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Bo H. Lindqvist and Francisco J. Samaniego

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Bo H. Lindqvist has homepage: http://www.math.ntnu.no/∼bo
E-mail: bo@math.ntnu.no
Address: Department of Mathematical Sciences, NTNU, N-7491 Trondheim, Norway.

ON THE STOCHASTIC DOMINATION OF SYSTEMS OF DIFFERENT SIZES, WITH APPLICATIONS TO RELIABILITY ECONOMICS

BO H. LINDQVIST,* *Norwegian University of Science and Technology*
FRANCISCO J. SAMANIEGO,** *University of California, Davis*

Abstract

The signature of a coherent system with independent and identically distributed component lifetimes is a useful tool in the study and comparison of lifetimes of systems. A key result is the representation of a system's survival distribution in terms of its signature vector, which enables comparison of lifetime distributions of different systems, possibly with differing number of components. The main result of the present paper is a characterization of systems of size n which stochastically dominate a given system of size $n+m$ for $m \geq 1$, where for simplicity of presentation we study in detail the case $m = 1$ only. The characterization is applied in a reliability economics setting to a corresponding comparison of performance-per-cost for systems of different sizes. We also obtain some new results on the comparison of performance-per-cost for systems with the same number of components.

Keywords: coherent system; system signature; k-out-of-n system; mixed system; ordering of random variables; performance-per-cost;

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* Postal address: Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

** Postal address: Department of Statistics, Mathematical Sciences Building, One Shields Avenue, University of California, Davis, CA 95616, USA

1. Introduction

Consider a coherent system with n binary components, as studied in the monograph by Barlow and Proschan [1]. We shall in the following say, for short, that a system with n components is an n-system. Suppose that the component lifetimes $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) with continuous distribution F, and let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be their ordered values. Samaniego [5] introduced the *signature vector*, $\mathbf{s} = (s_1, \ldots, s_n)$, of the system, defined by $s_k = P(T = X_{k:n})$; $k = 1, \ldots, n$. The signature of a system depends only on the system structure and does not depend on the distribution F of component lifetimes. Theorem 3.1 in Samaniego [6] states that the survival function of the lifetime T of the system can be represented as

$$\bar{F}_T(t) = P(T > t) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (F(t))^j (\bar{F}(t))^{n-j}, \tag{1}$$

where $\bar{F}(t) = 1 - F(t)$, that is, that the system lifetime distribution is completely determined by the pair $(\mathbf{s}, F)$.

Standard examples of coherent systems are series systems, which work if and only if all components are working, and parallel systems, which work if and only if at least one component is working. We shall also be interested in the so-called k-out-of-n systems which fail upon the k-th component failure, for $1 \leq k \leq n$. It is easy to see that the signature vector of a k-out-of-n system is $(0, \ldots, 1_k, \ldots, 0)$, where the subindex k refers to the kth element of the vector.

For practical as well as mathematical reasons, it has proven useful to extend the class of n-systems to include so-called *mixed* n-systems, see, e.g., [6, p.28-31]. A mixed n-system is a stochastic mixture of a finite number of coherent n-systems. It is easily verified that the result (1) continues to hold for mixed systems (see, e.g., remark in Samaniego et al. [7]). Note that any probability vector $(s_1, \ldots, s_n)$ can serve as the signature of a mixed system. One possible representation of such a mixed system is the one which gives weight s_k to a k-out-of-n system, for $k = 1, \ldots, n$.

Signature vectors have proven particularly useful in the comparison of lifetimes of different systems. Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be signature vectors of two mixed n-systems and let T_1 and T_2 be these systems' lifetimes. As is shown in ([6, Section 4.2]), certain ordering properties of signature vectors are preserved for the corresponding lifetime distributions. The following three orderings are considered in [6] and will be applied in Section 4 of this paper. The definitions below apply both for discrete and continuous pairs of random variables (X_1, X_2):

Definition 1. Let X_1 and X_2 be random variables with corresponding cumulative distribution functions F_1 and F_2, and let $\bar{F}_i = 1 - F_i$ for $i = 1, 2$. Then X_1 is smaller than X_2 in the *stochastic ordering,* denoted $X_1 \leq_{st} X_2$, if and only if $\bar{F}_1(x) \leq \bar{F}_2(x)$ for all x; in the *hazard rate ordering,* denoted $X_1 \leq_{hr} X_2$, if and only if $\bar{F}_2(x)/\bar{F}_1(x)$ is increasing in x; and in the *likelihood ratio ordering,* denoted $X_1 \leq_{lr} X_2$, if and only if $f_2(x)/f_1(x)$ is increasing in x (assuming that F_i is absolutely continuous with density f_i for $i = 1, 2$).

Theorems 4.3-4.5 of ([6]) state that when g denotes any of the three orderings of random variables, st, hr, lr, defined above we have that

$$\mathbf{s}_1 \leq_g \mathbf{s}_2 \text{ implies } T_1 \leq_g T_2. \tag{2}$$

Navarro et al. [3, Section 2.1] give the following definition of *equivalent systems*: Two systems with i.i.d. component lifetimes with distribution F, are said to be *equivalent* if the lifetime distributions of the systems are identical, for any component distribution F. It hence follows that two systems of equal sizes are equivalent if they have the same signature vector. The cited definition of equivalence is not, however, restricted to systems of the same size. In order to extend the use of comparison results like (2) in comparisons of lifetime distributions for systems of different sizes, Samaniego [6, page 32] suggested "converting" the smaller of two systems into an equivalent system of the same size as the larger one. Then, the signature vector of this derived system can be compared to the signature vector of the larger system so that the theorems cited above are applicable.

Theorem 3.2 of Samaniego [6] solves the problem of comparison of mixed n- and $(n+1)$-systems, for any n, by giving a formula for the signature of an $(n+1)$-system equivalent to a given n-system. By repeated use one is of course able to compare any two systems in this way. The cited theorem is displayed in Section 2.

One of the motivations for introducing mixed systems is from the application of signatures to reliability economics ([6, Ch. 7]. The problem considered is that of optimizing the performance of a system under given cost constraints. Here the performance of a system with signature vector $\mathbf{s} = (s_1, \ldots, s_n)$ is represented as a linear function of the signatures, $\sum_{i=1}^n h_i s_i$. A motivation for this choice of performance measure is that, e.g., either the expected lifetime of the system or the reliability function of the system can be written in this way (see [6, page 95]) and Section 3 of the present paper. Similarly, one may take the expected cost of building a system with the signature vector $\mathbf{s}$ to be $\sum_{i=1}^n c_i s_i$, where c_i is interpreted as the cost of building an i-out-of-n system, $i = 1, \ldots, n$. From this, [6, Ch. 7] defines the following measure of the relative value of performance and cost for a mixed n-system with signature vector $\mathbf{s}$,

$$m_r(\mathbf{s}, \mathbf{h}, \mathbf{c}) = \frac{\sum_{i=1}^n h_i s_i}{(\sum_{i=1}^n c_i s_i)^r}. \tag{3}$$

As explained in [6, p. 97], the power parameter $r > 0$ serves as a calibration parameter, determining the weight to be put on cost relative to performance in the criterion function (3). Thus $r = 1$ is the natural choice if equal weight is put on performance and cost.

The optimality problem considered in [6, Chapter 7] is the problem of maximizing the performance-per-cost criterion (3) with respect to the signature vector $\mathbf{s}$ among all mixed systems of the given size.

The main aim of the present paper is to explore the possibility of increasing performance-per-cost by building smaller systems, i.e., systems with fewer components. The motivation for this is that a smaller system, equivalent to a larger one, is expected to have a lower cost than the larger system, while performing exactly as well as the larger system. As noted in Lindqvist, Samaniego and Huseby [2], there may however not be equivalent systems of smaller sizes than the given system. In that case one may instead look for smaller systems which are approximately equivalent, or, as will be done in Section 3, look for smaller systems with better performance, and perhaps lower cost.

As a preparation for our study of optimal systems within a reliability economics framework, we give in Section 2, a characterization of the class of mixed n-systems with signature vectors which stochastically dominate the signature vector of a given coherent or mixed $(n+1)$-system. In Section 3 we will then consider the problem of optimizing the performance-per-cost function (3) within this set of n-systems, for a given $(n+1)$-system. Section 4 considers a slightly different topic, namely conditions under which a specific ordering of signature vectors of two n-systems leads to corresponding inequalities for the performance-per-cost criterion, particularly for the case $r = 1$ in (3). As a by-product, we obtain a new proof of one result and a strengthening of another result from [6, Ch. 7]. In the final section, we make some concluding remarks.

2. The set of n-systems that stochastically dominate a given $(n + 1)$-system

The approach of the present section has Theorem 3.2 of [6] as its point of departure. We state the theorem below, together with a corollary which proves useful in the sequel.

Theorem 1. *Let $\mathbf{s} = (s_i; i = 1, 2, \ldots, n)$ be the signature of a coherent or mixed system based on n components with i.i.d. lifetimes with common continuous distribution F. Then a (mixed) equivalent system with $n + 1$ components has the signature vector $\mathbf{s}^* = (s^*_1, s^*_2, \ldots, s^*_{n+1})$, where*

$$\begin{aligned} s^*_1 &= \frac{n}{n+1} s_1 \\ s^*_k &= \frac{k-1}{n+1} s_{k-1} + \frac{n-k+1}{n+1} s_k;\ k = 2, 3, \ldots, n \\ s^*_{n+1} &= \frac{n}{n+1} s_n. \end{aligned}$$

The following corollary to the theorem is formulated in terms of cumulative signature vectors. For an n-system and an equivalent $(n + 1)$-system, with signatures $\mathbf{s}$ and $\mathbf{s}^*$, respectively, we introduce the cumulative signature vectors, respectively, $\mathbf{b}$ and $\mathbf{b}^*$ given by $b_j = \sum_{i=1}^{j} s_i$ for $j = 1, \ldots, n$ and $b^*_j = \sum_{i=1}^{j} s^*_i$ for $j = 1, \ldots, n + 1$.

The result below is proved by summing the equations of Theorem 1.

Corollary 1. *Let $\mathbf{s} = (s_i; i = 1, 2, \ldots, n)$ be the signature of an n-system and let $\mathbf{b}$ be the corresponding cumulative signature vector. Then an equivalent coherent or mixed system with $n + 1$ components has the cumulative signature vector $\mathbf{b}^*$ given by*

$$b^*_j = \begin{cases} b_j - \frac{j}{n+1} s_j & \textit{for } j = 1, \ldots, n \\ 1 & \textit{for } j = n + 1 \end{cases}$$

Let an $(n+1)$-system with signature vector $\mathbf{s}^*$ and corresponding cumulative signature vector $\mathbf{b}^*$ be given. Suppose there is an n-system with signature $\mathbf{s} = (s_1, \ldots, s_n)$

that stochastically dominates the given $(n+1)$-system in the sense that the $(n+1)$-version of $\mathbf{s}$, here and in the following called $\tilde{\mathbf{s}}$ (given as in Theorem 1) satisfies $\tilde{\mathbf{s}} \geq_{st} \mathbf{s}^*$ (see Definition 1), or, equivalently,

$$\tilde{b}_j \leq b_j^* \text{ for } j = 1, \ldots, n,$$

where $\tilde{\mathbf{b}}$ is the cumulative signature corresponding to $\tilde{\mathbf{s}}$. Since necessarily $\tilde{\mathbf{b}}_{n+1} = \mathbf{b}_{n+1}^* = 1$, the above inequalities are, by Corollary 1, equivalent to

$$\begin{aligned}
\frac{n}{n+1}s_1 &\leq b_1^* \\
s_1 + \frac{n-1}{n+1}s_2 &\leq b_2^* \\
s_1 + s_2 + \frac{n-2}{n+1}s_3 &\leq b_3^* \\
&\cdots \\
s_1 + s_2 + \cdots + s_{n-2} + \frac{2}{n+1}s_{n-1} &\leq b_{n-1}^* \\
s_1 + s_2 + \cdots + s_{n-1} + \frac{1}{n+1}s_n &\leq b_n^*
\end{aligned}$$

Since we must have $s_1 + \ldots + s_n = 1$, we may in the last inequality substitute $s_n = 1 - s_1 - s_2 - \ldots - s_{n-1}$ and hence obtain the equivalent inequality

$$s_1 + s_2 + \cdots + s_{n-1} \leq \frac{n+1}{n}b_n^* - \frac{1}{n} \tag{4}$$

which in the following is assumed to replace the last inequality above.

This gives us a set of n inequalities for linear functions of $s_1, \ldots, s_{n-1}$, all of them with non-negative coefficients. Note than an additional inequality $s_1+s_2+\cdots+s_{n-1} \leq 1$ is not necessary since it is already implied by (4). Furthermore, since a nonnegative solution for $\mathbf{s}$ can only exist if all the right hand sides of the inequalities are nonnegative, (4) implies that an n-system that stochastically dominates an $(n+1)$-system with signature vector $\mathbf{s}^*$ can only exist if $b_n^* \geq 1/(n+1)$ or, equivalently, $s_{n+1}^* \leq n/(n+1)$. Informally this can be stated, "unless the $(n+1)$-system is a parallell system, or close to being so, we can find a better n-system". More precisely we can formulate the following result:

Theorem 2. *Let there be given an $(n+1)$-system with signature vector $\mathbf{s}^*$ and cumulative signature vector $\mathbf{b}^*$, satisfying $s_{n+1}^* \leq n/(n+1)$. Then there is a non-empty* *convex set of signature vectors* $\mathbf{s}$ *of* n*-systems which stochastically dominate the given* $(n+1)$*-system, where each such vector satisfies the* n *inequalities*

$$\begin{array}{rcl} \frac{n}{n+1} s_1 & \leq & b_1^* \\ s_1 + \frac{n-1}{n+1} s_2 & \leq & b_2^* \\ s_1 + s_2 + \frac{n-2}{n+1} s_3 & \leq & b_3^* \\ & \cdots & \\ s_1 + s_2 + \cdots + s_{n-2} + \frac{2}{n+1} s_{n-1} & \leq & b_{n-1}^* \\ s_1 + s_2 + \cdots + s_{n-1} & \leq & \frac{n+1}{n} b_n^* - \frac{1}{n} \end{array} \tag{5}$$

As a possible application, suppose we would like to build an n-system which is at least as good as a given $(n+1)$-system, at minimum cost. ("At least as good" here means with respect to stochastic ordering of signature vectors). Let the expected cost of a system with signature vector $\mathbf{s}$ be $\sum_{i=1}^{n} c_i s_i$, where $0 \leq c_1 \leq c_2 \leq \ldots \leq c_n$. (A discussion and motivation for this cost function is given in Section 3.) Then write

$$\sum_{i=1}^{n} c_i s_i = \sum_{i=1}^{n-1} c_i s_i + c_n(1 - s_1 - \ldots - s_{n-1}) = c_n - \sum_{i=1}^{n-1} (c_n - c_i) s_i.$$

Thus the problem of minimizing the cost of the n-system over the convex set of signature vectors satisfying (5) is equivalent to maximizing the linear combination $\sum_{i=1}^{n-1} (c_n - c_i) s_i$. By the theory of linear programming (see, e.g., Nering and Tucker [4]) this maximum will occur at an extreme point of the convex set defined in Theorem 2. Furthermore, since our setup in Theorem 2 involves $n-1$ variables, $s_1, \ldots, s_{n-1}$, it follows that any extreme point of the convex set defined in the theorem satisfies with equality at least $n-1$ of the inequalities which define the restrictions. These restrictions are first of all the n inequalities of (5), but also the inequalities $s_i \geq 0$ for $i = 1, \ldots, n-1$.

Example 1. (*The bridge system.*) The bridge system (see, e.g., [6, Page 9]) has 5 components and signature vector $\mathbf{s}^* = (0, 1/5, 3/5, 1/5, 0)$ and hence cumulative signature vector $\mathbf{b}^* = (0, 1/5, 4/5, 1, 1)$. Suppose we want a 4-system which is stochastically at least as strong as this system. Setting $n = 4$ in Theorem 2, we get the inequalities

$$\begin{aligned}
\frac{4}{5}s_1 &\leq_1 \quad b_1^* = 0 \\
s_1 + \frac{3}{5}s_2 &\leq_2 \quad b_2^* = \frac{1}{5} \\
s_1 + s_2 + \frac{2}{5}s_3 &\leq_3 \quad b_3^* = 4/5 \\
s_1 + s_2 + s_3 &\leq_4 \quad \frac{5}{4}b_4^* - \frac{1}{4} = 1 \\
s_1 &\geq_5 \quad 0 \\
s_2 &\geq_6 \quad 0 \\
s_3 &\geq_7 \quad 0
\end{aligned}$$

From inequalities $\leq_1$ and $\geq_5$ it is clear that $s_1 = 0$ for all the solutions to our problem. It can be shown that the extreme points of the set of (s_1, s_2, s_3) satisfying the above inequalities are

$$\begin{aligned}
(0,0,0) &\quad \text{with equalities at} \quad \geq_5,\ \geq_6,\ \geq_7 \\
(0,1/3,0) &\quad \text{with equalities at} \quad \leq_2,\ \geq_5,\ \geq_7 \\
(0,1/3,2/3) &\quad \text{with equalities at} \quad \leq_1,\ \leq_2,\ \leq_4 \\
(0,0,1) &\quad \text{with equalities at} \quad \leq_1,\ \leq_4,\ \geq_5,\ \geq_6
\end{aligned}$$

Since $s_4 = 1 - s_1 - s_2 - s_3$, these correspond to 4-systems with the following signature vectors,

$$(0,0,0,1),\ (0,1/3,0,2/3),\ (0,1/3,2/3,0),\ (0,0,1,0). \tag{6}$$

Thus, any 4-system which is stochastically as good as the bridge system, has a signature vector which can be written as a mixture of the four signature vectors in (6). Considering Table 2.1 of [6] it is seen that all of these signatures, except $(0,1/3,0,2/3)$ correspond to coherent systems.

Suppose as an example that the price of an i-out-of-4 system is i, $i = 1,\ldots,4$. In order to find the 4-system which stochastically dominates the bridge system, and which is of minimum cost, we need to check the cost only at the extreme points (6), and choose the one with the minimum cost. This turns out to be the system with signature $(0,1/3,2/3,0)$. The equivalent 5-system has the signature $(0,1/5,2/5,2/5,0)$ which is easily seen to be stochastically larger than the signature of the bridge system.

Example 2. Let $n = 3$ and let a coherent $(n+1)$-system have minimal cut sets $\{1\}, \{2,3,4\}$. The signature vector is then $\mathbf{s}^* = (1/4, 1/4, 1/2, 0)$, while $\mathbf{b}^* = (1/4, 1/2, 1, 1)$. Putting $n = 3$ in Theorem 2 we get the equations

$$\begin{aligned}
\frac{3}{4}s_1 &\leq \frac{1}{4} \\
s_1 + \frac{2}{4}s_2 &\leq \frac{1}{2} \\
s_1 + s_2 &\leq \frac{4}{3} - \frac{1}{3} = 1 \\
s_1 &\geq 0 \\
s_2 &\geq 0
\end{aligned}$$

It can be shown that the extreme points of the resulting convex set of (s_1, s_2) is

$$(0,0),\ (1/3,0),\ (1/3,1/3),\ (0,1). \tag{7}$$

so that the extreme points of the set of stochastically dominating signature vectors of size 3 are

$$(0,0,1),\ (1/3,0,2/3),\ (1/3,1/3,1/3),\ (0,1,0).$$

The first and last of these are signatures of coherent systems, while the other two are not. The signature $(1/3, 1/3, 1/3)$ is, however, the signature of a 3-system equivalent to a single component, i.e. a 1-system. It is intuitively clear that this system is stochastically better than the given system, since the latter has a minimal cut set *in addition to* the set $\{1\}$, which makes it more frail. Which of these systems might be considered best if system costs were taken into accouont remains undetermined. We examine such questions in the next section.

3. Reliability economics: Comparing performance-per-cost for n- and $(n+1)$-systems

As mentioned in the Introduction, we wish to compare performance-per-cost measures for systems of different sizes, in particular for n- and $(n+1)$-systems.

Consider the following situation. Suppose that there is given an $(n+1)$-system with signature vector $\mathbf{s}^*$. Let there also be given a cost vector $\mathbf{c}^* = (c_1^*, \ldots, c_{n+1}^*)$, where c_k^*

defines the expected cost of a k-out-of-$(n+1)$-systems for $k = 1, \ldots, n+1$. Suppose similarly that there is given a performance vector $\mathbf{h}^* = (h_1^*, \ldots, h_{n+1}^*)$. For the given system, the value of the criterion function (3) associated with the signature $\mathbf{s}^*$ is hence

$$A^* = m_r(\mathbf{s}^*, \mathbf{h}^*, \mathbf{c}^*) = \frac{\sum_{i=1}^{n+1} h_i^* s_i^*}{(\sum_{i=1}^{n+1} c_i^* s_i^*)^r}. \tag{8}$$

3.1. Expected cost of equivalent n and $(n+1)$-systems

Suppose now that only n components are at hand, and that one wishes to build an n-system with a performance-per-cost which is at least as large as A^*. For comparison, we then need to know the values c_i and h_i, for $i = 1, \ldots, n$, representing, respectively, the cost and the performance of a i-out-of-n-system. These may in practice be given from known sources, but we suggest below a reasonable relation between the c_i and h_i for n-systems and the corresponding values c_i^* and h_i^* for $(n+1)$-systems.

Proposition 1. *The expected cost of using an n-system is at most equal to (equal to) the cost of using the equivalent $(n+1)$-system, for all mixed n-systems, if and only if*

$$c_i \le (=)\ (1 - \frac{i}{n+1})c_i^* + \frac{i}{n+1}c_{i+1}^* \tag{9}$$

for $i = 1, \ldots, n$.

Proof: From Theorem 1 follows that $\sum_{i=1}^n c_i s_i \le \sum_{i=1}^{n+1} c_i^* s_i^*$ if and only if

$$\sum_{i=1}^n \left(c_i - \frac{n-i+1}{n+1}c_i^* - \frac{i}{n+1}c_{i+1}^* \right) s_i \ \le\ 0.$$

This implies (9) for each specific i by setting $s_i = 1$ and proves the 'only if' part of the proposition. The 'if' part is clear.

Samaniego [6, page 95] gives an example of how the linear representation $\sum_{i=1}^n c_i s_i$ of expected cost of an n-system might occur in practice, that is, in the so called "salvage model". Here, it is assumed that the cost c_i of an i-out-of-n system can be written as

$$c_i = C + nA - (n-i)B \text{ for } i = 1, \ldots, n, \tag{10}$$

where C is the initial fixed cost of manufacturing the system, A is the cost of an individual component, and B is the salvage value of a used but working component which is removed after system failure. Below we shall use a more convenient form of

(10), namely

$$c_i = C + n(A - B) + iB \text{ for } i = 1, \ldots, n. \tag{11}$$

Let us instead start with an $(n+1)$-system. Then under the salvage model (11) the expected cost of an i-out-of-$(n+1)$-system can be written

$$c_i^* = C^* + (n+1)(A^* - B^*) + iB^* \text{ for } i = 1, \ldots, n+1.$$

Assuming for a moment that the expected cost of any n-system should be the same as that of the equivalent $(n+1)$-system, leads to (by Proposition 1) the expected cost for an i-out-of-n system equal to

$$C^* + A^* - B^* + n(A^* - B^*) + iB^* + \frac{i}{n+1}B^*.$$

Comparing this with (11), and making the natural assumption that $A = A^*$ and $B = B^*$, we obtain the following result:

Proposition 2. *Suppose that the costs of n-systems and $(n+1)$-systems are given by the salvage model (11) with the same constants A and B, but with possibly different constants C, respectively C and C^* for the n-systems and $(n+1)$-systems. Then the cost reduction by using an i-out-of-n system compared to the equivalent $(n+1)$-system is*

$$C^* - C + (A - B) + \frac{iB}{n+1}.$$

Remark: Since naturally $A > B > 0$, the n-system hence has lower expected cost, unless C is too large in comparison with C^*.

Example 2 *(continued).* We consider the convex set of signatures for 3-systems which stochastically dominate the given 4-system. Suppose costs are given by a salvage model where the constants A, B, C are equal for 3- and 4-systems, with values $C = 1/10,\ A = 3/5,\ B = 1/2$. This gives the cost values for the 4-system and the 3-system, respectively,

$$\begin{aligned} c_i^* &= \frac{1}{10} + 4 \cdot \frac{1}{10} + i \cdot \frac{1}{2} = \frac{1}{2} + i \cdot \frac{1}{2} \\ c_i &= \frac{1}{10} + 3 \cdot \frac{1}{10} + i \cdot \frac{1}{2} = \frac{2}{5} + i \cdot \frac{1}{2} \end{aligned}$$

or, in vector form, $\mathbf{c}^* = (1, 3/2, 2, 5/2)$ and $\mathbf{c} = (9/10, 7/5, 19/10)$. We can now compare the latter values with the values we would get on the right hand side of the inequalities in Proposition 1. These would be, in the same order, $(9/8, 7/4, 19/8)$, which are all larger than the ones obtained by the salvage model. Alternatively, these values could been obtained by using Proposition 2.

Remark: It is interesting to note that the result of Proposition 2 can be obtained also when going from n-systems to $(n+1)$-systems. Thus consider an i-out-of-n-system with expected cost given by (11). To get an equivalent $(n+1)$-system we follow the idea of [2] and add an irrelevant component to the original n-system, which is independent of the components of the n-system and has the same lifetime distribution. The cost of this extra component is A units, but the component can be salvaged for B units if it does not fail before the i-out-of-n system fails. The probability of the latter case equals the probability that in the simultaneous ordering of the lifetimes $X_1, \ldots, X_n$ of the original components and the lifetime Y of the irrelevant component, Y is not among the i smallest. This probability is clearly $1 - i/(n+1)$. Thus, using the equivalent $(n+1)$-system obtained this way adds a cost $A - B(1 - i/(n+1)) = A - B + i/(n+1)B$ to the original system. But this is exactly the same amount that we obtained above when using Proposition 1, if the fixed costs C and C^* are assumed equal.

There are thus reasons to assume that the expected cost of equivalent systems is reduced when reducing the number of components. The salvage model (11) gives an explicit way of expressing this, when we assume that the fixed costs A, B, C are independent of n.

3.2. Performance measures of equivalent n- and $(n+1)$-systems

Now we turn to a more detailed study of the performance measure and how it differs between equivalent systems of different sizes. As we have already seen, the performance measure of an n-system with signature vector $\mathbf{s} = (s_1, \ldots, s_n)$, is given in the form $\sum_{i=1}^n h_i s_i$. Samaniego [6, page 95] motivates this by giving two natural examples of performace measures, the expected lifetime of the system, ET, and the reliability of the system, $P(T > t_0)$ for a specific value t_0. It is well known that $ET = \sum_{i=1}^n s_i EX_{i:n}$, while $P(T > t_0) = \sum_{i=1}^n s_i P(X_{i:n} > t_0)$, so both measures are of the form $\sum_{i=1}^n h_i s_i$.

More generally, for any function $\Phi(T)$, we can write $E\Phi(T) = \sum_{i=1}^{n} s_i E\Phi(X_{i:n})$, so there is a natural class of performance measures of the form $\sum_{i=1}^{n} h_i s_i$. An important property of the measures is that their values are unchanged among equivalent systems. Since an i-out-of-n system is equivalent to the mixture of an i-out-of-$(n+1)$ system and an $(i+1)$-out-of-$(n+1)$-system, with weight $1 - i/(n+1)$ to the former, we get the relation

$$h_i = (1 - \frac{i}{n+1})h_i^* + \frac{i}{n+1}h_{i+1}^* \text{ for } i = 1, \ldots, n,$$

between the performance vectors $\mathbf{h}^* = (h_1^*, \ldots, h_{n+1}^*)$ and $\mathbf{h} = (h_1, \ldots, h_n)$, respectively, of equivalent systems of size $n+1$ and n. This of course also follows from the same reasoning as in Proposition 1.

3.3. Comparison of performance-per-cost for n- and $(n+1)$-systems

Returning to the full criterion functions, (3) and (8), it follows from the above that if one starts with an $(n+1)$-system which has an equivalent n-system, then the numerators of (3) and (8) are equal. Thus the criterion function for the n-system will be the smaller of the two if and only if the n-system has a lower expected cost.

Suppose next that we start from an $(n+1)$-system for which there is no equivalent n-system. There may be reasons to search among the class of n-systems that are stochastically better than the given $(n+1)$-system. One motivation is that these may still have a lower cost than the $(n+1)$-system, and they will also have a better performance.

Theorem 2 defines the convex set of signature vectors of all n-systems that stochastically dominate the given $(n+1)$-system. We assume below that the required condition on s_{n+1}^* given in the theorem is satisfied, and we let R denote the convex set of $(s_1, \ldots, s_{n-1})$ defined by Theorem 2. We now seek to maximize the criterion function m_r from (3) over this set. Since $s_n = 1 - s_1 - \ldots - s_{n-1}$ we may rewrite (3) as

$$m_r(\mathbf{s}, \mathbf{h}, \mathbf{c}) = \frac{h_n - \sum_{i=1}^{n-1}(h_n - h_i)s_i}{(c_n - \sum_{i=1}^{n-1}(c_n - c_i)s_i)^r} \equiv \frac{h_n - \sum_{i=1}^{n-1}\tilde{h}_i s_i}{(c_n - \sum_{i=1}^{n-1}\tilde{c}_i s_i)^r}, \tag{12}$$

where $\tilde{c}_i = c_n - c_i$ and $\tilde{h}_i = h_n - h_i$ for $i = 1, \ldots, n-1$.

We claim that the maximum of (12) occurs on the boundary of the convex set R. To see this, assume, as in a proof by contradiction, that the maximum occurs at an interior point $\hat{\mathbf{s}}$ of R. At this point, consider the two hyperplanes of $(s_1, \ldots, s_{n-1})$ for which, respectively, the linear functions $\sum_{i=1}^{n-1} \tilde{h}_i s_i$ and $\sum_{i=1}^{n-1} \tilde{c}_i s_i$ have the same value as in the optimum point $\hat{\mathbf{s}}$. Let N be an open neighborhood of $\hat{\mathbf{s}}$ which is included in R, and consider the intersection N_0 of N and the above hyperplane defined by the $\tilde{c}_i$. On this set, the denominator of (12) is constant. The set N_0 will, however, contain points on both sides of the hyperplane defined by the $\tilde{h}_i$, meaning that $\sum_{i=1}^{n-1} \tilde{h}_i s_i$ on N_0 will have values both larger and smaller than its value at $\hat{\mathbf{s}}$. But then (12) will in N_0 take values larger than the value at $\hat{\mathbf{s}}$, which gives a contradiction since the maximum is assumed to be at $\hat{\mathbf{s}}$. Thus the maximum point of (12) is a boundary point of R.

The above argument clearly holds for all r. We now argue that for $r = 1$, the maximum value of (12) must be at an extreme point of R. Suppose, as in a proof by contradiction, that the maximum is at a boundary point of R which is not an extreme point. Let the maximum value of (12) be A. Then (12) equals A in a hyperplane in the space of $s_1, \ldots, s_{n-1}$ (the hyperplane will depend on A). But since the hyperplane contains a point on the boundary of R which is not an extreme point, the hyperplane will intersect the interior of R. Hence the interior of R will also contain an optimum point of (12). But this is impossible by what we have already seen for general r, so we get a contradiction. This shows that the maximum of (12) for $r = 1$ must be at an extreme point.

While it is shown in [6, Chapter 7] that, for $r \neq 1$, the maximum of m_r on the full simplex of signature vectors $\mathbf{s} = (s_1, \ldots, s_n)$ is attained for an $\mathbf{s}$ with at most two positive elements, it will be seen in an example below that the maximum of (12) restricted to the set R may well occur at boundary points with more than two positive entries.

Example 2 *(continued).* We have already computed the cost vectors $\mathbf{c}^* = (1, 3/2, 2, 5/2)$ and $\mathbf{c} = (9/10, 7/5, 19/10)$. Let the components' lifetimes X_i be exponential with expected value 1. It is well known that $EX_{i:n} = 1/n + 1/(n-1) + \ldots 1/(n-i+1)$ for $i = 1, \ldots, n$, so we have the performance vectors for the 4-system and 3-system given by, respectively, $\mathbf{h}^* = (1/4, 7/12, 13/12, 25/12)$ and $\mathbf{h} = (1/3, 5/6, 11/6)$.

It follows that A^* from (8) equals

$$A^* = \frac{6 \cdot 8^{r-1}}{13^r}.$$

Since we consider the set of 3-systems which stochastically dominate the given system, the function to maximize is

$$m_r(s_1, s_2) = \frac{\frac{11}{6} - \frac{3}{2}s_1 - s_2}{(\frac{19}{10} - s_1 - \frac{1}{2}s_2)^r}$$

on the convex set R with extreme points given in (7): $(0,0)$, $(1/3,0)$, $(1/3,1/3)$, $(0,1)$.

A rough grid search was used to find the optimal point in R for r ranging from 0.05 to 10.00 in steps of 0.05. The result is that for $r \leq 1.55$, the maximum is obtained at $(s_1, s_2, s_3) = (0,0,1)$, i.e. a parallell system of 3 components. For $1.60 \leq r \leq 1.75$ the optimum point changes continuously from $(0,0,1)$ to $(1/3,0,2/3)$ along the path $(p,0,1-p)$ for $0 \leq p \leq 1/3$. Next, for $1.80 \leq r \leq 2.35$ the optimum is at the single point $(1/3,0,2/3)$. When r increases further, $2.40 \leq r \leq 2.75$, the optimum changes continuously from $(1/3,0,2/3)$ to $(1/3,1/3,1/3)$ along the path $(1/3,p,2/3-p)$ for $0 \leq p \leq 1/3$, while for $r \geq 2.80$ the optimum is steady at $(1/3,1/3,1/3)$. The optimum value of the criterion function is for each r larger than the corresponding A^* for the original $(n+1)$-system.

We also maximized (3) in the full simplex of (s_1, s_2, s_3), i.e. over all possible mixed 3-systems. Then for $r \leq 1.55$ the optimum was at $(0,0,1)$ (parallel system) and for $r \geq 4.00$ at $(1,0,0)$ (series system). For r from 1.60 to 3.95 the optimum point changed continuously along $(p,0,1-p)$ for $0 \leq p \leq 1$.

A similar performance was seen when optimizing over all 4-systems. For $r \leq 1.45$ the optimum was at $(0,0,0,1)$ (parallel system). Then for $1.50 \leq r \leq 4.75$ the optimum changed continuously along $(p,0,0,1-p)$ to reach $(1,0,0,0)$ at $r = 4.80$ and is constant at that value for $r > 4.80$.

We finally compared the maximum value of the criterion functions for the optimal systems at different values of r. It turns out that for r smaller than approximately 0.4, the optimal 4-system is stronger than the optimal 3-system, while for all $r > 0.4$ the optimal 3-system is the stronger. The reason that the 4-system is stronger for small r is that then the performace is the most important part of the criterion function. Certainly a parallel system of 4 components has better performace than a parallel system with 3 components. On the other hand, for r large, the cost is becoming the most important aspect so a 3-system is preferred to a 4-system.

4. On the comparison of performance-per-cost for ordered systems

In this section we consider a slightly different problem than treated in the previous sections. Consider two systems of size n, where the signature vectors are ordered with respect to one of the order relations considered in Definition 1, *stochastic ordering* (st); *hazard rate ordering* (hr); or *likelihood ratio ordering* (lr).

Usually, we will have cost vectors $\mathbf{c} = (c_1, \ldots, c_n)$ and performance vectors $\mathbf{h} = (h_1, \ldots, h_n)$ with values which are increasing with the index. In this case, it follows that for two n-systems with respective signature vectors, $\mathbf{s} = (s_1, \ldots, s_n)$ and $\mathbf{t} = (t_1, \ldots, t_n)$ which are stochastically ordered, with $\mathbf{s} \leq_{st} \mathbf{t}$, then both the performance and cost will be larger for the system with signature $\mathbf{t}$ than for the one with $\mathbf{s}$. Since these appear, respectively, in the numerator and denominator of the performance-per-cost measure, it is still of interest to compare the criterion functions for such two systems. Intuitively, if in some way the performance increases proportionally more than the cost when going to a stochastically better system, then the value of the criterion function my also rise. To make this precise, we shall make use of the following theorem, which follows from results of Shaked and Shantikumar [8, Chapter 1].

Theorem 3. *Let X and Y be two independent random variables and let $\alpha(\cdot)$ and $\beta(\cdot)$ be arbitrary real-valued functions. Consider the follwing conditions:*

C1: $\beta(x) \geq 0$ *for all* x

C2: $\alpha(x)/\beta(x)$ *is nondecreasing in* x

C3: $\beta(x)$ *is increasing in* x

C4: $\alpha(y)\beta(x) - \alpha(x)\beta(y)$ *is nonincreasing in* x *on* $\{x \leq y\}$

C5: $E[\alpha(X)]E[\beta(Y)] \leq E[\alpha(Y)]E[\beta(X)]$ *(assuming that the expectations exist)*

The following implications hold:

(i) *If $X \leq_{lr} Y$, then C1,C2 together imply C5.*

(ii) *If $X \leq_{hr} Y$, then C1,C2,C3 together imply C5.*

(iii) *If $X \leq_{st} Y$, then C1,C2,C3,C4 together imply C5.*

Proof: The statement (ii) is part of Theorem 1.B.12 in Shaked and Shantikumar [8]. In their proof, (ii) is obtained by application of their Theorem 1.B.10 to a particular pair of functions ϕ_1, ϕ_2. The results (i) and (iii) are not stated in [8], but it is straightforward to check that (i) is obtained by using the same pair of functions as in their Theorem 1.C.22, while (iii) is obtained similarly from their Theorem 1.A.10.

Remark: Regarding (ii) above, Theorem 1.B.12 in [8] in fact shows that if C5 holds for all functions α, β satisfying C1,C2,C3, then $X \leq_{hr} Y$. We are not, however, able to show that corresponding results hold in (i) and (iii).

In our application of Theorem 3, we let X, Y be defined on $\{1, \ldots, n\}$ with probability distributions respectively given by the signature vectors $\mathbf{s}$ and $\mathbf{t}$. Further, the functions α and β will be defined on $\{1, \ldots, n\}$ with values given by, respectively, $\mathbf{h} = (h_1, \ldots, h_n)$ and $\mathbf{c} = (c_1, \ldots, c_n)$. Then it is seen that condition C5 of the theorem can be written as

$$\frac{\sum_{i=1}^n h_i s_i}{\sum_{i=1}^n c_i s_i} \leq \frac{\sum_{i=1}^n h_i t_i}{\sum_{i=1}^n c_i t_i} \tag{13}$$

which is to say that the performance-per-cost unit is higher for the system with signature vector $\mathbf{t}$ compared to the system with signature $\mathbf{s}$.

The conditions C1 and C3 now become, respectively, $c_i \geq 0$ and c_i is increasing in i, which are the usual assumptions for the cost function. Further, C2 can be written

$$\frac{h_1}{c_1} \leq \frac{h_2}{c_2} \leq \cdots \leq \frac{h_n}{c_n} \tag{14}$$

which expresses the intuition that performance needs to increase more than the cost when going from an i-out-of-n system to an $(i+1)$-out-of-n system. Now C4 can be viewed as a further strengthening of this.

First, C4 states that $h_j c_i - h_i c_j$ is decreasing in i for $i \leq j$. This means that $h_j c_i - h_i c_j \geq h_j c_{i+1} - h_{i+1} c_j$ or, equivalently,

$$\frac{h_{i+1} - h_i}{c_{i+1} - c_i} \geq \frac{h_j}{c_j} \text{ for } j = 2, \ldots, n;\ i = 1, 2, \ldots, j-1$$

Thus it follows from (14) that if we assume both C3 and C4, then in addition to (14) we have the condition

$$\frac{h_{i+1} - h_i}{c_{i+1} - c_i} \geq \frac{h_n}{c_n} \text{ for } i = 1, 2, \ldots, n-1$$

Corollary 2. *Let* $\mathbf{s}$ *and* $\mathbf{t}$ *be the signatures of two* n*-systems. Let* $\mathbf{c}$ *and* $\mathbf{h}$ *be, respectively, the cost vector and the performace vector. Consider the follwing conditions:*

C1': $c_i \geq 0$ *for all* i

C2': h_i/c_i *is increasing in* i

C3': c_i *is increasing in* i

C4': $\frac{h_{i+1}-h_i}{c_{i+1}-c_i} \geq \frac{h_n}{c_n}$ *for* $i = 1, 2, \ldots, n-1$

Then the following implications hold:

(i) *If* $\mathbf{s} \leq_{lr} \mathbf{t}$*, then C1',C2' together imply (13).*

(ii) *If* $\mathbf{s} \leq_{hr} \mathbf{t}$*, then C1',C2',C3' together imply (13).*

(iii) *If* $\mathbf{s} \leq_{st} \mathbf{t}$*, then C1',C2',C3',C4' together imply (13).*

Remark: Consider again (3), where we may have $r \neq 1$. We shall see that under the assumption that (13) holds, while c_i is increasing in i (C3') and $r \leq 1$, it will follow that

$$\frac{\sum_{i=1}^n h_i s_i}{(\sum_{i=1}^n c_i s_i)^r} \leq \frac{\sum_{i=1}^n h_i t_i}{(\sum_{i=1}^n c_i t_i)^r}. \tag{15}$$

This is because the inequality of (15) can be written

$$\frac{\sum_{i=1}^n h_i s_i}{\sum_{i=1}^n c_i s_i} \leq \frac{\sum_{i=1}^n h_i t_i}{\sum_{i=1}^n c_i t_i} \cdot \left(\frac{\sum_{i=1}^n c_i s_i}{\sum_{i=1}^n c_i t_i}\right)^{r-1}$$

which holds if (13) holds and the last factor is at least 1. But this latter fact is so provided that $r \leq 1$ and c_i is increasing in i, provided $\mathbf{s} \leq_{st} \mathbf{t}$.

The conclusion is that, for $r \leq 1$, the results (ii) and (iii) of Corollary 2 hold if the inequality of (13) is replaced by the inequality of (15).

As another corollary, we get the results of Theorem 7.1 and Corollary 7.1 of Samaniego [6], which concern optimizing (3) for $r = 1$.

Corollary 3. (Samaniego [6].) *Consider the class of mixed systems of size* n*. Assume that* $r = 1$ *in the criterion function (3), with* $\mathbf{h}$ *and* $\mathbf{c}$ *fixed. Let*

$$K^* = \left\{k \mid k = argmax_i \left\{\frac{h_i}{c_i}, i = 1, \ldots, n\right\}\right\}$$

Then (3) is maximized by any mixture of i*-out-of-*n *systems with* $i \in K^*$*.*

Proof: Let $\mathbf{s}$ be a given signature vector. Let $\{i_1, \ldots, i_n\}$ be a reordering of the components $\{1, 2, \ldots, n\}$ such that

$$\frac{h_{i_n}}{c_{i_n}} \geq \frac{h_{i_{n-1}}}{c_{i_{n-1}}} \geq \cdots \geq \frac{h_{i_1}}{c_{i_1}}$$

Let $\mathbf{s}' = (s_{i_1}, \ldots, s_{i_n})$. Then $\mathbf{s}' \leq_{lr} (0, \ldots, 0, 1)$ and it follows from (i) of Corollary 2 that

$$\frac{\sum_{j=1}^{n} h_{i_j} s_{i_j}}{\sum_{j=1}^{n} c_{i_j} s_{i_j}} \leq \frac{h_{i_n}}{c_{i_n}}$$

Here the left hand side equals $m_1(\mathbf{s}, \mathbf{h}, \mathbf{c})$, while the right hand side equals

$$\frac{\sum_{i=1}^{n} h_i t_i}{\sum_{i=1}^{n} c_i t_i}$$

for any signature (i.e. probability) vector $\mathbf{t} = (t_1, \ldots, t_n)$ for which $t_k > 0$ only if $k \in K^*$. Since $\mathbf{s}$ was arbitrarily chosen, this proves the corollary.

Remark: In view of Corollary 2 one might think that the result of Corollary 3 would hold also for $r < 1$. The following example shows, however, that this is not the case. Suppose $n = 3$ and

$$m_r(\mathbf{s}, \mathbf{h}, \mathbf{c}) = \frac{2s_1 + 3s_2 + 5.7s_3}{(s_1 + 2s_2 + 3s_3)^r}$$

Then for $r = 1$, this is uniquely maximized by $\mathbf{s} = (1, 0, 0)$ by Corollary 3. Now for $\mathbf{t} = (0, 0, 1)$ the m_r equals $5.7/3^r$, while for $\mathbf{s}$ it equals 2 for any r. Hence for $r \leq 0.9533$, $\mathbf{t}$ gives a higher value of the criterion function than $\mathbf{s}$. This suggests that for $r < 1$ one should no longer consider h_i/c_i, but rather h_i/c_i^r. In fact, by using Theorem 7.2 of Samaniego [6] we can prove the following result:

Theorem 4. *Assume that $r < 1$ in the criterion function (3), with $\mathbf{h}$ and $\mathbf{c}$ fixed with $0 < h_1 < h_2 < \ldots < h_n$ and $0 < c_1 < c_2 < \ldots < c_n$. Let*

$$K^* = \left\{ k \mid k = argmax_i \left\{ \frac{h_i}{c_i^r}, i = 1, \ldots, n \right\} \right\}$$

Then (3) is maximized by a system with signature vector $\mathbf{s}$ if and only if $\mathbf{s}$ puts mass 1 on one of the componentes $i \in K^*$, i.e. can be represented as an i-out-of-n system with $i \in K^*$.

Proof: By [6, Theorem 7.2], the signature vector maximizing (3) has at most two non-zero elements. Without loss of generality we can assume that these are the two first elements, s_1 and s_2. Thus by letting $s = s_1$ we may consider the function

$$m(s) = \frac{h_1 s + h_2(1-s)}{(c_1 s + c_2(1-s))^r} \text{ for } 0 \leq s \leq 1$$

Differentiation with respect to s gives that the sign of $m'(s)$ is the same as the sign of

$$g(s) = (h_1 - h_2)(c_1 s + c_2(1-s)) - (h_1 s + h_2(1-s))(c_1 - c_2)r \tag{16}$$

This is a linear function in s. We first rule out the possibility that $g(s) = 0$ for all $0 \leq s \leq 1$. In that case we would have $g(0) = g(1) = 0$, which gives the following two equations,

$$\begin{aligned} (h_1 - h_2)c_2 &= h_2(c_1 - c_2)r \\ (h_1 - h_2)c_1 &= h_1(c_1 - c_2)r \end{aligned}$$

By dividing the left hand sides and the right hand sides, we get $h_2/c_2 = h_1/c_1$. Substitution of this into the first equation gives, however, $r = 1$, which is a contradiction. Hence at least one of $g(0)$ and $g(1)$ is nonzero. From this we conclude that $g(s) = 0$ for an s strictly between 0 and 1 if and only if $g(0)$ and $g(1)$ are both nonzero and have different signs. Now if $g(0) < 0$ and $g(1) > 0$, $m(s)$ has necessarily a minimum in $(0, 1)$, in which case the maximum of $m(s)$ must occur for either $s = 0$ or $s = 1$. The only remaining case is hence that $g(0) > 0$ and $g(1) < 0$. If this is the case, $m(s)$ has a maximum value in the open interval $(0, 1)$. The key of the proof is, however, to show that this case is impossible when $r < 1$.

From (16) above we get that $g(0) > 0$ if and only if

$$\frac{h_2 - h_1}{c_2 - c_1} < \frac{h_2}{c_2} r \tag{17}$$

while $g(1) < 0$ if and only if

$$\frac{h_2 - h_1}{c_2 - c_1} > \frac{h_1}{c_1} r \tag{18}$$

Now since $r < 1$, (17) implies that

$$\frac{h_2 - h_1}{c_2 - c_1} < \frac{h_2}{c_2}$$

Reorganizing this inequality it is seen that it is equivalent to

$$\frac{h_1}{c_1} > \frac{h_2}{c_2}$$

Combining this with (18) we get

$$\frac{h_2 - h_1}{c_2 - c_1} > \frac{h_1}{c_1} r > \frac{h_2}{c_2} r$$

which contradicts (17).

Hence the function $m(s)$ cannot have a maximum for s strictly between 0 and 1, and hence only $s = 0$ or $s = 1$ are possible for its maximum. Returning to the original setup with $\mathbf{s}$ being n-dimensional and we want to maximize the function (3), it follows from the above that the maximum occurs when setting $s_i = 1$ for some i. This clearly implies the stated result.

Remark: Example 7.1, [6, page 102], shows that the result of Theorem 4 does not necessarily hold for $r > 1$. In this case there is hence not always a single i-out-of-n system that is optimal. The reason for the difference between the cases $r < 1$ and $r > 1$ is related to the fact that the denominator for $r < 1$ is a concave function, while for $r > 1$ it is convex. This is also the reason for the difference between Corollary 3 and Theorem 4, where mixtures of systems defined by K^* are not possible in the latter.

In the example preceding Theorem 4, it follows that for $r \leq 0.9533$, the optimal system is the parallel system (with 3 components). Note that the intuition of the case $r < 1$ is that cost is considered less important than performance.

5. Concluding remarks

In Section 2 we arrived at a characterization of the set of signature vectors of n-systems which stochastically dominate the signature of a given $(n + 1)$-system. The characterization was then used in Section 3 in a comparison of performance-per-cost of corresponding n- and $(n + 1)$-systems. We could as well have considered the more general case of characterizing the set of signature vectors of n-systems which stochastically dominate the signature of a given $(n + m)$ system for a general $m \geq 1$. In fact, by repeated application of Theorem 1 it would follow that the components of the signature vector of an $(n + m)$-system are linear functions of the components of the signature vector of an equivalent n-system. Explicit expressions for these linear functions are, furthermore, given in Navarro et al. [3] and Lindqvist et al. [2]. Thus the construction of a convex set of signature vectors of n-systems which stochastically dominate the $(n+m)$-system would be similar to the one performed for $m=1$.

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