Geometric vs. weakly geometric rough paths in infinite dimensions

Rough paths and SPDEs

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Rough paths on Banach spaces

Rough paths

A rough path on an interval $[0, T]$ with values in a Banach space E consists of continuous maps

$$X\colon [0,T] \to E, \quad \mathbb{X}\colon [0,T]^2 \to E \otimes E.$$

such that Chen's relation holds for all $s, u, t \in [0, T]$

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t}$$

and X is α-Hölder and $\mathbb{X}$ is 2α-Hölder. Denote by $\mathcal{C}^\alpha([0, T], E)$ the space of all α-Hölder rough paths.

Warning (Tensor norms)

All of this requires us to choose a tensor norm to make sense of the analytic conditions!

Geometric picture for rough paths

Choose a suitable tensor norm and construct $\mathcal{A} := \prod_{n\in\mathbb{N}} E^{\otimes_n}$.
Then $\mathcal{A}$ is a Fréchet algebra which contains interesting subalgebras.

Define the Lie polynomials as $\mathcal{P}^1 := E$ and recursively $\mathcal{P}^n \subseteq \mathcal{A}$ as the closure of the space

$$\mathcal{P}^{n-1} + \{[x,y] := x \otimes y - y \otimes x \mid x \in E, y \in \mathcal{P}^{n-1}\}.$$

$\mathcal{P}^\infty = \overline{\{\sum_{n\in\mathbb{N}} p_n \mid p_n \in E^{\otimes_n} \cap \bigcup_{m\in\mathbb{N}} \mathcal{P}^m\}}$ (Lie series).

1. $\mathcal{P}^n$ is a nilpotent Lie algebra (after truncation) and $\mathcal{P}^\infty$ is a Lie algebra.
2. The exponential series converges on $\mathcal{P}^n$ and yields a Lie group $G^n := \exp(\mathcal{P}^n), n \in \mathbb{N} \cup \{\infty\}$

We describe rough paths with $\alpha \in]1/3, 1/2[$ and will thus specialise to $n = 2$.

Geometric picture for rough paths II

Writing the rough path $(X, \mathbb{X})$ for $\alpha > 1/3$ additively we have

$$\mathbf{X}_{st} := 1 + X_t - X_s + \mathbb{X}_{s,t} \in G^2.$$

Chen's relation then becomes $\mathbf{X}_{st} = \mathbf{X}_{su}\mathbf{X}_{ut}$. To make sense of the Hölder condition we introduce the metric

$$|\mathbf{X}| = \max\{\|X\|, \|\mathbb{X}\|_2^{1/2}\} \quad d(\mathbf{x}.\mathbf{y}) := |\mathbf{x}^{-1}\mathbf{y}|$$
$$d_\alpha(\mathbf{X}, \mathbf{Y}) := \sup_{s \neq t} \frac{d(\mathbf{X}_{st}, \mathbf{Y}_{st})}{|t-s|^\alpha}.$$

Then $\mathcal{C}^\alpha([0,T], E) \cong C^\alpha([0,T], G^2)$

Weakly geometric vs geometric rough paths

For $\alpha \in (1/3, 1/2)$ let $\mathbf{X}_{st} = 1 + X_{st} + +mahtbbX_{st}$ be an α-rough path, it is called

- **geometric** rough path if it is in the closure of

$$S^2(X)_{st} = 1 + X_t - X_s + \int_s^t (X_r - X_s) \otimes dX_r$$

  of curves X_t of bounded variation
- **weakly geometric** if the symmetric part of $\mathbb{X}^{(2)}_{st}$ equals $\frac{1}{2} X_{st} \otimes X_{st}$

well known fact

Every geometric rough path is weakly geometric and if E is finite-dimensional every weakly geometric rough path is geometric up to Hölder tuning.

Geometric vs. weakly geometric in infinite dimensions

Theorem, Grong, Nilssen S. 2020

For $\alpha \in (1/3, 1/2)$, E a Hilbert space, let $\mathscr{C}_g^\alpha([0,T],E)$ and $\mathscr{C}_{wg}^\alpha([0,T],E)$ denote geometric rough paths and weakly geometric rough paths. Then for any $\beta \in (1/3, \alpha)$, we have inclusions

$$\mathscr{C}_g^\alpha([0,T],E) \subset \mathscr{C}_{wp}^\alpha([0,T],E) \subset \mathscr{C}_g^\beta([0,T],E).$$

Is actually the consequence of a more general theorem, for which we need to recall the Carnot-Caratheodory (CC) metric on G^2:

$$\rho(\mathbf{X},\mathbf{Y}) := \rho(1, \mathbf{X}^{-1}\mathbf{Y}),$$

$$\rho(1,\mathbf{Z}) := \inf\left\{ \int \|\dot{c}_t\| \mathrm{d}t \mid c \text{ bounded variation, cts. } c_0 = 0, S^2(c) = \mathbf{Y} \right\}$$

For $M_{cc} := \{z \in G^2 \mid \rho(1, z) < \infty\}$ we let $C([0, T], M_{cc})$ be the continuous curves with respect to ρ.

Theorem (Grong, Nilssen, S. 2020)

Let $\alpha \in]1/3, 1/2[$ and $\beta \in]1/3, \alpha[$. Assume that

1. For some $C > 0$ and all $z \in G^2$ we have $d(1, z) \leq C\rho(1, z)$,
2. The metric space (M_{cc}, ρ) is a complete, geodesic space.
3. The set $C^\alpha([0, T], G^2) \cap C([0, T], M_{cc})$ is dense in $C^\alpha([0, T], G^2)$ relative to the metric d_β.

Then for any $\mathbf{X} \in C^\alpha([0, T], G^2)$ there exists a sequence of bounded variation paths X_n such that

$$\mathbf{X}n = S^2(x_n) \to \mathbf{X} \text{ in } C^\beta([0, T], E).$$

Some remarks

Problems in generalising the result to Banach spaces

The proof exploits that in a Hilbert space projections onto closed subspaces are length shortening.Such projections are in general rare in Banach spaces.

We also use several specific identifications of tensor products of Hilbert spaces.

Applications... or why should you care?

1. Wong-Zakai type results for rough paths on Banach spaces
2. applicable to unbounded drivers (from rough PDE theory)

Thank you for your attention!

More information:
Grong, Nilssen, S.: Geometric rough paths on infinite dimensional spaces, arXiv:2006.06362.