteria used in optimal design theory.

Projectivity for two-level designs was introduced by Box and Tyssedal (1996). According to their definition, a $N \times k$ design with N runs and k factors each at two levels is said to be of projectivity P and called a (N, k, P) screen if every subset of P factors out of possible k contains a complete 2^P factorial design, possibly with some runs replicated.

Orthogonal two-level designs are of two types. Fractional factorial designs that are a regular $1/2^p$ fraction of a 2^k factorial are called geometric designs. The second type is called non-geometric designs or sometimes non-regular designs. The best known class of non-geometric is the Plackett-Burman designs. For geometric designs $P = R - 1$ where R is the resolution. However, for some non-geometric PB designs $P \geq 3$ while the resolution R=3 (Box and Bisgaard 1993, Box and Tyssedal 1996, Samset and Tyssedal 1999). Therefore design properties of non-geometric designs are not well described in terms of their resolution. However, the concept of projectivity is more universally useful for both types of designs. Designs with $P \geq 3$ have the important property that main-effects are not fully aliased with two-factor interactions. For geometric designs they are in fact free of aliasing assuming that 3rd and higher order interactions are negligible. This is also an important property for split plot experiments used for both factor screening and robust design experimentation, where specific *ws* interactions may be of practical interest; see Bisgaard (2000).

While the projective properties of completely randomized designs have been exhaustively studied in the literature, surprisingly little attention has been paid to these properties in split-plot designs. In the following we provide a detailed discussion of the projective properties of SPMIP designs constructed from both geometric and non-geometric designs. The focus will be on constructing designs of projectivity $P \geq 3$. In what follows we show how $P \geq 3$ SPMIP designs can be constructed from geometric designs.

5. CONSTRUCTION OF SPMIP DESIGNS FROM GEOMETRIC DESIGNS

Let $\mathbf{D}_{k-1}$ be a design matrix fully expanded with all two factor interaction columns and a column of $+1$'s for a 2^{k-1} design. Then the design matrix for the 2^k design with the same properties can always be written as

$$\mathbf{D}_k = \begin{bmatrix} \mathbf{D}_{k-1} & \mathbf{D}_{k-1} \\ \mathbf{D}_{k-1} & -\mathbf{D}_{k-1} \end{bmatrix} \qquad (6)$$

For instance let $\mathbf{i}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\mathbf{1}=\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}$, $\mathbf{2}=\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$, $\mathbf{12}=\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$ and $[\mathbf{D}_2]=[\mathbf{i}\quad \mathbf{1}\quad \mathbf{2}\quad \mathbf{12}]$ be the expanded design matrix for a 2^2 design. Then the expanded design matrix for the 2^3 design is given by:

$$\mathbf{D}_3=\begin{bmatrix}\mathbf{D}_2 & \mathbf{D}_2\\ \mathbf{D}_2 & -\mathbf{D}_2\end{bmatrix}=\begin{bmatrix}\mathbf{i} & \mathbf{1} & \mathbf{2} & \mathbf{12} & \mathbf{i} & \mathbf{1} & \mathbf{2} & \mathbf{12}\\ \mathbf{i} & \mathbf{1} & \mathbf{2} & \mathbf{12} & \mathbf{-i} & \mathbf{-1} & \mathbf{-2} & \mathbf{-12}\end{bmatrix} \qquad (7)$$

It is clear that the columns in any geometric design can be rearranged such that the design matrix has the structure of a SPMIP design. The corresponding design in 2^k-1 factors will be a $R=3$ and $P=2$ design. When used for screening it is natural to denote such designs as (N,nw,ns,P) screens where nw and ns give us the maximum number of whole-plot and sub-plot factors that can be screened in N runs in order to have projectivity P.

For N = 8, an 8 run $P=3$ SPMIP design in four factors is given by

$$\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{12} & \mathbf{i}\\ \mathbf{1} & \mathbf{2} & -\mathbf{12} & -\mathbf{i}\end{bmatrix} \qquad (8)$$

This design has two whole-plot factors and two sub-plot factors. In the Appendix it is shown that the maximum number of whole-plot factors as well as sub-plot factors for a $P=3$ SPMIP design is $N/4$. One way to obtain a $(N,N/4,N/4,3)$ screen is to start with a 2^{k-1} geometric design in $N/2$ runs fully expanded with main effects and interaction columns and do the following:

(a) Assign the whole-plot factors to the main-effects columns and odd $\left(3^{rd}\text{ order}, 5^{th}\text{ order}, \ldots\right)$ interaction effects columns of the $2^{k\text{-}1}$ design. There are at most $N/4$ such columns. This constitutes the $\mathbf{W}$ part of the design in (1).

(b) Assign the sub-plot factors to the even $\left(2^{nd}\text{ order}, 4^{th}\text{ order}, \ldots\right)$ interaction effects columns of the $2^{k\text{-}1}$ design and a column of plus 1's. There are at most $N/4$ such columns. This constitutes the $\mathbf{S}$ part of the design in (1).

If the number of either whole-plot or sub-plot factors is less than $N/4$, other ways of constructing SPMIP designs may give us $P = 3$ designs that better fit the experimental situation. If for instance only $n_w < k\text{-}1$ whole-plot columns are needed in $\mathbf{W}$ there are $k-1-n_w$ main effects columns that can be moved to $\mathbf{S}$.

In the following let **1, 2,**$\ldots$, **k** be the main effects columns in a 2^k design written in standard form. Interaction columns will then naturally be denoted with two or more of these numbers. For instance **12** will be the two-factor interaction column between factor 1 and factor 2**.**

*A discussion example****:*** Consider a split-plot design with three whole-plot factors and six sub-plot factors in 64 ($=2^6$) runs. Since we have a 64 ($=2^6$) run design, we will use $k-1=5$ main effect columns, namely **1, 2, 3, 4** and **5** together with the column of plus 1's of the 2^{6-1} design for the construction of the design. Furthermore since there are only three whole-plot factors, there are two main-effects columns that can be moved to $\mathbf{S}$ ($k-1-n_W = 6-1-3 = 2$). Let $[\mathbf{W}]=[\mathbf{1,2,3}]$ with corresponding whole-plot factors *A, B* and *C*. $\mathbf{S}$ will then consist of $\mathbf{i}$, **4** and **5**. We will denote the respective sub-plot factors as *P, Q* and *R*. Design columns for three more sub-plot factors *S, T* and *U* need to be constructed from the main effects columns $\mathbf{1,2,3,4}$ and $\mathbf{5}$. With 9 factors, it is only possible to obtain a $P = 3$ SPMIP design, see Table 4. In Table 1 three alternatives for the subplot design are given. Below we provide an argument for how to pick the "best" design alternative.

Minimum aberration (Fries and Hunter, 1984) is a popular criterion for discriminating between completely randomized designs of the same resolution. The idea behind minimum aberration is that among all designs of maximum resolution for given number of runs and factors, one should pick the one with the least amount of aliasing among low-order effects. However as pointed out in Kulahci et al. (2006), in split-plot experimentation it is often the case that not all two-factor interactions are of equal importance. Also some subset of factors may be of less interest than others. The minimum aberration criterion is not designed to distinguish among effects or subset of factors of unequal importance.

Table 1. Three alternatives for constructing the three additional sub-plot factor columns in a SPMIP design with 3 whole-plot factors and 6 sub-plot factors in 64 runs.

Subplot Design	Generators and defining relations
1. $[\mathbf{S}]=[\mathbf{i,4,5,123,1234,1235}]$	$S = ABCP, T = ABCQ$ and $U = ABCR$ $\Rightarrow I = ABCPS = ABCQT = ABCRU = PQST$ $= PSRU = QRTU = ABCPQRSTU$
2. $[\mathbf{S}]=[\mathbf{i,4,5,12,145,2345}]$	$S = ABP, T = APQR$ and $U = BCPQR$ $\Rightarrow I = ABPS = APQRT = BCPQRU = BSQRT$ $= ACQRSU = ABCTU = CPSTU$
3. $[\mathbf{S}]=[\mathbf{i,4,5,123,2345,145}]$	$S = ABCP, T = ABCQ$ and $U = ABCR$ $\Rightarrow I = ABCPS = ABCQT = ABCRU = PQST$ $= PSRU = QRTU = ABCPQRSTU$

For the three design alternatives given in Table 1, the first one has three words of length four in its defining relation and based on the minimum aberration criterion may seem inferior to design alternatives two and three that have only one. These designs are minimum aberration designs for which we have the fewest 4-letter words in the defining relation. But there are other important differences. In robust design experimentation *ws* interactions are normally of particular interest and if design factors are assigned to the whole-plot factors, *ss* interactions are of minor importance. We notice that the first and the third design allow us to estimate *ws* interactions free of aliasing with all other two-factor interactions. Also if a screening is performed and only four factors are active, the only projections onto four factors that are not complete 2^4 designs are the projections onto subplot factors for design alternatives 1 and 3. Split-plot experimentation where only sub-plot factors come out active is quite unexpected. The second design does not have these properties. In fact if *ss* interactions are of no interest and the possibility that only sub-plot factors are active can be ignored, design alternative 1 may be the best choice. For more discussion about the use of the minimum aberration criterion in split plot experimentation as well as other criteria we refer to Kulahci et al. (2006).

In Tables 2, 3 and 4, lists of geometric $P \geq 3$ designs for the number of runs $N = 16, 32$ and 64 are given. In these tables the design columns are columns in a 2^{k-1} design where $2^k = N$. If $n_w \leq k\text{-}1$, the columns in $\mathbf{W}$ are always assumed to be $\mathbf{1}, \ldots, \mathbf{n_w}$. Table 2 contains SPMIP designs for $N = 16$ and 32. To construct these designs, in $\mathbf{S}$ we put $\mathbf{i}$, main effects columns not used in $\mathbf{W}$ and interaction effects columns of the first $k\text{-}1$ factors of highest possible order under the restriction that two-factor interaction columns between one column allocated to $\mathbf{W}$ and one allocated to $\mathbf{S}$ are avoided. As an example consider the (32, 3, $n_s \le 4$, 3) screen in Table 2. A total of 32 runs means that columns $\mathbf{1, 2, 3}$ and $\mathbf{4}$ together with $\mathbf{i}$ are the ones that can be used for the construction. Columns **1**, **2** and **3** will be used to allocate the three whole plot factors. Therefore columns **i** and **4** are free to be used for the subplot factors. With **4** allocated to $\mathbf{S}$, **14**, **24** and **34** must be avoided. If the interaction column of highest possible order, **1234**, is put in $\mathbf{S}$, then the **234, 134,** and **124** columns cannot be used either. Similarly if **123** is chosen to be in $\mathbf{S}$, **12, 13** and **23** can no longer be used.

The fourth column of Table 2 gives a way of constructing possible $P = 3$ SPMIP designs from geometric designs. Consider for example the first design in Table 1 with 3 whole-plot (*A*, *B*, *C*) and 6 sub-plot factors (*P*, *Q*, *R*, *S*, *T*, *U*). Recall that for that design, we used $S = ABCP$, $T = ABCQ$ and $U = ABCR$ to construct *S*, *T* and *U*. Since in constructing these columns we use all three main effects (*A*, *B*, *C*) and only one of the remaining subplot factors (*P*, *Q*, *R*), we denote this construction scheme with $\binom{3}{3}\binom{3}{1}$. In general let $\mathbf{1}, \cdots, \mathbf{k-1}$ and $\mathbf{i}$ be denoted as main effects columns and let w_m of the columns $\mathbf{1}, \cdots, \mathbf{k-1}$ be allocated to $\mathbf{W}$ and $s_m = k - w_m$ of the rest be allocated to $\mathbf{S}$. Further let the notation

$$\binom{w_m}{p}\binom{s_m}{q} \tag{9}$$

represent all possible ways of entry-wise multiplying p main-effects columns in $\mathbf{W}$ out of a total of $w_m \le k-1$ with q main-effects columns in $\mathbf{S}$ out of a total of $s_m = k - w_m$. We should note that it is necessary that q is odd, otherwise the fold-over property of the sub-plot columns is lost, and also in order to avoid two-factor interactions, *p* and *q* cannot

TABLE 2. A list of geometric projectivity $P = 3$ SPMIP designs for the numbers of runs $N \le 32$.

N	Screens	Columns in $\mathbf{W}$ and $\mathbf{S}$	Construction of sub-plot columns from main effects columns
16	(16, $n_w \le 4$, $n_s \le 4$, 3)	$[\mathbf{W}] = [\mathbf{1,2,3,123}]$ $[\mathbf{S}] = [\mathbf{i,12,13,23}]$	$\binom{3}{2}\binom{1}{1}$
32	(32, 1, $n_s \le 8$, 3)	$[\mathbf{S}] = [\mathbf{i,2,3,4,123,124,134,1234}]$	$\binom{1}{1}\binom{4}{3}$
	(32, 2, $n_s \le 8$, 3)	$[\mathbf{S}] = [\mathbf{i,3,4,134,234,124,123,12}]$	$\binom{2}{1}\binom{3}{3}+\binom{2}{2}\binom{3}{1}$
	(32, 3, $n_s \le 4$, 3)	$[\mathbf{S}] = [\mathbf{i,4,123,1234}]$	$\binom{3}{3}\binom{2}{1}$
	(32, 3, $n_s \le 8$, 3)	$[\mathbf{S}] = [\mathbf{i,4,12,134,234,13,124,23}]$	$\binom{3}{2}\binom{2}{1}$
	(32, 4, $n_s \le 5$, 3)	$[\mathbf{S}] = [\mathbf{i,123,124,134,234}]$	$\binom{4}{3}\binom{1}{1}$
	(32, 4, $n_s \le 8$, 3)	$[\mathbf{S}] = [\mathbf{i,1234,12,34,13,24,14,23}]$	$\binom{4}{4}\binom{1}{1}+\binom{4}{2}\binom{1}{1}$
	(32, 5, $n_s \le 5$, 3)	$[\mathbf{W}] = [\mathbf{1,2,3,4,1234}]$, $[\mathbf{S}] = [\mathbf{i,123,124,134,234}]$	$\binom{4}{3}\binom{1}{1}$
	(32, 5, $n_s \le 7$, 3)	$[\mathbf{W}] = [\mathbf{1,2,3,4,1234}]$, $[\mathbf{S}] = [\mathbf{i,12,34,13,24,14,23}]$	$\binom{4}{2}\binom{1}{1}$
	(32, 5, $n_s \le 8$, 3)	$[\mathbf{W}] = [\mathbf{1,2,3,4,123}]$, $[\mathbf{S}] = [\mathbf{i,1234,12,34,13,24,14,23}]$	$\binom{4}{4}\binom{1}{1}+\binom{4}{2}\binom{1}{1}$
	(32, 6, $n_s \le 4$, 3)	$[\mathbf{W}] = [\mathbf{1,2,3,4,123,124}]$, $[\mathbf{S}] = [\mathbf{i,134,234,12}]$	
	(32, 6, $n_s \le 8$, 3)	$[\mathbf{W}] = [\mathbf{1,2,3,4,123,124}]$, $[\mathbf{S}] = [\mathbf{i,1234,12,34,13,24,14,23}]$	$\binom{4}{4}\binom{1}{1}+\binom{4}{2}\binom{1}{1}$
	(32, $n_w \le 8$, $n_s \le 8$, 3)	$[\mathbf{W}] = [\mathbf{1,2,3,4,123,124,134,234}]$, $[\mathbf{S}] = [\mathbf{i,1234,12,34,13,24,14,23}]$	$\binom{4}{4}\binom{1}{1}+\binom{4}{2}\binom{1}{1}$

both be one. Also if in the construction of the additional sub-plot factor columns two terms of type (9) with the same p (or q) occur, the corresponding q's (or p's) must differ with at least 2 for the same reason. Hence we could augment the first design in Table 1 with the $\binom{3}{3}\binom{3}{3}$ sub-plot column and still preserve the property that full aliasing among two-factor interactions only happens for the *ss* interactions. It is possible to extend this design with the three $\binom{3}{1}\binom{3}{3}$ columns, but then this property will be lost. We observe that the (32, $6, n_s \le 4$, 3) screen in Table 2 cannot be represented in this way. Here $n_w = 6$ and two of the four $\binom{4}{3}\binom{1}{1}$ possible sub-plot factor columns will become interaction columns between a whole-plot factor and a sub-plot factor column. Leaving these columns out allows us to add exactly one particular $\binom{4}{2}\binom{1}{1}$ column. In general, whenever $n_w > k-1$, two-factor interaction columns between one column allocated to $\mathbf{W}$ and one allocated to $\mathbf{S}$ must be avoided when using this method.

In Table 3 we list $P = 3$ SPMIP design for N=64. For these designs, *nw* and *ns* can both be 16 and a complete listing in a table will be rather large. Therefore only designs represented by means of the $\binom{w_m}{p}\binom{s_m}{q}$ notation are given.

SPMIP designs of projectivity $P \ge 4$ need to be fractional factorial designs of resolution 5 or higher and they can therefore estimate main effects and two-factor interactions free of aliasing given that third and higher order interactions are inert, Box, Hunter and Hunter (1978). The total number of factors allowed in order to have resolution greater than four for *N* = 16, 32 and 64 is 5, 6 and 8 respectively (Box et al. 2005). For *N* = 16 and 32, except when *N* = 16 and n_w = 2, using one interaction column of as high an order as possible together with the main effect columns of the corresponding 2^{k-1} design will always give us a $P \geq 4$ SPMIP designs, see Table 4.

TABLE 3. A list of $P = 3$ SPMIP designs for N=64 .

Screens	Columns in **S** used for construction of sub-plot columns	Construction of additional sub-plot columns
(64, 1, $n_s \leq 16$, 3)	**i, 2, 3, 4, 5**	$\binom{1}{1}\binom{5}{5}+\binom{1}{1}\binom{5}{3}$
(64, 2, $n_s \leq 8$, 3)	**i, 3, 4, 5**	$\binom{2}{2}\binom{4}{3}$
(64, 2, $n_s \leq 16$, 3)	**i, 3, 4, 5**	$\binom{2}{1}\binom{4}{3}+\binom{2}{2}\binom{4}{1}$
(64, 3, $n_s \leq 7$, 3)	**i, 4, 5**	$\binom{3}{3}\binom{3}{1}+\binom{3}{3}\binom{3}{3}$, see also example
(64, 3, $n_s \leq 16$, 3)	**i, 4, 5**	$\binom{3}{1}\binom{3}{3}+\binom{3}{2}\binom{3}{1}+\binom{3}{3}\binom{3}{3}$
(64, 4, $n_s \leq 10$, 3)	**i ,5**	$\binom{4}{3}\binom{2}{1}$
(64, 4, $n_s \leq 16$,3)	**i ,5**	$\binom{4}{2}\binom{2}{1}+\binom{4}{4}\binom{2}{1}$
(64, 5, $n_s \leq 6$,3)	**i**	$\binom{5}{4}\binom{1}{1}$
(64, 5, $n_s \leq 16$,3)	**i**	$\binom{5}{4}\binom{1}{1}+\binom{5}{2}\binom{1}{1}$
(64, 6, $n_s \leq 6$, 3) $[\mathbf{W}]=[\mathbf{1,2,3,4,5,12345}]$	**i**	$\binom{5}{4}\binom{1}{1}$
(64, 10, $n_s \leq 6$, 3) $[\mathbf{W}]=[\mathbf{1,2,3,4,5,1234,1235,\ldots,2345}]$	**i**	$\binom{5}{5}\binom{1}{1}+\binom{5}{3}\binom{1}{1}$
(64, $n_w \leq 16$, $n_s \leq 16$, 3)	**i**	$\binom{5}{4}\binom{1}{1}+\binom{5}{2}\binom{1}{1}$

For N = 64 there is room for constructing two columns in $[\mathbf{W} \quad \mathbf{S}]$ that are not main-effects columns. If these two columns are in $\mathbf{S}$, both of them as well as their interaction column need to be at least a three-factor interaction column in order to have projectivity $P = 4$. If one is in $\mathbf{W}$ and one in $\mathbf{S}$, their interaction column can be a two-

TABLE 4. A list of geometric projectivity $P \geq 4$ SPMIP designs for the numbers of runs $N \leq 64$.

Number of runs $N =$	Screens	Way of construction
16	(16, 1, 4, 4)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{2}, \mathbf{3}, \mathbf{123}]$
	(16, 2, 2, 4)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{3}]$
	(16, 3, 2, 4)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{123}]$
32	(32, 1, 5, 4)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{1234}]$
	(32, 2, 4, 5)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{3}, \mathbf{4}, \mathbf{1234}]$
	(32, 3, 3, 4)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{4}, \mathbf{1234}]$
	(32, 4, 2, 5)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{1234}]$
	(32, 5, 1, 4)	$[\mathbf{W}] = [\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{1234}]$ $[\mathbf{S}] = [\mathbf{i}]$
64	(64, 1, 6, 6)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{12345}]$
	(64, 1, 7, 4)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{123}, \mathbf{1345}]$
	(64, 2, 5, 5)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{1234}]$
	(64, 2, 6, 4)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{134}, \mathbf{2345}]$
	(64, 3, 4, 6)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{4}, \mathbf{5}, \mathbf{12345}]$
	(64, 3, 5, 4)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{4}, \mathbf{5}, \mathbf{145}, \mathbf{2345}]$
	(64, 4, 3, 5)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{5}, \mathbf{1234}]$
	(64, 4, 4, 4)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{5}, \mathbf{123}, \mathbf{2345}]$
	(64, 5, 2, 6)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{12345}]$
	(64, 5, 3, 4)	$[\mathbf{S}] = [\mathbf{i}, \mathbf{123}, \mathbf{345}]$
	(64, 6, 1, 5)	$[\mathbf{W}] = [\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{12345}]$, $[\mathbf{S}] = [\mathbf{i}]$
	(64, 6, 2, 4)	$[\mathbf{W}] = [\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{12345}]$, $[\mathbf{S}] = [\mathbf{i}, \mathbf{123}]$

factor interaction column, but the column in $\mathbf{W}$ must at least be a four-factor interaction column since $\mathbf{W}$ is not folded over. It is interesting to note that the design projectivity depends on the number of whole-plot factors and sub-plot factors as well as the total number of factors.

The lists provided do not cover all possibilities. We believe, however, that the designs given will be useful in many experimental situations and the guidelines given in order to obtain them will be useful in cases where the experimenter finds it more satisfactory to construct his/her designs.

The $P = 3$ SPMIP designs are of practical importance whenever the experimental plan follows a split-plot structure, especially when a screening is performed and some factors are harder to change compared to others. However, all the $(N, n_w, n_s, 3)$ screens based on geometric designs suffer from the same problem as geometric $P = 3$ designs. For three factors to be uniquely identified, the three-factor interaction has to be assumed negligible and the fact that two-factor interactions are fully aliased with other two-factor interactions complicates the allocation of active contrasts to the individual effects. If the projectivity is to be increased, the number of runs will get large. Another drawback might be the restriction on the number of whole-plot and sub-plot factors that can be included in the design. The non-geometric designs will therefore be valuable alternatives to use as building blocks for designs with the above structure.

6. CONSTRUCTION OF SPMIP DESIGNS FROM NON-GEOMETRIC DESIGNS

We will here only handle non-geometric PB designs. In Box and Tyssedal (1996), these were classified according to projectivity $P = 2$ and $P = 3$ for number of runs less than or equal to 84, and in Samset and Tyssedal (1999) they were also classified with respect to projectivity $P = 4$. Some of these PB designs are constructed by doubling, a technique that is illustrated using the 12 run PB design below:

$$\begin{bmatrix} \text{PB12} & \text{PB12} & \mathbf{i} \\ \text{PB12} & -\text{PB12} & -\mathbf{i} \end{bmatrix} \tag{12}$$

where PB12 represents the 11 columns of the 12 run PB design. This is an orthogonal design with 23 factors in 24 runs and follows the structure for split plot designs given in (1). However, as shown in Box and Tyssedal (1996), this design has $P = 2$ unless the column $\begin{bmatrix} \mathbf{i} \\ -\mathbf{i} \end{bmatrix}$ is removed. Then this design constitutes a $(24,11,11,3)$ screen which is impressive in terms of allowable number of factors and number of runs of same projectivity compared to geometric designs. Most non-geometric PB designs are $(n, n-1, 3)$ screens from which $(2n, n-1, n-1, 3)$ screens can be constructed. The exceptions are for the number of runs $n = 40, 56, 88$ or 96 where only n- 2 of the main-effects columns can be used. From these $(2n, n-2, n-2, 3)$ screens can be constructed. Note that this technique can also be used to create $P = 3$ split-plot designs from geometric designs as long as $[\mathbf{W}]$ is a $P = 3$ design.

Despite the impressive number of whole-plot and sub-plot factors that can be included in a $(N, nw, ns, 3)$ screen obtained from a non-geometric design using the doubling technique, these designs have a problem in common with $P = 3$ SPMIP designs constructed from geometric designs. Some whole-plot interactions are fully aliased with sub-plot interactions which may complicate the identification of the active effects. This can be avoided by using distinct columns in $\mathbf{W}$ and $\mathbf{S}$; for instance different columns from non-geometric $P = 3$ PB design as shown in Table 5.

The design in Table 5 has 11 factors $A, B, \cdots, K$ in 24 runs. It is constructed from a 12 run PB design. To obtain the 24 run SPMIP design, the first six columns of PB12 are replicated and the remaining 5 columns are folded over. In the split-plot structure the first 6 columns represent whole-plot columns and the last 5 are sub-plot columns. While a P = 3 split-plot design constructed from geometric designs exists for our industrial example (see Table 2) the design in Table 5 represents an alternative design in only 24 runs where complete aliasing between two-factor interactions is avoided.

Adding a column of the form $\begin{bmatrix} \mathbf{i} \\ -\mathbf{i} \end{bmatrix}$ gives us the following design matrix :

$$\begin{bmatrix} \mathbf{i} & \mathbf{W} & \mathbf{S} & \mathbf{i} \\ \mathbf{i} & \mathbf{W} & -\mathbf{S} & -\mathbf{i} \end{bmatrix}. \tag{13}$$

The resulting design has no restriction on n_w and n_s except that $n_w + n_s \leq 12$. Let us denote these designs as $(N, n_w + n_s \leq m, P)$ screens where m is the maximum number of factors allowed in order to have a projectivity P design. For $N = 16$ it is, as pointed out earlier, possible to have $(16, 4, 4, 3)$ screens or $(16, 8, 7, 2)$ screens (Huang et al (1998), Bingham and Sitter (1999)). At the expense of increasing the number of runs to 24, in the $(24, n_w + n_s \leq 12, 3)$ design, n_w and n_s can be anything between 1 and 11 as long as their sum does not exceed 12.

TABLE 5. A 24 run SPMIP design constructed from a 12 run PB design. The 6 first columns are repeated twice. The last 5 are folded over.

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
1	-1	1	-1	-1	-1	1	1	1	-1	1
1	1	-1	1	-1	-1	-1	1	1	1	-1
-1	1	1	-1	1	-1	-1	-1	1	1	1
1	-1	1	1	-1	1	-1	-1	-1	1	1
1	1	-1	1	1	-1	1	-1	-1	-1	1
1	1	1	-1	1	1	-1	1	-1	-1	-1
-1	1	1	1	-1	1	1	-1	1	-1	-1
-1	-1	1	1	1	-1	1	1	-1	1	-1
-1	-1	-1	1	1	1	-1	1	1	-1	1
1	-1	-1	-1	1	1	1	-1	1	1	-1
-1	1	-1	-1	-1	1	1	1	-1	1	1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	1	-1	-1	-1	-1	-1	-1	1	-1
1	1	-1	1	-1	-1	1	-1	-1	-1	1
-1	1	1	-1	1	-1	1	1	-1	-1	-1
1	-1	1	1	-1	1	1	1	1	-1	-1
1	1	-1	1	1	-1	-1	1	1	1	-1
1	1	1	-1	1	1	1	-1	1	1	1
-1	1	1	1	-1	1	-1	1	-1	1	1
-1	-1	1	1	1	-1	-1	-1	1	-1	1
-1	-1	-1	1	1	1	1	-1	-1	1	-1
1	-1	-1	-1	1	1	-1	1	-1	-1	1
-1	1	-1	-1	-1	1	-1	-1	1	-1	-1
-1	-1	-1	-1	-1	-1	1	1	1	1	1

Compared to the $(16,8,7,2)$ screen, the projectivity is increased by one and hence no full aliasing exists between main effects and two-factor interactions or between any two two-factor interactions. Complete aliasing between two-factor interactions will only happen if the number of runs in the base design equals 40, 56, 88 or 96 (Samset and Tyssedal 1999). The $\begin{bmatrix} \mathbf{i} \\ -\mathbf{i} \end{bmatrix}$ column is orthogonal to all main effects and two-factor interaction columns. Hence a factor assigned to this column has the special property that its main effect and interaction effects with all other factors can be estimated free of aliasing directly. This column is an obvious candidate for a sub-plot factor of particular interest.

Fold-over of non-geometric PB designs can be used to construct SPMIP designs for which main effects can be estimated free of aliasing from two-factor interactions. For instance the design created from the 12 run PB design by fold-over:

$$\begin{bmatrix} \text{PB12} & \mathbf{i} \\ \text{- PB12} & -\mathbf{i} \end{bmatrix} \tag{14}$$

is a $(24,12,4)$ screen, with all main effect columns orthogonal to two-factor interaction columns. Picking $\mathbf{W}$ and $\mathbf{S}$ columns from this design expanded by a column of the form $\begin{bmatrix} \mathbf{i} \\ \mathbf{i} \end{bmatrix}$ allow us to construct a design with up to 13 whole-plot and sub-plot factors in 48 runs, with the desired property. In this case some two-factor interactions will be partially confounded. Table 6 provides a list of SPMIP designs with good projective properties constructed from non-geometric PB designs.

TABLE 6. A list of orthogonal two-level projectivity $P=3$ and $P=4$ SPMIP designs constructed from non-geometric designs for the number of runs $N \leq 96$.

Number of runs $N=$	$P=3$ screens	Way of construction	$P=4$ screens	Way of construction
24	$(24,11,11,3)$ $(24, n_w + n_s \leq 12, 3)$	From PB12 From PB12		
40	$(40,19,19,3)$ $(40, n_w + n_s \leq 20, 3)$	From PB20 From PB20		
48	$(48,23,23,3)$ $(48, n_w + n_s \leq 24, 3)$	From PB24 From PB24	$(48, nw + ns \leq 13, 4)$	From fold-over of PB12
56	$(56,27,27,3)$ $(56, n_w + n_s \leq 28, 3)$	From PB28 From PB28		
64	$(64, n_w + n_s \leq 32, 3)$	From PB32		
72	$(72,35,35,3)$ $(72, n_w + n_s \leq 36, 3)$	From PB36 From PB 36		
80	$(80,38,38,3)$ $(80, n_w + n_s \leq 40, 3)$	From PB40 From PB40	$(80, n_w + n_s \leq 21, 4)$	From fold-over of PB20
88	$(88,43,43,3)$ $(88, n_w + n_s \leq 44, 3)$	From PB44 From PB 44		
96	$(96,47,47,3)$ $(96, n_w + n_s \leq 48, 3)$	From PB48 From PB48	$(96, n_w + n_s \leq 25, 4)$	From fold-over of PB24

7. DISCUSSION

Split-plot designs, where for each level combination of the whole-plot factors only two runs as mirror images are used, are very run efficient. These designs are useful in many practical situations. Typical examples include treatments to be applied to subjects with two eyes, two ears, or two feet, or animal litters of two being easier to obtain than of four or more. In these situations the proposed designs will yield useful and effective alternatives. Moreover these designs split *s* and *ws* effects and *w*, *ww* and *ss* effects into two orthogonal sub-spaces, which simplify the identification of active factors and also the estimation of active effects. This is discussed further in Tyssedal and Kulahci (2005).

For screening and model estimation it is an advantage to be able to separate main effects from two-factor interactions. Designs of projectivity $P=3$ or higher avoid full aliasing between main effects and two-factor interactions. For geometric designs they even ensure these effects to be free of aliasing. If many factors need to be investigated the $(N, nw, ns, P \geq 3)$ screens constructed from non-geometric designs offer the advantage of allowing more factors to be investigated in designs with a fixed projectivity compared to designs constructed from geometric designs. To be able to clearly estimate the two-factor interactions, geometric designs will need to be of projectivity $P=4$ or higher. This corresponds to more runs or fewer factors in the investigation. Interesting run-efficient alternatives are the $(N, n_w + n_s \leq m, P)$ screens constructed from non-geometric PB designs by repeating some columns and taking a fold-over of the others. Except when the number of runs in the baseline designs is equal to 40, 56, 88 or 96, these designs have only partial confounding between any two effects as long as these are main effects or two-factor interactions, see Samset and Tyssedal (1999). This makes them very appealing split-plot designs for both screening and robust design experimentation.

APPENDIX: A PROOF OF THE NUMBER OF ALLOWED WHOLE-PLOT AND SUB-PLOT FACTOR COLUMNS IN A *P*=3 SPMIP DESIGN CONSTRUCTED FROM A GEOMETRIC DESIGN

For matrices defined as in (6) the following general properties hold.

1. Entry-wise multiplying together two columns gives a new column in the matrix.
2. For any factor exactly half of the columns are involved with that factor either as a main effect column or as interaction effects columns.

Result: If $N = 2^k$, $k \geq 3$ *is the number of runs in a SPMIP design constructed from a geometric design, the maximum number of whole-plot factors as well as sub-plot factors allowed in order to have a* $P = 3$ *SPMIP design is* $N/4$.

Proof : For *N*=8 an 8 run $P = 3$ SPMIP is given in (8). Clearly we can not augment on any of the numbers of whole-plot and sub-plot factors in this design without adding interaction columns.

Assume $N > 8$. The columns in $\begin{bmatrix} \mathbf{W} \\ \mathbf{W} \end{bmatrix}$ have to be taken from $\begin{bmatrix} \mathbf{D}_{\text{k-1}} \\ \mathbf{D}_{\text{k-1}} \end{bmatrix}$. $\begin{bmatrix} \mathbf{W} \\ \mathbf{W} \end{bmatrix}$ is of projectivity $P = 3$ if and only if $\mathbf{W}$ is of projectivity $P = 3$ which is only possible for $N/4$ of the columns or less, Box and Tyssedal (1996). Now let $\mathbf{i}$ be a column of 2^{k-1} +1's. The columns in $\begin{bmatrix} \mathbf{D}_{\text{k-1}} \\ -\mathbf{D}_{\text{k-1}} \end{bmatrix}$ are obtained by entry-wise multiplying the columns in

$\begin{bmatrix} \mathbf{D}_{k-1} \\ \mathbf{D}_{k-1} \end{bmatrix}$ by $\begin{bmatrix} \mathbf{i} \\ \mathbf{-i} \end{bmatrix}$. The following construction will always give a $(N, N/4, N/4, 3)$ screen. Choose $N/4$ columns from $\begin{bmatrix} \mathbf{D}_{k-1} \\ \mathbf{D}_{k-1} \end{bmatrix}$ such that $\begin{bmatrix} \mathbf{W} \\ \mathbf{W} \end{bmatrix}$ is of projectivity $P = 3$ and let $\begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{W} \end{bmatrix}^*$ be the remaining columns in $\begin{bmatrix} \mathbf{D}_{k-1} \\ \mathbf{D}_{k-1} \end{bmatrix}$. Note that the columns in $\begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{W} \end{bmatrix}^*$ can be obtained by entrywise multiplying the columns in $\begin{bmatrix} \mathbf{W} \\ \mathbf{W} \end{bmatrix}$ by an arbitrary factor column in $\begin{bmatrix} \mathbf{W} \\ \mathbf{W} \end{bmatrix}$. Let $\begin{bmatrix} \mathbf{S} \\ \mathbf{-S} \end{bmatrix}$ be the columns obtained by entrywise multiplying the columns in $\begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{W} \end{bmatrix}^*$ by $\begin{bmatrix} \mathbf{i} \\ \mathbf{-i} \end{bmatrix}$. Then $\begin{bmatrix} \boldsymbol{W} & \boldsymbol{S} \\ \boldsymbol{W} & -\boldsymbol{S} \end{bmatrix}$ is of projectivity $P = 3$ since no entrywise product of two of its columns are in $\begin{bmatrix} \boldsymbol{W} & \boldsymbol{S} \\ \boldsymbol{W} & -\boldsymbol{S} \end{bmatrix}$.

Now for any one of the whole plot factors in $\begin{bmatrix} \mathbf{D}_{k-1} \\ \mathbf{D}_{k-1} \end{bmatrix}$, say factor A, exactly $N/4$ of the columns in $\begin{bmatrix} \mathbf{D}_{k-1} \\ \mathbf{D}_{k-1} \end{bmatrix}$ is involved with that factor, either as a main-effect column or as interaction effects columns. Therefore exactly $N/4$ of the columns in $\begin{bmatrix} \mathbf{D}_{k-1} \\ \mathbf{-D}_{k-1} \end{bmatrix}$ are interaction effects columns where factor A is involved. These columns entrywised multiplied by the column for factor A will then necessarily give the remaining $N/4$

columns in $\begin{bmatrix} \mathbf{D}_{\text{k-1}} \\ \mathbf{-D}_{\text{k-1}} \end{bmatrix}$. Since two-factor interaction columns have to be excluded we can have at most $N/4$ columns in $\begin{bmatrix} \mathbf{S} \\ \mathbf{-S} \end{bmatrix}$ in order to have a $P = 3$ SPMIP design.

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