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« Renormalisation from non-geometric to geometric rough Paths »

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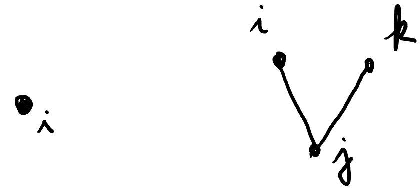
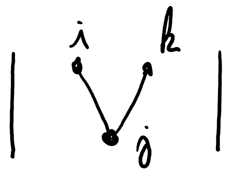
Branched Rough Paths.

$\mathcal{T}$ = rooted trees, nodes decorated by $\{0,\dots,d\}$



$|\cdot| : \mathcal{T} \to \mathbb{N}$ number of nodes



$= 3$

$\mathcal{F}$ = Forest over $\mathcal{T}$.

$\mathcal{H} = \langle \mathcal{F} \rangle$ $\quad$ $\mathcal{H}_N$ = Forest of size N.

$(\mathcal{H}, \Delta, \cdot)$ Hopf algebra $\quad$ $\mathcal{G}$ = group of characters

Connes-Kreimer coproduct $\quad$ forest product.

Branched-γ rough Path:

$$X : [0,1]^2 \to \mathcal{G}$$

Chen's relation $\quad X_{su} * X_{ut} = X_{st} \quad s,t,u \in [0,1]$

convolution product associated to Δ.

$|\langle X_{st}, \tau \rangle| \lesssim |t-s|^{\gamma|\tau|}$ $\quad$ Truncation: N $\quad$ $\gamma N \leq 1$

$\mathcal{G}_N$ $\quad$ BRP^{γ}

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Geometric Rough Paths:

A = alphabet. $T(A)$ words

$(T(A), \Delta, ⧢)$

Δ ↑ deconcatenation, ⧢ ↑ shuffle product.

grading = length of the word.

Group of characters $\mathcal{G}_A$

<u>Anisotropic Rough Paths:</u> $\gamma = (\gamma_a, a \in A)$

$X : [0,1]^2 \to \mathcal{G}_A$.

$X_{su} * X_{ut} = X_{st}$

$*$ ↑ convolution for deconcatenation.

$|\langle X_{st}, v\rangle| \lesssim |t-s|^{\sum_i \gamma_{a_i}}$ $v = a_1 \dots a_n$.

<u>Ex:</u> $A = \mathcal{T}_N$ $T_N(\mathcal{T}_N)$

$v = \tau_1 \dots \tau_n$ $\sum_i |\tau_i| \leq N$.

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Move from non-geometric to geometric.

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Hairer-Kelly map:

$$\Psi_{HK} : \mathcal{H} \to T(\mathcal{T}) \text{ algebra morphism.}$$

+ Lyons-Victoir: $(x^a)_{a \in A}$ $x^a \in \mathcal{C}^{\gamma_a}([0,1])$

$\exists \bar{X}$ over $(x^a)_{a \in A}$ $\quad x^a_t - x^a_s = \langle \bar{X}_{st}, a \rangle$

$\bar{X} \in ARP^\gamma$

Thm (Hairer-Kelly) (Tapia-Zambotti):

$\forall X \in BRP^\gamma$, $\exists \bar{X} \in ARP^\gamma$ $\quad \langle X, \tau \rangle = \langle \bar{X}, \Psi_{HK}(\tau) \rangle$.

Construction 2:

Chapoton-Foissy isomorphism:

$$\Psi_{CF} : \mathcal{H}_N^* \to T_N(\mathcal{B}_N).$$

$\mathcal{B}_N$ = basis of trees.

$\tau \in \mathcal{H}_N^*$ $\quad \tau = \sum_{R=(r_1,\dots,r_m)} \lambda_R \, \tau_{r_1} * \cdots * \tau_{r_m} \mapsto \sum_R \lambda_R \, \tau_{r_1} \cdots \tau_{r_m}$

convolution product.

$\tau_{r_i} \in \mathcal{B}_N$.

Thm: $X \in BRP^\gamma$ $\quad \tilde{X} = \Psi_{CF}(X) \in ARP^\gamma$.

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Understand how renormalisation maps M interact with these constructions.

- $M : \mathcal{H} \to \mathcal{H}$ multiplicative for $\cdot$
- $\langle \underset{\sim}{X^M_{St}}, \tau \rangle = \langle X_{St}, M\tau \rangle = \langle M^* X_{St}, \tau \rangle$.
- Chen's: $(M \otimes M)\Delta = \Delta M$.
- Analysis: $M\tau = \sum_i \lambda_i \tau_i \qquad \gamma_{\tau_i} \geq \gamma_\tau \qquad |\tau_i| \leq |\tau|$

<u>Two examples:</u>

- BPHZ renormalisation.
- local products renormalisation.

<u>Results:</u>

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\varphi_{HK}} & T(\mathcal{T}) \\ {\scriptstyle M}\big\downarrow & & \big\downarrow{\scriptstyle \bar{M}} \\ \mathcal{H} & \xrightarrow{\varphi_{HK}} & T(\mathcal{T}) \end{array}$$

$\bar{M}$ translation map.

$$\begin{array}{ccc} \mathcal{H}_N^* & \xrightarrow{\varphi_{cF}} & T_N(\mathcal{B}_N) \\ {\scriptstyle M^*}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{M}^*} \\ \mathcal{H}_N^* & \xrightarrow{\varphi_{cF}} & T_N(\mathcal{B}_N) \end{array}$$

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$$\langle X^M, \tau \rangle = \langle X, M\tau \rangle$$

Haines-Kelly.

$$= \langle \bar{X}, \Psi_{HK}(M\tau) \rangle$$

$$= \langle \bar{X}, \bar{M}\, \Psi_{HK}(\tau) \rangle$$

$$= \langle \bar{M}^* \bar{X}, \Psi_{HK}(\tau) \rangle$$

$$= \langle \overline{M^* X}, \Psi_{HK}(\tau) \rangle$$

$$\langle X^M, \tau \rangle = \langle M^* X, \tau \rangle$$

Do we have $\overline{M^* X} = \bar{M}^* \bar{X}$?

$$\Psi_{CF}(M^* X) = \tilde{M}^* \, \Psi_{CF}(X)$$ nice interaction.

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+ Topics - Zambotti parametrisation:

$$\mathcal{E}^{\gamma} = \{ (g^{\tau})_{\tau} : g_0^{\tau} = 0, \ g^{\tau} \in C^{\gamma_{\tau}}([0,1]). \}.$$

Thm: $\exists$ an action $\mathcal{E}^{\gamma}$ on BRP^{γ}.

$$(g, X) \longmapsto gX$$

$\forall X, X' \in BRP^{\gamma} \quad \exists! g \quad gX = X'.$

$$g(g'X) = (g+g')X.$$

$$\langle gX, \tau \rangle := \langle g\bar{X}, \Psi_{Hk}(\tau) \rangle$$

$$\langle g\bar{X}_{st}, \tau \rangle = g_t^{\tau} - g_s^{\tau} + \underbrace{\langle \bar{X}_{st}, \tau \rangle}_{= x_t^{\tau} - x_s^{\tau}} .$$

$\exists! g \quad \langle X, M\tau \rangle = \langle gX, \tau \rangle.$

Prop: $M^* X$ and gX coincide then

$\overline{M^* X}$ and $g\bar{X}$ coincide.

Thm: $g_t^{\tau} - g_s^{\tau} = \langle \underset{\sim}{\overline{M^*X}}_{st}, \tau \rangle - \langle \bar{X}_{st}, \tau \rangle$

Remark: $\overline{M^* X_{st}} = \bar{M}^* \bar{X}_{st}$?

$$g_t^{\tau} - g_s^{\tau} = x_t^{M\tau - \tau} - x_s^{M\tau - \tau}.$$

<u>Extension:</u> Action on the parametrisation of models (Singular SPDEs)