NORGES TEKNISK-NATURVITENSKAPELIGE UNIVERSITET

On the Equivalence of Systems of Different Sizes

by

Bo H. Lindqvist, Francisco J. Samaniego, Arne B. Huseby

PREPRINT
STATISTICS NO. 2/2014

NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
TRONDHEIM, NORWAY

This report has URL
http://www.math.ntnu.no/preprint/statistics/2014/S2-2014.pdf
Bo H. Lindqvist has homepage: http://www.math.ntnu.no/∼bo
E-mail: bo@math.ntnu.no
Address: Department of Mathematical Sciences, NTNU, N-7491 Trondheim, Norway.

ON THE EQUIVALENCE OF SYSTEMS OF DIFFERENT SIZES

BO H. LINDQVIST,* *Norwegian University of Science and Technology*

FRANCISCO J. SAMANIEGO,** *University of California, Davis*

ARNE B. HUSEBY,*** *University of Oslo*

* Postal address: Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

** Postal address: Department of Statistics, Mathematical Sciences Building, One Shields Avenue, University of California, Davis, CA 95616, USA

*** Postal address: Department of Mathematics, University of Oslo, P.O.Box 1053 Blindern, N-0316, Oslo, Norway

Abstract

The signature of a coherent system with independent and identically distributed component lifetimes has been found to be a useful tool in the study and comparison of lifetimes of engineered systems. A key result is the representation of a system's survival distribution in terms of its signature vector, which leads to several results on stochastic comparison of system lifetimes. In order to compare two coherent systems of different sizes with respect to their signatures, the smaller system needs to be represented by an equivalent system of the same size as the larger system. Here equivalence between systems means that their lifetime distributions are identical for any component distribution. While such equivalent systems are usually represented as mixtures of coherent systems (so called mixed systems), in the present paper we demonstrate that they can be obtained in a simpler fashion by addition of irrelevant components to the smaller system, thereby representing them by monotone systems. In addition to making the formulas for signatures of equivalent systems more transparent, the new representation aids in the usual interpretation of mixed systems. We also consider the opposite problem of whether, for a given mixed system, there can be found equivalent systems of smaller sizes. While there is always an equivalent mixed system of larger size, there need not be equivalent systems of smaller sizes. We finally study the problem of equivalence of systems of different sizes when we restrict to coherent systems. A sufficient condition for equivalence of coherent systems of sizes respectively n and $n+1$, for general n, is given; it follows, as a special case, that any k-out-of-n -system with $1 < k < n$ has an equivalent coherent system of size $n+1$. The proof is based on first adding an irrelevant component to the smaller system, and then obtaining an equivalent coherent system by manipulating the minimal cut sets of the original system.

Keywords: coherent system; system signature; k-out-of-n system; mixed system; reliability polynomial; irrelevant component; cut set; critical path vector

2010 Mathematics Subject Classification: Primary 60K10

Secondary 62N05

1. Introduction. Systems and equivalent systems

Consider a coherent system with n components, as defined in the well-known monograph of Barlow and Proschan [1]. (We shall in the following say, for short, that a system with n components is an n-system.) Suppose now that the component lifetimes are i.i.d. with continuous distribution F and let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the ordered lifetimes of the n components. Samaniego [7] introduced the *signature vector*, $\mathbf{s} = (s_1, \ldots, s_n)$, of the system, defined by $s_k = P(T = X_{k:n})$; $k = 1, \ldots, n$. The signature of a system is a topological invariant that depends only on the system's design and does not depend on the distribution of component lifetimes. A key result is Theorem 3.1 in Samaniego [8], stating that the survival function of the lifetime T of the system can be represented as

$$\bar{F}_T(t) = P(T > t) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (F(t))^j (\bar{F}(t))^{n-j}, \tag{1}$$

where $\bar{F}(t) = 1 - F(t)$.

Standard examples of coherent systems are series systems, which work if and only if all components are working, and parallel systems, which work if and only if at least one component is working. We shall also be concerned with so-called k-out-of-n systems, which fail upon the k-th component failure. Thus a series system is a 1-out-of-n system while a parallel system is an n-out-of-n system. It is furthermore easy to see that the signature vector of a k-out-of-n system is $(0, \ldots, 1_k, \ldots, 0)$, where the subindex k refers to the kth element of the vector.

It is easy to see that (1) continues to hold if the system is *monotone*. Recall that a monotone system is one where the structure function is monotone as a function of the component states, but that one or more of the components may be *irrelevant*, meaning that they do not contribute to the functioning of the system (see, e.g., [1]). Note that any coherent system is a monotone system by this definition, and that a monotone system may be reduced to a coherent system by removing the irrelevant components.

Further, as argued in [8, p.28–31] it may be convenient, for practical as well as mathematical reasons, to extend the class of n-systems to include so-called *mixed* n-systems, which are stochastic mixtures of coherent n-systems. We easily conclude that the result (1) continues to hold for mixed systems (see, e.g., remark in Samaniego et al. [9]).

Of course, any coherent system is trivially a mixed system. Note that while, for a given n, there are finitely many coherent or monotone n-systems and therefore also finitiely many possible signature vectors corresponding to coherent or monotone n-systems, any probability vector $(s_1, \ldots, s_n)$ can serve as the signature of a mixed system. One possible representation of such a mixed system is the one which draws a k-out-of-n system with probability s_k.

Also note that for a fixed n and a fixed signature vector $\mathbf{s}$, there may exist several coherent n-systems. For example, if $n = 4$, the systems with minimal cut sets, respectively, $\{1,2\},\{3,4\}$ and $\{1,2\},\{1,3\},\{2,3,4\}$ have the same signature vector $\mathbf{s} = (0, 1/3, 2/3, 0)$ ([8, Table 3.2]). Allowing also mixed systems, these two systems are in turn equivalent to, e.g., the mixed 4-system which gives weight 1/3 to a 2-out-of-4 system and weight 2/3 to a 3-out-of-4 system.

A typical use of signature vectors is in the comparison of lifetimes of different systems. Let $\mathbf{s}_1 = (s_{11}, \ldots, s_{1n})$ and $\mathbf{s}_2 = (s_{21}, \ldots, s_{2n})$ be signature vectors of two mixed n-systems and let T_1 and T_2 be these systems' lifetimes. Suppose $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$, which means that

$$\sum_{i=j}^{n} s_{1i} \leq \sum_{i=j}^{n} s_{2i} \text{ for } j = 1, 2, \ldots, n.$$

Then ([8, p. 44]) $T_1 \leq_{st} T_2$, where $\leq_{st}$ here stands for the stochastic ordering of random variables, defined as $P(T_1 \geq t) \leq P(T_2 \geq t)$ for all $t \geq 0$.

It follows from (1) that we may write

$$\bar{F}_T(t) = h(\bar{F}(t)), \tag{2}$$

with

$$h(p) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (1-p)^j p^{n-j}, \tag{3}$$

where $h(p)$ is the *reliability polynomial* corresponding to the system (see [1]).

In [8, Chapter 6] is given an explicit recipe for computation of the coefficients of the reliability polynomial from the signature vector. Assume that

$$h(p) = \sum_{r=1}^{n} d_r p^r,$$

where $\mathbf{d} = (d_1, \ldots, d_n)$ is the so-called vector of *dominations*, and let the vector $\mathbf{a} = (a_1, \ldots, a_n)$ be defined from the signature vector $\mathbf{s} = (s_1, \ldots, s_n)$, as

$$a_j = \sum_{i=n-j+1}^{n} s_i; \ j = 1, \ldots, n. \tag{4}$$

Then

$$\mathbf{d} = \mathbf{M}_n \mathbf{a} \tag{5}$$

where $\mathbf{M}_n$ is the lower triangular $n \times n$ matrix with entries

$$\mathbf{M}_n(i,j) = (-1)^{i-j} \binom{n}{j} \binom{n-j}{i-j} \text{ for } 1 \le j \le i \le n, \tag{6}$$

(see [8, (6.15)]).

Conversely, from

$$\mathbf{a} = \mathbf{M}_n^{-1} \mathbf{d} \tag{7}$$

it follows that *for a given system size* n, the vector $\mathbf{a}$ and hence the signature vector $\mathbf{s}$ can be uniquely found from the reliability polynomial. An explicit expression for the (i,j)th term of $\mathbf{M}_n^{-1}$ is given in [8, (6.16)],

$$\mathbf{M}_n^{-1}(i,j) = \frac{i(i-1)\cdots(i-j+1)}{n(n-1)\cdots(n-j+1)} \text{ for } 1 \le j \le i \le n.$$

Note that in (5) and (7) we treat $\mathbf{a}$ and $\mathbf{d}$ as column vectors.

Navarro et al. [5, Section 2.1] give the following definition of *equivalent systems*: Two systems with i.i.d. component lifetimes with distribution F, are said to be *equivalent* if the lifetime distributions of the systems are identical, for any component distribution F. In view of (2) we state the following alterntative definition:

Definition 1. Two systems are said to be equivalent if their reliability polynomials (3) are equal as functions of p, $0 \le p \le 1$.

It is important to note that two systems may be equivalent even if they do not have the same number of components. An example is given by the two systems in Figure 1. The two systems have the same reliability polynomial,

$$h(p) = 3p^2 - 2p^3$$

and are hence equivalent according to Definition 1. Their signature vectors are, on the other hand, $(0, 1, 0)$ and $(0, 1/2, 1/2, 0)$, respectively for the left hand and right hand

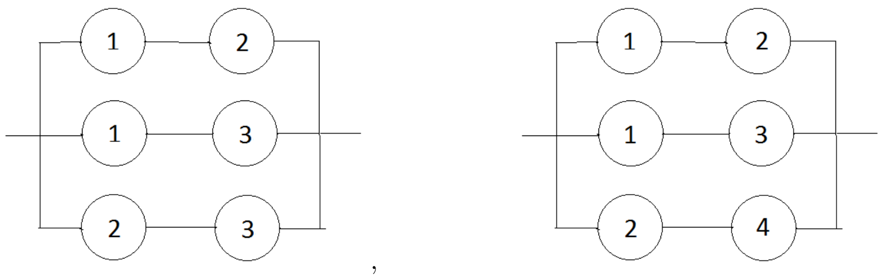

FIGURE 1: Two equivalent systems, with common reliability poynomial $h(p) = 3p^2 - 2p^3$, but with different signature vectors, $(0, 1, 0)$ (left system), $(0, 1/2, 1/2, 0)$ (right system).

system. Thus it would appear that signatures are not well suited for the comparison of lifetime distributions of systems of different sizes. In this connection, recall that the result stated earlier on stochastic ordering of lifetime distributions required signature vectors of the same size. Samaniego [8, page 32] therefore suggested "converting" the smaller of two systems into an equivalent system of the same size as the larger one. The signature vector of this derived system can then be compared to the signature vector of the larger system.

The following theorem solves the problem of comparing systems of size n and $n+1$ by giving the formula for the signature of an $(n+1)$-system equivalent to a given n-system. By repeated use one is then of course able to compare any two systems in this way.

Theorem 1. *(Samaniego [8, Theorem 3.2]) Let* $\mathbf{s} = (s_i; i = 1, 2, \ldots, n)$ *be the signature of a coherent or mixed system based on* n *components with i.i.d. lifetimes with common continuous distribution* F*. Then a (mixed) equivalent system with* $n+1$ *components has the signature vector* $\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_{n+1}^*)$*, where*

$$\begin{aligned} s_1^* &= \frac{n}{n+1} s_1 \\ s_k^* &= \frac{k-1}{n+1} s_{k-1} + \frac{n-k+1}{n+1} s_k;\ k = 2, 3, \ldots, n \\ s_{n+1}^* &= \frac{n}{n+1} s_n. \end{aligned}$$

The proof in [8] uses the representation of mixed systems as mixtures of k-out-of-n systems. A different and shorter proof of the resulting formula has been given by Navarro et al. [5]. It is the purpose of the Section 2 below to give an even shorter and more elementary proof, which furthermore sheds new light on the nature of the creation of equivalent systems, which is here done by adding new components. The idea is simply that the addition of irrelevant components to a coherent or mixed system does not change its lifetime distribution. Also in Section 2 we prove, from first principles, the extension of the above for equivalent systems of arbitrary sizes n and $n+r$.

Section 3 considers the complementary scenario where, for a given system, one is interested in the possible existence of an equivalent system of smaller size. This version of the problem is motivated by the simple fact that the cost of the smaller system is lower than that of the original one. It is demonstrated that not every mixed $(n+1)$-system has an equivalent mixed n-system; a characterization is given for $(n+1)$-systems which have an equivalent n-system.

In Section 4 we restrict attention to coherent systems and give sufficient conditions for a coherent n-system to be equivalent to a coherent $(n+1)$-system. The idea is to first add an irrelevant component to the n-system and then by a certain modification of the minimal cut sets obtain an equivalent coherent system. In particular it is shown that any k-out-of-n -system with $1 < k < n$ has an equivalent coherent system of size $n+1$.

Some concluding remarks are given in Section 5.

2. Obtaining equivalent systems of larger sizes by the addition of irrelevant components

The purpose of this section is to give a new proof of Theorem 1. Before proceeding with technical arguments, we provide the following heuristics.

First, it is clear that one may interpret the reliability polynomial $h(p)$ as the probability that the system is working if the components are independent and each is working with probability p. Adding to the system a new component, independent of the others, with probability p of working, but not influencing the state of the system (working or non-working), of course does not change the probability that the system will work. Hence this addition of such a component defines an equivalent system with $n+1$ components according to Definition 1. The added component is *irrelevant*, as defined in the Introduction. Formulated differently, starting with an n-system, constructing an $(n+1)$-system by adding an independent component with the same lifetime distribution as the ones in the n-system, and running the $n+1$ components until failure of the n-(sub)system, leads to an $(n+1)$-system which has components with i.i.d. lifetimes and has the same failure time distribution as the original n-system.

It should then also be clear that one may obtain the signature of an $(n+r)$-system equivalent to a given n-system, for any positive integer r, simply by adding r irrelevant components to the original n-system. We will also present a formal statement and proof of this fact below, obtaining a new and simpler proof of the result of Corollary 2.8 of Navarro et al. [5], which was based on mixtures of k-out-of-n systems. For a more transparent illustration of the ideas, we prove the result for $r = 1$ first.

Proof of Theorem 1: Let $X_1, \ldots, X_n$ be the lifetimes of the n components of the system, assumed i.i.d. with distribution F. Further, let Y have distribution F and be independent of $X_1, \ldots, X_n$. Here Y is the lifetime of the additional component, which is assumed to be irrelevant to the original system in the sense that the lifetime of the new $(n+1)$-system is still T. Now, let

$$X^*_{1:n+1} < X^*_{2:n+1} < \cdots < X^*_{n+1:n+1}$$

be the order statistics corresponding to the $(n+1)$-vector $(X_1, X_2, \ldots, X_n, Y)$ of the lifetimes of the components of the new $(n+1)$-system.

For compactness of notation, define $X_{0:n} = 0, X_{n+1:n} = \infty$. Then, from the law of total probability, the signature of the new $(n+1)$-system can be found from the expressions

$$s^*_k = P(T = X^*_{k:n+1}) = \sum_{i=1}^{n+1} P(T = X^*_{k:n+1} | X_{i-1:n} < Y < X_{i:n}) P(X_{i-1:n} < Y < X_{i:n}), \tag{8}$$

valid for $k = 1, \ldots, n+1$.

Now it can be seen that

$$P(T = X^*_{k:n+1} | X_{i-1:n} < Y < X_{i:n}) = \begin{cases} P(T = X_{k-1:n}) & \text{for } i = 1, 2, \ldots, k-1 \\ 0 & \text{for } i = k \\ P(T = X_{k:n}) & \text{for } i = k+1, k+2, \ldots, n+1, \end{cases}$$

where we have used the fact that the events $\{T = X_{k:n}\}$ and $\{X_{i-1:n} < Y < X_{i:n}\}$ are independent. This follows since the former event depends only on the permutation of the indices $1, \ldots, n$ corresponding to the ordering of the $X_1, \ldots, X_n$, while the latter event depends on the values of the X_i (and Y) actually observed. It is well known that these two functions of the data are independent (see e.g. Randles and Wolfe [6, Lemma 8.3.11]). The theorem now follows directly from (8) by using the simple fact that all the orderings $X_{0:n} < \cdots < X_{i-1:n} < Y < X_{i:n} < \cdots < X_{n+1:n}$ for $i = 1, \ldots, n+1$ are equally probable, a fact which implies

$$P(X_{i-1:n} < Y < X_{i:n}) = \frac{1}{n+1} \quad \text{for } i = 1, 2, \ldots, n+1.$$

We may then rewrite (8) as

$$\begin{aligned} P(T = X^*_{k:n+1}) &= \frac{1}{n+1}\left\{\sum_{i=1}^{k-1} P(T = X_{k-1:n}) + \sum_{i=k+1}^{n+1} P(T = X_{k:n})\right\} \\ &= \frac{1}{n+1}\left\{(k-1)P(T = X_{k-1:n}) + (n-k+1)P(T = X_{k:n})\right\} \\ &= \frac{k-1}{n+1}s_{k-1} + \frac{n-k+1}{n+1}s_k \end{aligned}$$

for $k = 1, \ldots, n+1$.

We now present a corollary to Theorem 1 which is formulated in terms of cumulative signature vectors. More precisely, for an n-system and an equivalent $(n+1)$-system, with signatures $\mathbf{s}$ and $\mathbf{s}^*$, respectively, we introduce the cumulative signature vectors, respectively, $\mathbf{b}$ and $\mathbf{b}^*$ given by $b_j = \sum_{i=1}^{j} s_i$ for $j = 1, \ldots, n$ and $b_j^* = \sum_{i=1}^{j} s_i^*$ for $j = 1, \ldots, n+1$.

This alternative summary of system signatures may, in particular, simplify calculations of the signature vector, as demonstrated by the example after the corollary.

Corollary 1. *Let* $\mathbf{s} = (s_i; i = 1, 2, \ldots, n)$ *be the signature of an* n*-system and let* $\mathbf{b}$ *be the corresponding cumulative signature vector. Then an equivalent coherent or mixed system with* $n+1$ *components has the cumulative signature vector* $\mathbf{b}^*$ *given by*

$$b_j^* = \begin{cases} b_j - \frac{j}{n+1}s_j & \text{for } j = 1, \ldots, n \\ 1 & \text{for } j = n+1 \end{cases}$$

Proof: Using the expressions for the s_i^* in Theorem 1 we get, for $j = 1, \ldots, n$,

$$\begin{aligned} b_j^* &= \sum_{i=1}^{j} s_i^* \\ &= \frac{n}{n+1} s_1 + \sum_{i=2}^{j} \left\{ \frac{i-1}{n+1} s_{i-1} + \frac{n-i+1}{n+1} s_i \right\} \\ &= \sum_{i=1}^{j-1} \frac{i}{n+1} s_i + \sum_{i=1}^{j} \frac{n-i+1}{n+1} s_i \\ &= \frac{n-j+1}{n+1} s_j + \sum_{i=1}^{j-1} \left\{ \frac{i}{n+1} + \frac{n-i+1}{n+1} \right\} s_i \\ &= \frac{n-j+1}{n+1} s_j + \sum_{i=1}^{j-1} s_i \\ &= \frac{-j}{n+1} s_j + \sum_{i=1}^{j} s_i \\ &= b_j - \frac{j}{n+1} s_j \end{aligned}$$

Example 1. Suppose $n = 3$ and $\mathbf{s} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Then $\mathbf{b} = (\frac{1}{2}, \frac{3}{4}, 1)$ and it is readily seen that

$$\mathbf{b}^* = \left(\frac{1}{2} - \frac{1}{4} \cdot \frac{1}{2},\ \ \frac{3}{4} - \frac{2}{4} \cdot \frac{1}{4},\ \ 1 - \frac{3}{4} \cdot \frac{1}{4},\ \ 1 \right) = \left(\frac{3}{8}, \frac{5}{8}, \frac{13}{16}, 1 \right)$$

so

$$\mathbf{s}^* = \left(\frac{3}{8}, \frac{2}{8}, \frac{3}{16}, \frac{3}{16} \right).$$

Now we turn to the generalization where one goes from n to $n + r$ components. The result is stated and proved in Navarro et al. [5], but our proof here is based on the addition of irrelevant components, thus generalizing the proof of the case $r = 1$ given above.

Theorem 2. (Navarro et al. [5].) *Let* $\mathbf{s}$ *be the signature vector of a mixed or coherent system of order* n. *Then for any positive integer* r *there is an equivalent system of order* $n + r$ *with signature* $\mathbf{s}^*$ *given by*

$$s_k^* = \frac{n}{n+r} \frac{1}{\binom{n+r-1}{k-1}} \sum_{i=\max(1,k-r)}^{\min(k,n)} \binom{n-1}{i-1} \binom{r}{k-i} s_i$$

for $k = 1, 2, \ldots, n + r$.

Proof: Let $X_1, \ldots, X_n$ be the lifetimes of the components of the n-system, assumed to be i.i.d. with distribution F. Let $Y_1, \ldots, Y_r$ be an independent set of i.i.d. variables from F (representing r irrelevant components).

First, place all the $n+r$ variables $X_1, \ldots, X_n, Y_1, \ldots, Y_r$ in increasing order as follows:

$$X^*_{1:n+r} < X^*_{2:n+r} < \cdots < X^*_{n+r:n+r}.$$

Then for $j = 1, 2, \ldots, n+r$, define

$$U_j = \begin{cases} 1 & \text{if } X^*_{j:n+r} \text{ originates from an } X, \\ 0 & \text{if } X^*_{j:n+r} \text{ originates from an } Y. \end{cases}$$

Let T be the lifetime of the n-system defined above, and let this also be the lifetime of the $n+r$-system obtained by adding the r irrelevant components with lifetimes Y_i.

Then, since $T = X^*_{k:n+r}$ is impossible when $U_k = 0$, we may write

$$\begin{aligned} s^*_k &\equiv P(T = X^*_{k:n+r}) \\ &= \sum_{i=1}^{k} P(T = X^*_{k:n+r} | U_k = 1, \sum_{m=1}^{k-1} U_m = i-1)\; P(U_k = 1, \sum_{m=1}^{k-1} U_m = i-1) \\ &= \sum_{i=1}^{k} P(T = X_{i:n})\; P(\sum_{m=1}^{k-1} U_m = i-1)\; P(U_k = 1 | \sum_{m=1}^{k-1} U_m = i-1). \end{aligned}$$

Here we have used the independence of the event $\{T = X_{i:n}\}$ and the $\{U_1, \ldots, U_{n+r}\}$, which is similar to the independence result used in Theorem 1. More precisely, independence holds since the former event depends on the permutation of the indices $1, \ldots, n$ in the ordering of the Xs, while the latter set depends on the values of the X_i and Y_i that are actually observed.

We then continue, noting that $\{U_1, \ldots, U_{n+r}\}$ are distributed as independent draws without replacement from an urn containing n X and r Y, giving result 1 to an X and 0 to a Y. Thus in particular, $\sum_{m=1}^{k-1} U_m$ is hypergeometrically distributed. From this we get

$$\begin{aligned} s^*_k &= \sum_{i=1}^{k} s_i \frac{\binom{n}{i-1}\binom{r}{k-i}}{\binom{n+r}{k-1}} \cdot \frac{n-i+1}{n+r-k+1} \\ &= \frac{n}{n+r} \frac{1}{\binom{n+r-1}{k-1}} \sum_{i=1}^{k} \binom{n-1}{i-1} \binom{r}{k-i} s_i \end{aligned}$$

Finally, noting that $\binom{r}{k-i}$ is 0 if $i < k - r$ and that s_i is defined for $i \leq n$, we can redefine the limits of the summing variable as in the statement of the theorem.

Example 2. Let $n = 3$ and $k = 2$. Then the theorem gives that a 3-system with signature vector $\mathbf{s} = (s_1, s_2, s_3)$ is equivalent to a 5-system with signature vector

$$\mathbf{s}^* = \left(\frac{3s_1}{5}, \frac{3s_1 + 3s_2}{10}, \frac{s_1 + 4s_2 + s_3}{10}, \frac{3s_2 + 3s_3}{10}, \frac{3s_3}{5} \right).$$

3. When is there an equivalent system of lower order?

Section 2 was concerned with the problem of modifying the signature vector of the smaller of two systems in order that their signature vectors be of equal size. In the present section we consider the opposite situation. The issue of whether there exists a smaller system that is equivalent to a given system is of evident practical importance, as the smaller system, consisting of components with identical lifetime distributions, can invariably be constructed at a lower cost.

Thus, consider a coherent or mixed $(n + 1)$-system with signature vector $\mathbf{s}^* = (s_1^*, \ldots, s_{n+1}^*)$. The question we pose is whether there is an equivalent n-system, and if so, what is its signature vector $\mathbf{s} = (s_1, \ldots, s_n)$.

By Definition 1, it may be natural to start by computing the reliability polynomial $h(p)$ corresponding to $\mathbf{s}^*$ and let its corresponding domination vector be $\mathbf{d}^*$ (see Section 1). If h has degree $n + 1$, i.e., the coefficient of the term p^{n+1} is nonzero, then it follows from (3) that there cannot be an equivalent system of lower size. Thus, because of the relation (5) we have

$$\mathbf{d}^* = \mathbf{M}_{n+1}\mathbf{a}^*$$

(where $\mathbf{a}^*$ is similar to (4)), and hence a necessary condition for the existence of an equivalent n-system is that the $(n+1)$th entry of $\mathbf{d}^*$ equals zero. From (6) we readily get

$$d_{n+1}^* \equiv \sum_{j=1}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} a_j^*, \tag{9}$$

and from this we get the following result where, for later use the result is written in terms of the cumulative signature vector $\mathbf{b}^*$.

Theorem 3. *For an $(n+1)$-system with signature vector $\mathbf{s}^* = (s_1^*, \ldots, s_{n+1}^*)$, a necessary condition for the existence of an equivalent n-system is*

$$\sum_{j=1}^{n+1} (-1)^{j-1} \binom{n+1}{j} b_j^* = 0. \tag{10}$$

Proof: The result follows by substituting $a_j^* = 1 - b_{n+1-j}^*$ for $j = 1, \ldots, n$ in (9), which gives the left hand side of (10) as an alternative formula for d_{n+1}^*.

The condition of the theorem is not sufficient, however, as is seen from the following example.

Example 3. Suppose $n = 5$ and let a mixed 6-system have signature vector

$$\mathbf{s}^* = (0, \frac{4}{10}, \frac{1}{10}, \frac{1}{10}, \frac{4}{10}, 0)$$

so that

$$\mathbf{b}^* = \mathbf{a}^* = (0, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, 1, 1).$$

Then the left hand side of (10) is

$$\binom{6}{1} \cdot 0 - \binom{6}{2} \cdot \frac{4}{10} + \binom{6}{3} \cdot \frac{5}{10} - \binom{6}{4} \cdot \frac{6}{10} + \binom{6}{5} \cdot 1 - \binom{6}{6} \cdot 1 = 0$$

so the necessary condition of the theorem is satisfied.

Of course we may compute the full reliability polynomial using (5),

$$\mathbf{d}^* = \mathbf{M}_6 \mathbf{a}^* = (0, 6, -14, 15, -6, 0),$$

giving the reliability polynomial

$$h(p) = 6p^2 - 14p^3 + 15p^4 - 6p^5, \tag{11}$$

which is of degree $n = 5$. The question is still, however, whether it is the reliability polynomial of some 5-system. We therefore let $\mathbf{d} = (0, 6, -14, 15, -6)$, and solve for $\mathbf{a}$ using (7),

$$\mathbf{a} = \mathbf{M}_5^{-1} \mathbf{d} = (0, \frac{3}{5}, \frac{2}{5}, 1, 1).$$

This is *not* a legitimate $\mathbf{a}$-vector since the elements are not nondecreasing. This shows that there is no 5-system equivalent to the given 6-system, and hence that the condition of Theorem 3 is *not* sufficient.

The above example suggests the following procedure for finding, if it exists, the signature vector of an n-system which is equivalent to a given $(n+1)$-system:

Let $\mathbf{a}^*$ *correspond to the given* $(n+1)$*-system and compute* $\mathbf{d}^* = \mathbf{M}_{n+1}\mathbf{a}^*$*. If* $d^*_{n+1} = 0$*, let* $\mathbf{d} = (d^*_1, \ldots, d^*_n)$ *and compute* $\mathbf{a} = \mathbf{M}_n^{-1}\mathbf{d}$*. If the elements of* $\mathbf{a}$ *are non-decreasing, then* $\mathbf{a}$ *corresponds to the signature vector of a (mixed)* n*-system equivalent to the given* $(n+1)$*-system.*

Instead of using the reliability polynomial directly, one may seek to "invert" Theorem 1 by looking for solutions for $\mathbf{s} = (s_1, \ldots, s_n)$ of the equations given there, given the signatures s^*_i of the $(n+1)$-system.

Equivalently, using Corollary 1, one may seek to solve the equations

$$\begin{aligned}
b^*_1 &= \frac{n}{n+1} s_1 \\
b^*_2 &= s_1 + \frac{n-1}{n+1} s_2 \\
b^*_3 &= s_1 + s_2 + \frac{n-2}{n+1} s_3 \\
&\vdots \\
b^*_n &= s_1 + s_2 + \cdots + s_{n-1} + \frac{1}{n+1} s_n \\
1 &= s_1 + s_2 + \cdots + s_n
\end{aligned} \tag{12}$$

The first n equations of (12) clearly has a unique solution for $s_1, \ldots, s_n$. However, it may happen that these s_i do not sum to 1, in which case the $(n+1)$th equations will not be satisfied. Further, even if $\sum_{i=1}^n s_i = 1$, it may happen that there are some negative s_i in the solution, thus leading of course to the conclusion that there is no equivalent system. Some examples of these possibilities are given later in the section.

Somewhat unexpectedly, we are able to show in Proposition 1 that the condition of Theorem 3, which is equivalent to the reliability polynomial of the $(n+1)$-system being of order n, is a necessary and sufficient condition for $\sum_{i=1}^n s_i = 1$ when $(s_1, \ldots, s_n)$ is the solution of the first n equations of (12).

Proposition 1. *For an* $(n+1)$*-system, the reliability polynomial* $h^*(p)$ *is of degree* n *if and only if the solutions* $(s_1, \ldots, s_n)$ *to the first* n *equations of (12) sum to 1.*

Proof: We denote the coefficient matrix of the equations (12) by $\mathbf{C}$, i.e.

$$\mathbf{C} = \begin{bmatrix} \frac{n}{n+1} & 0 & 0 & \cdots & 0 \\ 1 & \frac{n-1}{n+1} & 0 & \cdots & 0 \\ 1 & 1 & \frac{n-2}{n+1} & \cdots & 0 \\ \vdots & & & & \\ 1 & 1 & 1 & \cdots & \frac{1}{n+1} \end{bmatrix}$$

It can be shown that the inverse matrix $\mathbf{C}^{-1}$ is the lower triangular matrix with entries given by, for $i = 1, \ldots, n$,

$$\begin{aligned} \mathbf{C}^{-1}(i,i) &= \frac{n+1}{n-i+1} \\ \mathbf{C}^{-1}(i,i-1) &= \frac{-(n+1)^2}{(n-i+2)(n-i+1)} \\ \mathbf{C}^{-1}(i,j) &= \frac{(-1)^{i-j}(i-1)(i-2)\cdots(j+1)(n+1)^2}{(n-j+1)(n-j)\cdots(n-i+1)} \text{ for } j = 1,2,\ldots,i-2 \end{aligned}$$

The solution to the first n equations of (12) is now given by

$$\mathbf{s} = \mathbf{C}^{-1}\mathbf{b}^{(*)},$$

where $\mathbf{b}^{(*)}$ is the vector of the n first entries of $\mathbf{b}^*$. Thus the sum of the s_i is

$$\mathbf{1}'\mathbf{s} = \mathbf{1}'\mathbf{C}^{-1}\mathbf{b}^{(*)}.$$

Let

$$v_j = (-1)^{j+n}\binom{n+1}{j}, \;\; j = 1, \ldots, n. \tag{13}$$

We will first show that

$$\mathbf{1}'\mathbf{C}^{-1} = (v_1, \ldots, v_n),$$

or equivalently that

$$\mathbf{1}' = (v_1, \ldots, v_n)\mathbf{C}. \tag{14}$$

To prove (14) we need to prove that for $k = 1, 2, \ldots, n$ we have

$$\frac{n-k+1}{n+1}v_k + v_{k+1} + \cdots + v_n = 1. \tag{15}$$

For $k = n$ the above equation is $v_n/(n+1) = 1$, which holds since $v_n = n+1$ by (13). We now prove the result (14) by backward induction for $k = n, n-1, n-2, \ldots$.

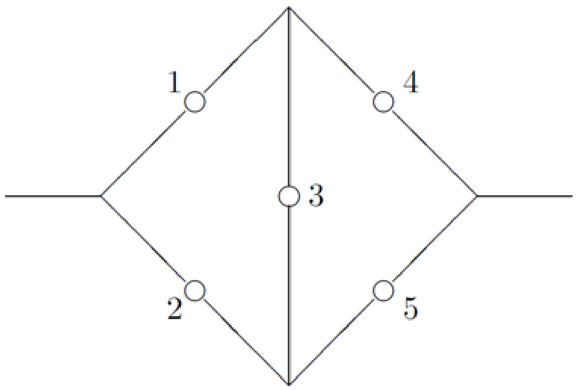

FIGURE 2: The bridge system

Suppose therefore that (15) holds for a particular k. The left hand side of (15) for $k-1$ is

$$\frac{n-k+2}{n+1}v_{k-1}+v_k+v_{k+1}+\cdots+v_n. \tag{16}$$

Subtracting the left hand side of (15) from (16), we get

$$\frac{n-k+2}{n+1}v_{k-1}+\frac{k}{n+1}v_k,$$

and by the induction hypothesis we are done if we can show that this equals 0. But this is a routine task using (13).

Now

$$\mathbf{1}'\mathbf{s}=(v_1,\ldots,u_n)\mathbf{b}^{(*)}=\sum_{j=1}^{n}(-1)^{j+n}\binom{n+1}{j}b_j^*$$

and it is straightforward to show that this equals 1 if and only if (10) holds, which by the proof of Theorem 3 is equivalent to $h^*(p)$ being of degree n. This completes the proof.

Example 3 (continued) By solving the equations (12) we get the solution

$$\mathbf{s}=(0,\frac{3}{5},-\frac{1}{5},\frac{3}{5},0)'.$$

This is not a legitimate signature vector, but still sums to 1, as it should by Proposition 1 and computations done in the first part of the example.

Example 4. (*The bridge system.*) The bridge system (see Figure 2) is a standard example in textbooks of system reliability. This is a 5-system with signature vector $\mathbf{s}^* = (0, 1/5, 3/5, 1/5, 0)$ ([8, Example 4.1]). The solution (s_1, s_2, s_3, s_4) of (12) is $(0, 1/3, 7/6, -5/2)$, which neither sums to 1 nor is non-negative. Thus there is no 4-system equivalent to the bridge system, which would also be clear from computation of the domination vector, in which $d_5 = 2 \neq 0$. The result is, furthermore, consistent with Propostion 1. (We shall, however, see in Section 4 that there is in fact an equivalent *coherent* 6-system).

Example 5. This example shows that we may have a strictly positve solution of the first n equations in (12), which does not sum to 1. Here let $n = 3$ and let the original system have signature $\mathbf{s}^* = (3/20, 1/4, 1/4, 7/20)$, so $\mathbf{b}^* = (3/20, 2/5, 13/20, 1)$. The solution is then found to be $s_1 = 1/5$, $s_2 = 2/5$, $s_3 = 1/5$, thus being positve, but not summing to 1. Further, the condition of Theorem 3 is

$$-\binom{4}{1}\cdot\frac{3}{20} + \binom{4}{2}\cdot\frac{2}{5} - \binom{4}{3}\cdot\frac{13}{20} + \binom{4}{4}\cdot 1 = \frac{4}{20} \neq 0$$

which is of course consistent with Proposition 1.

4. Equivalence of coherent systems of different sizes

In this section we restrict attention to coherent systems. The question we pose is, for a given coherent n-system, does there exist an equivalent *coherent* $(n+1)$-system?

Recall from Section 2 that for any n-system, an equivalent $(n+1)$-system can be obtained by the addition of an irrelevant component, with a lifetime which is independent of the lifetimes of the other components, and has the same distribution. While such a system is not coherent, we shall see that in some cases it may be modified in such a way that it becomes a coherent $(n+1)$-system equivalent to the original n-system.

As a simple example, consider the 2-out-of-3-system given in the left panel of Figure 1. This system has minimal path sets $\{1,2\}, \{1,3\}, \{2,3\}$. We have already noted that this 2-out-of-3 system is equivalent to the 4-system appearing to the right in the figure. The 4-system is here simply obtained by replacing component 3 in the minimal path set $\{2,3\}$ by a new component 4. Alternatively, the right hand system could be obtained by replacing the minimal cut set $\{1,3\}$ of the 2-out-of-3 system by the set $\{1,4\}$.

In a similar way we may obtain coherent 5-systems which are equivalent to, respectively, 2-out-of-4 and 3-out-of-4-systems, simply by modifying one of the minimal path or minimal cut sets of the respective 4-system. These findings suggest the general result for k-out-of-n systems given below. The proof is based on the following Lemma which is essentially given in Boland [2], by noting that Boland's proof still holds if there are irrelevant components.

The lemma gives a formula for the signature of a monotone n-system in terms of the *cut sets* of the system. Recall that a cut set is a subset of the component set $\{1,2,\ldots,n\}$ such that the system fails if all components in the set have failed. Note that a cut set necessarily contains at least one minimal cut set, but needs not itself be minimal.

Lemma 1. (Boland [2].) *Consider a monotone n-system with cumulative signature vector $\mathbf{b}=(b_1,\ldots,b_n)$. Then for $i=1,\ldots,n$,*

$$b_i = \frac{\#\ \textit{cut sets of size } i \textit{ for the system}}{\binom{n}{i}}.$$

Theorem 4. *For any k-out-of-n system, where $n \geq 3$ and $1 < k < n$, there exists an equivalent coherent $(n+1)$-system.*

Proof: The cut sets of a k-out-of-n system are all subsets of $\{1,2,\ldots,n\}$ of size k or larger. Consider now the equivalent $(n+1)$-system obtained by adding an irrelevant component, $n+1$, to the n components. The new system constructed this way has the following cut sets:

Size k: All the $\binom{n}{k}$ subsets of $\{1,2,\ldots,n\}$ of size k.

Size $k+1$: All the $\binom{n}{k}$ subsets of $\{1,2,\ldots,n\}$ of size k, with the component $n+1$ added to them, plus all the $\binom{n}{k+1}$ subsets of $\{1,2,\ldots,n\}$ of size $k+1$.

Size $r \in \{k+2,\ldots,n+1\}$: All the $\binom{n+1}{r}$ subsets of $\{1,2,\ldots,n,n+1\}$ of size r.

The minimal cut sets of this new system are exactly the cut sets of size k as given above. Hence the system is not coherent, since the union of the minimal cut sets is a strict subset of $\{1,2,\ldots,n,n+1\}$ (see [1, Exercise 5(a), p. 15]). The modification where the minimal cut set $\{n-k+1, n-k+2, \ldots, n-1, n\}$ is replaced by $\{n-k+1, n-k+2, \ldots, n-1, n+1\}$, and the other minimal cut sets are unchanged, will however be coherent. We are done if we can prove that this system is equivalent to the incoherent $(n+1)$-system constructed above. To see this, consider the cut sets of the constructed coherent $(n+1)$-system. One can readily verify that these are:

Size k: The $\binom{n}{k}-1$ unmodified subsets of $\{1, 2, \ldots, n\}$ plus the modified cut set $\{n-k+1, n-k+2, \ldots, n-1, n+1\}$.

Size $k+1$: All the $\binom{n}{k}-1$ unmodified subsets of $\{1, 2, \ldots, n\}$ of size k, with the component $n+1$ added to them, plus the modified set $\{n-k+1, n-k+2, \ldots, n-1, n+1\}$ with the component n added to it. In addition, all the $\binom{n}{k+1}$ subsets of $\{1, 2, \ldots, n\}$ of size $k+1$.

Size $r \in \{k+2, \ldots, n+1\}$: All the $\binom{n+1}{r}$ subsets of $\{1, 2, \ldots, n, n+1\}$ of size r.

It is seen that the number of cut sets of each size are unchanged when we modify the system. Hence by Lemma 1, the **b**-vector is not changed and hence the systems are equivalent.

We will now prove that the result of the theorem does not hold for $k=1$ or $k=n$. First, by examining the proof of Theorem 4 it is seen that the construction breaks down if $k=1$ or $k=n$. Next, observe that it is enough to prove this for $k=1$. The case $k=n$ will follow since it corresponds to the *dual* system (see, e.g., [1]). The $(n+1)$-system equivalent to a 1-out-of-n system (i.e. a series n-system) has signature vector $\mathbf{s}^* = (n/(n+1), 1/(n+1), 0, \ldots, 0)$ by Theorem 1. Assume, as in a proof by contradiction, that this corresponds to a coherent system. By Lemma 1 the number of cut sets of order 1 must then be n, while all sets of size 2 or larger must be cut sets. There is hence one component, say $n+1$, which does not form a cut set of size 1. Since the system is coherent, there must however be a minimal cut set which contains the component $n+1$. This set is necessarily of size at least 2, and hence must contain at least one of the components $1, 2, \ldots, n$. But this is impossible since these components already form minimal cut sets of size 1. This contradiction proves that the statement of Theorem 4 does not hold for $k=1$ or $k=n$.

We shall next see that Theorem 4 can be extended to more general classes of coherent n-systems for which equivalent coherent $(n+1)$-systems can be constructed. For example, consider again the bridge system shown in Figure 2. The minimal *path* sets of this system are $\{1,4\}$, $\{2,5\}$, $\{1,3,5\}$ and $\{2,3,4\}$. Consider the system as a network where the source node is to the left and the target node is to the right in the figure. Component 3 is then the "bridge" in the network and turns out to have a role as a twoway-relevant component. More precisely, in the minimal path set $\{1,3,5\}$, component 3 is relevant "downwards", while in the minimal path set $\{2,3,4\}$, it is relevant "upwards". Thus, if we replace component 3 with a *directed* component, then this component will be relevant both if directed upwards and if directed downwards. All the other components have just one relevant direction. If we, for example, replace one of the components 1,2,4,5 with a directed component, it will be relevant only if the direction is from the source node to the target node.

The above suggests that, based on the undirected bridge system, we can construct a 6-system where we replace component 3 by two components, 3' and 3", say, where 3' is directed downwards and 3" is directed upwards. In this case, 3' and 3" are said to be connected in anti-parallel. It is easy to check that the new 6-system is equivalent to the original bridge system with five components. Furthermore, the new system is also coherent.

This type of construction turns out to be generally valid for undirected two-terminal network systems containing at least one twoway-relevant component. An equivalent coherent $(n+1)$-system can then be constructed by replacing a twoway-relevant component by two directed components connected in anti-parallel. This is the motivating idea behind Theorem 5 to be given below.

We have found it convenient below to study coherent systems in terms of their minimal cut sets. Since the minimal cut sets of a coherent system are exactly the minimal path sets of the dual system (see [1, p.12]), all stated assumptions and results that involve minimal cut sets have equivalent versions for path sets.

Consider a coherent n-system, below denoted system Φ, for which we seek an equivalent coherent $(n+1)$-system. One component of Φ plays a special role in the construction; without loss of generality we may assume that it is component n. For the construction to work, component n must be a member of at least two minimal cut sets of Φ. Now denote by Φ^* the $(n+1)$-system obtained by changing component n to $n+1$ in some, but not all of these minimal cut sets of Φ. The system Φ^* is now coherent, since the union of its minimal cut sets equals the full set of components.

As demonstrated in Section 2, we may construct a non-coherent $(n+1)$-system equivalent to Φ by adding an irrelevant component, $n+1$, to Φ. We shall denote the resulting (non-coherent) system by Ψ. The cut sets of Ψ are all cut sets of Φ plus the sets obtained by adding component $n+1$ to each of these cut sets.

We shall prove that under conditions to be given, the system Φ^* is equivalent to Ψ and hence to Φ. We shall do this by proving that there is a 1-1 mapping from the cut sets of Φ^* onto the cut sets of Ψ, where this mapping preserves the size of the sets. By Lemma 1, this will imply the equivalence of Φ^* and Ψ.

Theorem 5. *Let the systems Φ and Φ^* be given as described above. For any cut set K of Φ^*, define $K' = K \setminus \{n, n+1\}$; the set of elements of K different from n and $n+1$. If each K is of one of the following three types, then the systems Φ and Φ^* are equivalent:*

Type A: *K' is not a cut set; $K' \cup \{n\}$ is a cut set; $K' \cup \{n+1\}$ is not a cut set,*

Type B: *K' is not a cut set; $K' \cup \{n\}$ is not a cut set; $K' \cup \{n+1\}$ is a cut set,*

Type C: *K' is a cut set.*

Proof of the theorem: We shall prove that system Φ^* and system Ψ are equivalent by showing that they have the same number of cut sets of each size. The equivalence will then follow from Lemma 1. It is clearly enough to show that each cut set of system Φ^* can be mapped to a cut set of system Ψ of the same size, such that no two cut sets of system Φ^* are mapped to the same cut set of system Ψ, and that the map is *onto.*

Thus consider a cut set K of system Φ^*. We distinguish between the following disjoint cases:

1. $n \notin K$, $n+1 \notin K$. The set K is then of type C and is clearly also a cut set for system Ψ. Thus K is mapped to itself.

2. $n \in K$, $n+1 \notin K$. Now K is either of type A or type C, but is in any case a cut set of Ψ. Thus again K is mapped to itself.

3. $n \notin K,\ n+1 \in K$. Here K is either of type B or type C. If it is of type C, then it is clearly also a cut set of Ψ and K is mapped to itself.

   If K is of type B, then it is not a cut set of Ψ. This is because by the definition of type B, K must contain at least one minimal cut set of Φ^* which includes $n+1$, and K contains no other minimal cut set. K will be a cut set of Ψ, however, if component $n+1$ is replaced by component n in K, thereby changing K to $K'' = K' \cup \{n\}$. To see that K'' is indeed a cut set of Ψ, recall that K contains at least one minimal cut set of Φ^* which includes $n+1$. By changing $n+1$ to n, this minimal cut set becomes instead a minimal cut set of the system Ψ, and hence K'' must be a cut set of Ψ. We hence map K to K''. To ensure that our mapping of cut sets is 1-1, we then need to check that K'' is not one of the sets that is obtained in case 2 above. This is however not so since, by defintion of type B, K'' is not a cut set of Φ^*.

4. $n \in K,\ n+1 \in K$. Now K can be any of the types A, B or C. If it is of type C, then K' is a cut set of Φ^* and it is clear that K then is also a cut set of system Ψ. Thus K is mapped to itself.

   If K is of type A, then it is also a cut set of Ψ, so again K is mapped to itself. To see this, note that by definition of type A, K must contain at least one minimal cut set of Φ^* which includes n. But this set is also a minimal cut set of Ψ, and hence K is a cut set of Ψ as well.

   Finally, if K is of type B, then K contains at least one minimal cut set of Φ^* which includes $n+1$. But then K is also a cut set for system Ψ, since the present K contains n, and by our construction of Φ^*, n replaces $n+1$ when going from minimal cut sets of Ψ^* to minimal cut sets of Ψ. Thus K is again mapped to itself.

We finally need to prove that the mapping of cut sets defined above is *onto* the collection of cut sets of Ψ. Thus, suppose for contradiction that there is a cut set L of Ψ which is not mapped from a cut set of Φ^* in the way considered in the proof.

Assume first that L is a cut set of Φ^*. Then, by the proof of the 1-1 property, it would be mapped to itself, which is impossible, unless $n+1 \in L, n \notin L$ and L is of type B. But as stated in case 3 of the proof, then L is not a cut set of Ψ, which gives a contradiction.

The only possibility is hence that L is not a cut set of Φ^*. Since it is a cut set of Ψ, it must hence contain at least one minimal cut set of the original Φ where n was changed to $n+1$ in the creation of Φ^*, and contain no other minimal cut set. Further, L can not contain $n+1$. But then if n was changed to $n+1$ in L, the set would be a cut set of Ψ^*, or more precisely, $L \setminus \{n\} \cup \{n+1\}$ is a cut set of Φ^* of type B. But then it is seen from case 3 in the proof that L was already obtained by the mapping.

Thus the proof is complete.

Remark 1. To see that the conditions in Theorem 5 are not always met, consider the 3-system Φ with minimal cut sets $\{1,3\},\{2,3\}$. Then change the second minimal cut set to $\{2,4\}$, thus defining the system Φ^* with minimal cut sets $\{1,3\},\{2,4\}$, and 3 and 4 playing the roles of n and $n+1$, respectively. Now $K = \{1,2,3,4\}$ is a cut set of Φ^*, while $K' = K \setminus \{3,4\} = \{1,2\}$ is not. But since both $K' \cup \{3\}$ and $K' \cup \{4\}$ are cut sets, K is not of any of the types A,B,C in Theorem 5.

Conditions for the equivalence of the systems Φ and Φ^* which are equivalent to the conditions of Theorem 5 can be given in terms of the structure function of the system Φ^*. Thus, let the structure function be given by $\Phi^*(y_1,\dots,y_{n+1})$, which equals 1 if the system functions when the component states are $(y_1,\dots,y_{n+1})$, and equals 0 if it is failed. Here y_i equals 1 if component i is working and 0 otherwise.

Define the following sets:

$$\begin{aligned}
A &= \{(y_1,\dots,y_{n-1}) \mid \Phi^*(y_1,\dots,y_{n-1},1,0) > \Phi^*(y_1,\dots,y_{n-1},0,1)\},\\
B &= \{(y_1,\dots,y_{n-1}) \mid \Phi^*(y_1,\dots,y_{n-1},0,1) > \Phi^*(y_1,\dots,y_{n-1},1,0)\},\\
C &= \{(y_1,\dots,y_{n-1}) \mid \Phi^*(y_1,\dots,y_{n-1},1,1) = \Phi^*(y_1,\dots,y_{n-1},0,0)\}.
\end{aligned}$$

Now if $A \cup B \cup C = \{0,1\}^{n-1}$, the conditions of Theorem 5 will be satisfied, and vice versa. A reformulation and proof of Theorem 5 using the above assumptions, with a proof based on computation of reliability polynomials, is given in the Appendix.

Remark 2. By the definition of *a critical path vector* in Barlow and Proschan [1, p.13-14], it follows that A is the set of $(y_1,\dots,y_{n-1})$ such that $(y_1,\dots,y_{n-1},1,y_{n+1}))$

is a critical path vector for component n (i.e., if the 1 in place n is changed to 0, then Φ^* changes from 1 to 0), whatever be y_{n+1}, and such that $(y_1, \ldots, y_{n-1}, y_n, 1)$ is *not* a critical path vector for component $n+1$, whatever be y_n.

A similar interpretation can be given for the set B, if components n and $n+1$ are interchanged in the above explanation for A.

Finally, C is the set of $(y_1, \ldots, y_{n-1})$ such that the state of the system is not influenced by the states of components n and $n+1$. Hence no critical path vectors for n or $n+1$ can be formed from a vector $(y_1, \ldots, y_{n-1})$ in C.

Remark 3. It is easy to verify that Theorem 4 is a consequence of Theorem 5. In fact, if Φ is the k-out-of-n system, we can define Φ^* by changing component n to $n+1$ in exactly one minimal cut set of the k-out-of-n system, for example the set $\{n-k+1, \ldots, n-1, n\}$ (as we also did in the proof of Theorem 4). It is then straightforward to verify that the assumptions of Theorem 5 are satisfied.

Example 6. Let us reconsider the bridge system (Figure 2), which we used as a motivation for Theorem 6. The minimal cut sets are $\{1,2\}$, $\{4,5\}$, $\{1,3,5\}$ and $\{2,3,4\}$. Introduce a new component, 6, which replaces component 3 in the last minimal cut set. We claim that the equivalent conditions for Theorem 5, using the structure function, are satsified when respectively 3 and 6 play the roles of n and $n+1$ in the theorem. To see this, note that in the modified system, called Φ^* in the theorem, the vector $(0, 1, 1_3, 1, 0, y_6)$ is the only critical path vector for state 3, for any value of y_6, so that $A = \{(0, 1, 1, 0)\}$ gives the set of states for components 1,2,4 and 5 under this condition. By symmetry in the problem, $B = \{(1, 0, 0, 1)\}$ gives the states of components 1,2,4 and 5 for which component 6 is critical, which happens independently of the state of component 3. Finally, the set C contains the remaining 14 state vectors of components 1,2,4 and 5, and it is seen that the system state is not influenced by the states of components 3 and 6 for these state vectors. For example, for the state vector $(1, 1, 1, 0)$, components 1,2 and 4 are working while 5 is failed. In this case the system is working whatever be the state of components 3 and 6.

We close the section by giving two examples where the question posed in the beginning of this section is reversed, so that we ask instead the following question: For a given coherent $n+1$-system, does there exist an equivalent coherent n-system?

In Section 3 we considered this question in general for mixed systems. We are not able to give a complete answer to the question when restricting to coherent systems, but we note that Theorem 5 clearly solves some special cases of the problem. We illustrate this by giving an example where, for a given coherent 6-system, we seek an equivalent coherent 5-system.

Example 7. Consider the coherent 6-system, which we shall call Φ^*, with minimal cut sets $\{1,2\}$, $\{1,3,5\}$, $\{2,3,4,6\}$. The coefficient vector of the reliability polynomial of Φ^* is $\mathbf{d}^* = (0,6,-9,5,-1,0)$. It is hence seen that the reliability polynomial is of order 5, which indicates the possibility of existence of an equivalent 5-system. Using the procedure described in Section 3 we may show that there is in fact an equivalent *mixed* 5-system. However, in order to investigate the possibility of an equivalent coherent 5-system, we shall instead apply Theorem 5, using components 5 and 6 as n and $n+1$, respectively. Considering the minimal cut sets of Φ^*, it is seen that the critical path vectors for component 5 are the ones where 1,3 have failed and 2 is working, while 4 is either working or failed; all this independent of the state of component 6. This defines the set A among the alternative conditions for Theorem 5. On the other hand, the critical path vectors for component 6 are the ones where 2,3,4 have failed and 1 is working; independently of the state of component 5. This defines the set B. For all other combinations of states of components 1,2,3,4, it can be verified that the state of Φ^* not influenced by the states of components 5 and 6. It follows that the equivalent conditions of Theorem 5 are satisfied, and hence that the given 6-system is equivalent to the 5-system with minimal cut sets $\{1,2\}$, $\{1,3,5\}$, $\{2,3,4,5\}$.

In the next example we consider a coherent 6-system Φ^*, corresponding to a directed network system, for which the conditions of Theorem 5 are not satisfied, but where there is still an equivalent coherent 5-system. This shows that the conditions of Theorem 5 for an $(n+1)$-system are not necessary for the existence of an equivalent n-system. Somewhat surprising, however, we are in this example able to find a different coherent 6-system, equivalent to Φ^*, for which the conditions of Theorem 5 hold.

Example 8. Consider the 6-system Φ^* with the following seven minimal path sets: $\{1,2,3\},\{1,2,4\},\{1,3,5\},\{2,3,6\},\{1,4,5\},\{2,4,6\},\{3,5,6\}$. This system can be represented as the directed network system depicted in Figure 3, with one source node,

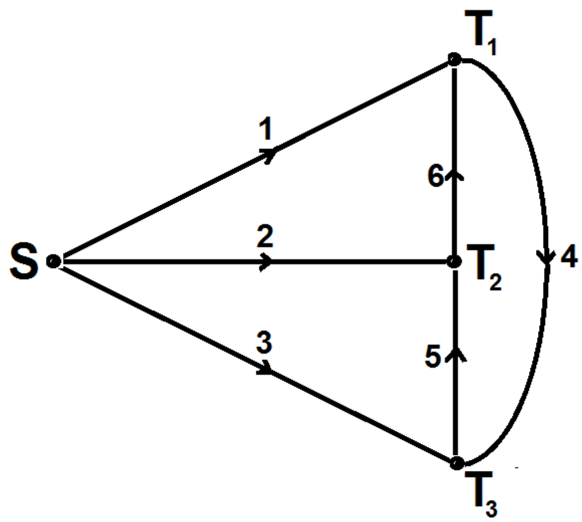

FIGURE 3: Directed network system with source node S and terminal nodes T_1, T_2, T_3.

S, and three terminal nodes, T_1, T_2, T_3, where the system is said to be working if the source S can send signals through the network to all three terminal nodes.

The special feature of this system is that there is no pair of components that satisfies the condition in Theorem 5, i.e. can play the roles of n and $n+1$. To check this, one can start by looking for pairs of components included in at least one common minimal path set. Such component pairs will obviously not satisfy the condition in (i). In the system, almost all component pairs could be excluded precisely for this reason. The only pairs we are left with as possible candidates are $(1,6), (2,5)$ and $(3,4)$. Critical path vectors for component 6 depend, however, on the condition of component 1, so the pair $(1,6)$ is not usable. The same applies to the two remaining pairs.

We then calculate the reliability polynomial of the system, $h(p) = 7p^3 - 9p^4 + 3p^5$. This polynomial is of degree 5 (which is a direct consequence of the fact that this is a directed cyclic network system), giving hope that there is an equivalent 5-system. In turns out, in fact, that there are four different coherent 5-systems equivalent to this system, namely systems 50-53 from the complete list of coherent 5-systems in Navarro and Rubio [4].

Finally, performing a search among 6-systems with reliability polynomials of order

5, using the file containing all 16,145 coherent 6-systems, referred to by Navarro and Rubio [4], we find that the 6-system with minimal path sets $\{1,2,3\},\{1,2,4\}$, $\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,6\}$ is equivalent to Φ^*, and furthermore satisfies the conditions of Theorem 5 with n and $n+1$ given as, respectively, components 4 and 6.

To sum up, in this example we start by a 6-system which does not satisfy the conditions of Theorem 5 for any pair of components. It is, however, equivalent to a coherent 6-system for which the conditions *are* satisfied, thus guaranteeing a coherent equivalent 5-system. A natural question is then whether this is a general fact, i.e., that any coherent $(n+1)$-system which has an equivalent n-system, also has an equivalent $(n+1)$-system for which the conditions of Theorem 5 hold. This remains an open question which we will not pursue further here.

5. Concluding remarks

The paper is concerned with the possible existence of equivalent systems of different sizes, where equivalence means having the same lifetime distribution. Within the class of mixed systems one can always find equivalent mixed systems of larger sizes, but not necessarily if the size is decreased. As pointed out in Section 3, an obvious condition for an $(n+1)$-system to have an equivalent n-system, is that the reliability polynomial of the former is of degree n. Example 3 shows, however, that this condition is not sufficient in general for the existence of equivalent mixed systems of lower size.

In the previous section we investigated the existence of equivalent *coherent* systems of different sizes. A sufficient condition for the equivalence of coherent systems of size $n+1$ and n was presented, together with several examples. In view of the above mentioned example of mixed systems, it is interesting to note that in a complete search among all possible 5-systems with reliability polynomials of order 4, we found that all of the polynomials correspond to *coherent* systems of order 4. We have not, however, performed a corresponding search among the possible 6-systems, but note that our Examples 7 and 8 in fact consider cases where we, for coherent 6-systems with reliability polynomials of order 5, are able to find equivalent coherent 5-systems.

In the paper we have mostly studied the problem of finding pairs of equivalent systems of sizes that differ by *one* component. Referring to our motivating example in the previous section involving twoway-relevant components, it may be possible to derive equivalence results also for coherent systems that differ in size by more than one component by considering directed systems containing more than one twoway-relevant component. We have not pursued such a task. It is notable, however, that the 2-out-of-3 system with minimal cut sets $\{1,2\},\{1,3\},\{2,3\}$ is equivalent to the coherent 5-system with minimal cut sets $\{1,2\},\{1,3\},\{1,4\},\{2,3,5\}$. (This is in fact the only coherent 5-system which has reliability polynomial of order 3).

We have seen that, for coherent systems as well as for mixed systems, it is not always possible to find equivalent systems of lower sizes. Still, for a given system, there may be reasons to look for interesting lower sized systems, for example due to the possible lower cost of building a smaller system. If there are no equivalent systems of lower size, one may instead look for smaller systems which in some sense perform approximately as well as the given one. Lindqvist and Samaniego [3] study the class of mixed n-systems with signature vector which stochastically dominate the signature of a given coherent or mixed $(n+1)$-system, and they consider the problem of optimizing a certain performance per cost criterion within this set.

Acknowledgement

This work was initiated while the first author was visiting Department of Statistics, University of California, Davis in the academic year 2011/2012. The kind hospitality of the faculty and staff is gratefully acknowledged.

References

[1] Barlow, R.E. and Proschan, F. (1981). *Statistical theory of reliability and life testing.* To Begin With, Silver Spring, MD.

[2] Boland, P.J. (2001). Signatures of indirect majority systems. *Journal of Applied Probability* **38,** 597–603.

[3] Lindqvist, B. H. and Samaniego, F. J. (2014). On the stochastic domination of systems of different sizes, with applications to reliability

economics. Statistics No. 3/2014, Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim. http://www.math.ntnu.no/preprint/statistics/2014/S3-2014.pdf

[4] Navarro, J. and Rubio, R. (2010). Computations of signatures of coherent systems with five components. *Communications in Statistics - Simulation and Computation* **39,** 68–84.

[5] Navarro, J., Samaniego, F. J., Balakrishnan, N. and Bhattacharya, D. (2008). On the application and extension of system signatures in engineering reliability. *Naval Research Logistics* **55,** 313–327.

[6] Randles, R. and Wolfe, D. (1991). *Introduction to the theory of nonparametric statistics*. Wiley, New York.

[7] Samaniego, F. J. (1985). On closure of the IFR class under formation of coherent systems. *IEEE Transactions on Reliability* **TR-34,** 69-72.

[8] Samaniego, F. J. (2007). *System signatures and their applications in engineering reliability*. International Series in Operations Research & Management Science, Vol. 110, Springer, New York.

[9] Samaniego, F. J., Balakrishnan, N. and Navarro, J. (2008). Dynamic signatures and their use in comparing the reliability of new and used systems. *Naval Research Logistics* **56,** 577–591.

Appendix A. Alternative formulation and proof of Theorem 5

Theorem 6 below is equivalent to Theorem 5, but uses a different formulation of the assumptions and is proven by comparing reliability polynomials instead of signature vectors. Some notation and a definition from the theory of coherent systems are given first.

Consider a monotone n-system with components $\{1, 2 \dots, n\}$ and structure function $\Phi(y_1, \dots, y_n)$, defined for $(y_1, \dots, y_n) \in \{0, 1\}^n$, which equals 1 if the system is working when the component states are $(y_1, \dots, y_n)$, and equals 0 if it is failed. Let $Y_1, \dots, Y_n$

be i.i.d. Bernoulli(p). Then the reliability polynomial of the system is given by $h(p) = P(\Phi(Y_1, \ldots, Y_n) = 1)$.

Theorem 6. *Given a coherent n-system where the component n is a member of at least two minimal cut sets. Construct an $(n+1)$-system by changing the component n with a new component $n+1$ in at least one of these minimal cut sets. Then the new $(n+1)$-system is coherent. Denote its structure function by $\Phi^*(y_1, \ldots, y_{n+1})$ and define the following sets:*

$$\begin{aligned}
A &= \{(y_1, \ldots, y_{n-1}) \mid \Phi^*(y_1, \ldots, y_{n-1}, 1, 0) > \Phi^*(y_1, \ldots, y_{n-1}, 0, 1)\} \\
B &= \{(y_1, \ldots, y_{n-1}) \mid \Phi^*(y_1, \ldots, y_{n-1}, 0, 1) > \Phi^*(y_1, \ldots, y_{n-1}, 1, 0)\} \\
C &= \{(y_1, \ldots, y_{n-1}) \mid \Phi^*(y_1, \ldots, y_{n-1}, 1, 1) = \Phi^*(y_1, \ldots, y_{n-1}, 0, 0)\}
\end{aligned}$$

If $A \cup B \cup C = \{0, 1\}^{n-1}$, then the given coherent n-system Φ and the constructed coherent $(n+1)$-system Φ^* are equivalent.

Proof: In the following we write for simplity $\Phi^* \equiv \Phi^*(Y_1, \ldots, Y_{n+1})$. The reliability polynomial of the $(n+1)$ system is hence given by $h^*(p) = P(\Phi^* = 1)$, where $Y_1, \ldots, Y_{n+1}$ are i.i.d. Bernoulli(p).

By abuse of notation we shall for simplicity denote by A the event that $(Y_1, \ldots, Y_{n-1}) \in A$, and similarly for B and C.

The following partial results are clear from the definitions and will be used below:

$$\begin{aligned}
P(\Phi^* = 1 | A, Y_n = 0, Y_{n+1} = 0) &= 0, \\
P(\Phi^* = 1 | B, Y_n = 0, Y_{n+1} = 0) &= 0, \\
P(\Phi^* = 1 | C, Y_n = 0, Y_{n+1} = 0) &= P(\Phi^* | C).
\end{aligned}$$

Now we get

$$\begin{aligned}
P(\Phi^* = 1) &= P(\Phi^* = 1|A)P(A) + P(\Phi^* = 1|B)P(B) + P(\Phi^* = 1|C)P(C) \\
&= P(\Phi^* = 1|A)P(A) + P(\Phi^* = 1|A, Y_n = 0, Y_{n+1} = 0)P(A) \\
&\quad + P(\Phi^* = 1|B)P(B) + P(\Phi^* = 1|B, Y_n = 0, Y_{n+1} = 0)P(B) \\
&\quad + P(\Phi^* = 1|C, Y_n = 0, Y_{n+1} = 0)P(C) \\
&= p\,[P(A) + P(B)] + P(\Phi^* = 1|Y_n = 0, Y_{n+1} = 0) \\
&= p\,[P(\Phi = 1|Y_n = 1) - P(\Phi = 1|Y_n = 0)] + P(\Phi = 1|Y_n = 0) \\
&= p\,P(\Phi = 1|Y_n = 1) + (1-p)P(\Phi = 1|Y_n = 0) \\
&= P(\Phi = 1)
\end{aligned}$$

Here we have used that $P(\Phi^* = 1|A) = P(\Phi^* = 1|B) = p$, which follow from the definitions of the events A and B. We have furthermore used that, referring now to sets,

$$A \cup B = \{(y_1, \ldots, y_{n-1}) \mid \Phi^*(y_1, \ldots, y_{n-1}, 1, 1) > \Phi^*(y_1, \ldots, y_{n-1}, 0, 0)\}$$

so that $P(A) + P(B) = P(A \cup B)$ is the probability that component n is critical in the original n-system. By the theory of coherent systems ([1]) this probability equals $P(\Phi = 1|Y_n = 1) - P(\Phi = 1|Y_n = 0)$.

This proves that the reliability polynomials of the systems Φ and Φ^* are equal, and hence that the corresponding systems are equivalent by Definition 1.