Regularisation by noise and nonlinear Young integrals

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Regularisation by noise in a nutshell

Regularisation by noise refers to a class of phenomena in which ill-posed equations become well-posed under the introduction of noise; frequently observed in ODEs and PDEs.

Why is it important?
The model is a mathematical idealisation, obtained by ignoring less relevant factors in the dynamics; real systems are always affected by a background noise
$\rightsquigarrow$ physically observed solutions should be the ones stable under small noisy perturbations
$\rightsquigarrow$ **vanishing noise selection** criteria for mathematical solutions.

The “regularisation by noise program” ideated by Flandoli amounts to:

1. Find a physically meaningful, arbitrarily small noise restoring well-posedness in the equation.
2. Identify the zero noise limit (hopefully unique in some reasonable sense) **NB: this step is still mostly open!**

Probabilistic literature on Step 1.

There is now an extensive literature on regularisation for SDEs of the form

$$x_t = x_0 + \int_0^t b(s, x_s)\mathrm{d}s + \varepsilon W_t$$

with W suitable stochastic process (BM, Lévy, fBm), $\varepsilon > 0$. First results go back to Zvonkin (1974) and Veretennikov (1981).

A (very incomplete!) list of references:

- Krylov–Röckner (PTRL 2005): singular $b \in L^q_t L^p_x$;
- Flandoli–Gubinelli–Priola (Invent. 2010): flow and link with transport equation; later improved in works by Flandoli and collaborators;
- Works by Priola and collaborators for W Lévy noise;
- Works by Da Prato and collaborators for infinite dimensional ODEs with cylindrical BM;
- Nualart–Ouknine (SPA 2002) first considered W fBm; improved by Lê (EJP 2020).
- Proske and collaborators: very rough noises and C^∞ regularisation.

Path-by-path uniqueness

In all of the above: W is a stoch. proc. on filtered $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, uniqueness among solutions adapted to $\mathcal{F}_t$. But in principle the equation **pathwise meaningful**, i.e. we can fix the realization $W(\omega)$ and regard

$$x_t = x_0 + \int_0^t b(s, x_s)\mathrm{d}s + W_t(\omega) \tag{1}$$

as an ODE with random coefficients. For $x_0 \in \mathbb{R}^d$, denote by $C(x_0, \omega)$ the random set of solutions to (1).

Theorem (Davie, 2007)

Let W sampled as BM, b measurable and bounded. Then for any $x_0 \in \mathbb{R}^d$

$$\mathbb{P}(\omega \in \Omega : \# C(x_0, \omega) = 1) = 1$$

namely **path-by-path uniqueness** *holds.*

After Davie, several path-by-path uniqueness results have been established by Catellier–Gubinelli (2016), Shaposhnikov (2016) and more recently Harang–Perkowski (2020). Still, we are (partially) in a probabilistic setting.

An abstract analytic formulation of the problem

For given $x_0 \in \mathbb{R}^d$ and $w \in C([0,T];\mathbb{R}^d)$, we aim at studying the equation

$$x_t = x_0 + \int_0^t b(s, x_s)\mathrm{d}s + w_t \quad \forall\, t \in [0,T] \tag{2}$$

with w a given continuous path (possibly very rough) and b possibly distributional (for which the Lebesgue integral in (2) is not clearly defined). Suppose we can find the following "ingredients":

a) A Banach space $E \subset C([0,T];\mathbb{R}^d)$ containing $\{x_0 + w : x_0 \in \mathbb{R}^d\}$.

b) A good definition for $T : E \to C([0,T];\mathbb{R}^d)$ formally given by

$$(Tx)_t = \int_0^t b(s, x_s)\mathrm{d}s \quad \forall\, t \in [0,T]$$

in such a way that $I - T$ leaves E invariant (I identity map).

c) All of the above consistent with the classical setting: for continuous b, T defined as usual and E includes all classical solutions to (2).

Then under the change of variable $\gamma = x_0 + w \in E$, eq. (2) becomes

$$x_t - (Tx)_t = \gamma_t \Leftrightarrow (I - T)x = \gamma \Leftrightarrow x \in (I - T)^{-1}(\gamma).$$

so that wellposedness amounts to $\#(I - T)^{-1}(\gamma) = 1$.

Mathematical tools available

We have reduced ourselves to the study of the set $(I - T)^{-1}(\gamma)$ for a map $T : E \to E$ defined map on a Banach space.

- **Leray–Schauder–Tychonoff**, **Schaefer**'s theorems: if T is a continuous compact map satisfying suitable conditions, then $\#(I - T)^{-1}(\gamma) \geq 1$ for all $\gamma \in E$.
- **Banach–Caccioppoli**: if T is Lipschitz with constant $c < 1$, then $I - T$ is a bijection with Lipschitz inverse satisfying

$$\|(I - T)^{-1}(\gamma_1) - (I - T)^{-1}(\gamma_2)\| \leq \frac{1}{1 - c}\|\gamma_1 - \gamma_2\|.$$

- **Inverse function theorem**: if $x \in (I - T)^{-1}(\gamma)$, T is differentiable in U neighb. U of x and $D_x(I - T)$ is an isomorphism of E, then there exists V neighb. of γ s.t. $I - T$ has a differentiable inverse in V.
- **Sard's theorem**: if E is finite dimensional and T is C^1, then $(I - T)$ is locally invertible around Lebesgue-a.e. $\gamma \in E$.

Old and new results in this formalism

Consider now the case $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $Tx = \int_0^\cdot b(s, x_s)\mathrm{d}s$.

- **Cauchy–Lipschitz**: if b Lipschitz, then $(I - T)$ is invertible on $E = C([0,T];\mathbb{R}^d)$ with Lipschitz inverse.
  Sharp: cannot be weakened to any Hölder condition.
- **Peano**: if b continuous, T is continuous compact on $E = C([0,T];\mathbb{R}^d)$ and $\#(I - T)^{-1}(\gamma) \geq 1$ for every $\gamma \in E$.
  Sharp: cannot be weakened to b measurable bounded.
- If $b \in W^{1,p}$ with $p > d$ (so $W^{1,p} \hookrightarrow C^0$), then $\#(I - T)^{-1}(x_0) = 1$ for a.e. $x_0 \in \mathbb{R}^d$ [Caravenna,Crippa 2018]. Sharp: cannot be weakened to $b \in W^{1,p}$ with $p < d$ [Brué, Colombo, de Lellis 2020].
- **Catellier–Gubinelli**: even for distributional b, if w is a "sufficiently irregular" path, one can define T on a suitable space $E = E^w$ and show that $(I - T)$ has a Lipschitz inverse on E.
- **Natural question** in view of Sard's theorem: what happens for generic γ? Is $(I - T)$ invertible around a.e. $\gamma \in C^0$?

Catellier and Gubinelli's idea revisited

For "sufficiently irregular" w, a good choice of "ingredients" is:

a) $E^w = w + C_t^\alpha \simeq C_t^\alpha$ for some $\alpha > 1/2$. E^w is made of paths which "look like" w at first order and corresponds to CG's solution Ansatz.

b) For such E^w, T is defined by means of **nonlinear Young integrals.**

The "irregularity" of w amounts to requiring that the **averaged field**

$$T^w b(t,z) := \int_0^t b(r, z + w_r)\mathrm{d}r, \quad T^w_{s,t} b(z) := T^w b(t,z) - T^w b(s,z)$$

is well defined with good space-time regularity. Intuitive picture: wild oscillations of w allows to trade between space and time regularity for $T^w b$, even for distributional b. For b continuous, $x = \theta + w$, then

$$\int_0^t b(r, \theta_r + w_r)\mathrm{d}r \approx \sum_i \int_{t_i}^{t_{i+1}} b(r, \theta_{t_i} + w_r)\mathrm{d}r = \sum_i T^w_{t_i, t_{i+1}} b(\theta_{t_i}).$$

Dependending on $T^w b$ and θ, the Riemann–Stjeltes sums on the r.h.s. might converge even if b distributional $\rightsquigarrow$ nonlinear Young integral.

Nonlinear Young integration

Notation: $A_{s,t}(x) = A(t,x) - A(s,x)$. We say that $A : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is $C^\alpha_t C^\beta_x$ if

$$|A_{s,t}(x) - A_{s,t}(y)| \lesssim |t-s|^\alpha |x-y|^\beta, \quad |A_{s,t}(x)| \lesssim |t-s|^\alpha, \quad |A(t,x) - A(t,y)| \lesssim |x-y|^\beta;$$

optimal constants $[\![A]\!]_{\alpha,\beta}.$, $\|A\|_{\alpha,\beta}$; same $A \in C^\alpha_t C^{n+\beta}_x$.

Theorem (Catellier, Gubinelli, 2012)

Let $A \in C^\alpha_t C^\beta_x$, $\theta \in C^\gamma([0,T];\mathbb{R}^d)$ such that $\alpha + \beta\gamma > 1$. Then the following limit of Riemann-Stjeltes sums is well defined:

$$\int_0^T A(\mathrm{d}s, \theta_s) := \lim_{|\Pi| \to 0} \sum_{i=1} A_{t_i, t_{i+1}}(\theta_{t_i})$$

Moreover $\int_0^\cdot A(\mathrm{d}s, \theta_s) \in C^\alpha_t$ and the map $(A, \theta) \mapsto \int_0^\cdot A(\mathrm{d}s, \theta_s)$ is continuous in respective topologies.

Idea of proof: Apply the sewing lemma to $\Gamma_{s,t} := A_{s,t}(\theta_s)$. Indeed:

$$\begin{aligned} |\delta\Gamma_{s,u,t}| &= |A_{s,t}(\theta_s) - A_{s,u}(\theta_s) - A_{u,t}(\theta_u)| = |A_{u,t}(\theta_s) - A_{u,t}(\theta_u)| \\ &\leq [\![A]\!]_{\alpha,\beta} |t-u|^\alpha |\theta_{s,u}|^\beta \leq [\![A]\!]_{\alpha,\beta} [\![\theta]\!]_\gamma |t-s|^{\alpha+\beta\gamma}. \quad \square \end{aligned}$$

Perturbed ODE as nonlinear YDE

Therefore if $T^w b \in C^\alpha_t C^1_x$ for some $\alpha > 1/2$ and $x = \theta + w$ with $\theta \in C^\alpha_t$,

$$(Tx)_\cdot = \int_0^\cdot T^w b(\mathrm{d}s, \theta_s) =: \int_0^t b(s, x_s)\mathrm{d}s$$

is a well defined element of C^α_t and $I - T$ maps E^w into itself. If $x, \gamma \in E^w$ are given by $x = \theta + w, \gamma = \tilde{\gamma} + w$, then $x \in (I - T)^{-1}(\gamma)$ is equivalent to

$$\theta_t = \int_0^t T^w b(\mathrm{d}s, \theta_s) + \tilde{\gamma}_t \tag{YDE}$$

which is a **nonlinear Young differential equation** for the choice $A = T^w b$; $\gamma = x_0 + w$ corresponds to $\tilde{\gamma}_t \equiv x_0 = \theta_0$ initial data.

- Nonlinear YDEs are a generalization of classical YDEs and can be applied in other classes of problems (see later).
- The solution theory we will develop allows to invert $I - T$ on E^w, thus not on a neighb. of w in C^0_t but rather along "special directions" $\rightsquigarrow$ **analogy with Malliavin calculus** and Cameron–Martin space.

Solution theory for nonlinear YDEs

Theorem

Given $A \in C^\alpha_t C^1_x$ and $\gamma \in C^\alpha_t$, $\alpha > 1/2$, let $C(A,\gamma)$ denote the set of solutions to the NYDE associated to (A,γ).

1) The set $C(A,\gamma)$ is non-empty, compact, simply connected in $C^{1/2}$; the map $(A,\gamma) \mapsto C(A,\gamma)$ is measurable in suitable topologies
   $\rightsquigarrow$ if A is random, then $C(A,\gamma)$ is a random compact set
   $\rightsquigarrow$ can construct measurable selections for $(A,\gamma) \mapsto \theta \in C(A,\gamma)$.
2) If $A \in C^\alpha_t C^2_x$, then $\# C(A,\gamma) = 1$ and the solution map $(A,\gamma) \mapsto \theta(A,\gamma)$ is Frechét differentiable.
3) Restricting to $\gamma = x_0 \in \mathbb{R}^d$, to any $A \in C^\alpha_t C^2_x$ we can associated a flow of diffeormorphisms $\Phi^A(s,t,x_0)$; $A \mapsto \Phi^A$ continuous.
4) Higher regularity: if $A \in C^\alpha_t C^{n+1}_x$, then $\Phi^A(s,t,\cdot)$ belongs to C^n_x.

Proofs sparse in the literature: Catellier–Gubinelli, G.–Gubinelli., Hu–Lê, G.–Harang; everything can be found in the review arXiv:2009.12884 .

Solution theory for nonlinear YDEs - II

Global existence of solutions holds under more general growth conditions. We say that $A \in C^\alpha_t C^{\beta,\lambda}_x$ if

$$|A_{s,t}(x) - A_{s,t}(y)| \lesssim |t-s|^\alpha |x-y|^\beta (1 + |x|^\lambda + |y|^\lambda);$$

similar concepts for $C^\alpha_t C^{n+\beta,\lambda}_x$, $C^\alpha_t C^{n+\beta}_{loc}$. Then:

- Point 1) holds for $A \in C^\alpha_t C^{\beta,\lambda}_x$ with $\alpha(1+\beta) > 1$, $\beta + \lambda \leq 1$;
- Point 2) holds for $A \in C^\alpha_t C^{\beta,\lambda}_x \cap A \in C^\alpha_t C^{1+\beta}_{loc}$, α, β, λ as above.

Explicit expression for the differential of $\gamma \mapsto \theta(A,\gamma)$ in point 2): Consider wlog the differential around $\gamma \equiv 0$, set $\bar\theta := \theta(A, 0)$ and let $M \in C^\alpha_t([0,T]; \mathbb{R}^{d\times d})$ be the unique solution to

$$M_t = I_d + \int_0^t DA(\mathrm{d}s, \bar\theta_s) M_s;$$

let N_t denote its matrix inverse. Then

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \theta(A, \varepsilon\gamma)|_{\varepsilon=0} = M_t \gamma_0 + \int_0^t M_t N_s \mathrm{d}\gamma_s$$

$\rightsquigarrow$ link with **Malliavin derivative**.

Back to ODEs and solvability for a.e. w

With these results at hand we can now solve the perturbed ODE

$$x_t = x_0 + \int_0^t b(s, x_s)\mathrm{d}s + w_t \tag{3}$$

for distributional b and generic w. **Analogy with rough paths**: enhance the data of the problem from (b, w) to $(b, w, T^w b)$ with $T^w b \in C_t^\alpha C_x^2$, so

$$(b, w) \mapsto (b, w, T^w b) \mapsto (\Phi^{T^w b}, w) \mapsto x_t = \Phi^{T^w b}(0, t, x_0) + w_t$$

where the first step can be done in a measurable way and all the remaining one are analytically defined continuous mappings.

Theorem (G.,Gubinelli, 2020)

Let $b \in L_t^q C_x^{-n}$ for some $q > 2$ and $n \in \mathbb{N}$; then for almost every $w \in C([0, T]; \mathbb{R}^d)$ it holds $T^w b \in C_t^\alpha C_x^2$, where "almost every" must be understood in the sense of ***prevalence****; as a consequence, existence and uniqueness holds for (3) and $I - T$ is invertible "around" w in the sense of the set of special directions given by E^w.*

Other applications of nonlinear YDE theory

A) **Modulated equations** treated by Chouk, Gubinelli (2014-2015):

$$\partial_t \varphi_t = A\varphi_t \dot{w}_t + \mathcal{N}(\varphi_t)$$

where A is the generator of a group $\{e^{tA}\}_{t\in\mathbb{R}}$ acting isometrically on all H^α and $\mathcal{N}$ is a nonlinearity. Setting $U^w_t = e^{w_t A}$ and $\psi_t = (U^w_t)^{-1}\varphi_t$, formally ψ satisfies the mild formulation

$$\psi_t = \psi_0 + \int_0^t (U^w_s)^{-1}\mathcal{N}(U^w_s \psi_s)\mathrm{d}s =: \psi_0 + \int_0^t A(\mathrm{d}s, \psi_s)$$

which can be recast as a nonlinear YDE driven by

$$A_{s,t}(\chi) := \int_s^t (U^w_r)^{-1}\mathcal{N}(U^w_r \chi)\mathrm{d}r.$$

B) **McKean–Vlasov** nonlinear YDEs treated by Harang, Mayorcas (2020):

$$X_t = \xi + \int_0^t K * \mathcal{L}(X_s)(X_s + Z_s)\mathrm{d}s + B_t$$

for Z deterministic irregular path; setting $A = T^Z K$ it is of the form

$$X_t = \xi + \int_0^t A_{\mathrm{d}s} * \mathcal{L}(X_s)(X_s) + B_t.$$

Can we regularise SDEs driven by multiplicative fBm?

Consider now an SDE driven by B^H fBm of parameter $H > 1/2$ of the form

$$\mathrm{d}x_t = b_1(t, x_t)\mathrm{d}t + b_2(t, x_t)\mathrm{d}B^H_t;$$

take for simplicity of exposition $b_1 \equiv 0$, $b_2(t,x) = b(x)$ autonomous.

Since $H > 1/2$, for regular b the equation is pathwise well-posed in the Young sense; if $b \in C^\alpha_x$ with $\alpha < 1$, explicit counterexamples to uniqueness are known; if $\alpha \le 1/H - 1$, the Young interpretation breaks down.

Question: can noise cure these pathologies? Leads to consider

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}B^H_s + w_t \tag{4}$$

New difficulties:

- If w is very rough, we expect a strong regularising effect, but at the same time (4) is not pathwise meaningful anymore even for smooth b!
- In analogy with above, we expect a central role to be played by

$$\Gamma^w b(t,z) \text{ “}=\text{” } \int_0^t b(z + w_r)\mathrm{d}B^H_r$$

Wiener integral for fBm

Given $h \in C([0,T];\mathbb{R}^d)$ and $\kappa \in (0,1)$, define $\|h\|_{-\kappa} := \| \int_0^{\cdot} h_s \mathrm{d}s\|_{1-\kappa}$.

Theorem (Hairer, Li, AP 2020)

Let B^H fBm with $H > 1/2$, $\kappa \in (0, H-1/2)$, h deterministic, then

$$\Big\| \int_s^t h_r \,\mathrm{d}B^H_r \Big\|_{L^p_\Omega} \lesssim_{p,H,T} \|h\|_{-\kappa} |t-s|^{H-\kappa} \quad \forall\, [s,t] \subset [0,T].$$

By a density argument, the Wiener integral $\int_s^t h_r \mathrm{d}B^H_r$ is then well defined for all $h \in C^{-\kappa}$; moreover by Garsia-Rodemich-Rumsay

$$\mathbb{E} \Big\| \int_0^{\cdot} h_r \,\mathrm{d}B^H_r \Big\|^p_{H-\kappa-\varepsilon} \lesssim_{p,H,T,\varepsilon} \|h\|^p_{-\kappa}$$

where by assumption we can take $H - \kappa - \varepsilon > 1/2$.

Actually already known: [Jolis, JMAA 2007] for $h \in W^{1/2-H,2}$.

Intuitive idea: $\mathrm{d}B^H$ "=" $I^{H-1/2}\mathrm{d}B$ where B standard Bm and I^α fractional integral, so $\langle h, \mathrm{d}B^H\rangle$ "=" $\langle I^{H-1/2}h, \mathrm{d}B\rangle$, it suffices $I^{H-1/2}h \in L^2$. $h \in C^{-\kappa} \Rightarrow I^{H-1/2}h \in C^{H-1/2-\kappa-}$, so it's enough to require $\kappa < H - 1/2$.

Construction of multiplicative averaged field

We can now construct $\Gamma^w b$ by relying on the regularity of $T^w b$!
If $T^w b \in C_t^\alpha C_x^1$ with $\alpha + H > 3/2$, taking $\kappa = 1 - \alpha$, $H - \kappa > 1/2$, then

$$\begin{aligned}
\|\Gamma^w_{s,t} b(x) - \Gamma^w_{s,t} b(y)\|_{L^p_\Omega} &= \Big\| \int_s^t [b(x + w_r) - b(y + w_r)]\, \mathrm{d}B^H_r \Big\|_{L^p_\Omega} \\
&\lesssim_p |t-s|^{H-\kappa} \Big\| \int_0^\cdot b(x + w_r)\mathrm{d}r - \int_0^\cdot b(y + w_r)\mathrm{d}r \Big\|_{\alpha} \\
&\lesssim_p \| T^w b\|_{C_t^\alpha C_x^1} |t-s|^{1/2+} |x-y|
\end{aligned}$$

and now applying a (suitably modified) version of GRR lemma we obtain

Proposition

Suppose $T^w b \in C_t^\alpha C_x^{n+\beta}$ for $\alpha > 3/2 - H$ and $\beta \in (0,1)$. Then for any $\alpha' \in (1/2, \alpha + H - 1)$, $\beta' < \beta$ and $\lambda > 0$ it holds

$$\Gamma^w b \in C_t^{\alpha'} C_x^{n+\beta',\lambda} \quad \mathbb{P}\text{-a.s.}$$

The results from [CG16], [GG20], [HP20] give plenty examples of deterministic w for which $T^w b$ has the desired regularity.

Solution theory for perturbed SDEs

Let's go back to the study of

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}B^H_s + w_t \tag{5}$$

Definition

We say that $x : \Omega \to C([0,T];\mathbb{R}^d)$ is a *pathwise solution* of (5) if there exist α, β, λ with $\alpha > 1/2$, $\alpha(1+\beta) > 1$, $\beta + \lambda \leq 1$ such that

a) $\Gamma^w b(\omega)$ is a well defined element of $C^\alpha_t C^{\beta,\lambda}_x$ for $\mathbb{P}$-a.e. ω;

b) $x(\omega) \in E^w$ for $\mathbb{P}$-a.e. ω;

c) $\theta(\omega) := x(\omega) - w \in C(\Gamma^w b(\omega), x_0)$ for $\mathbb{P}$-a.e. ω.

Definition

We say that *path-by-path wellposedness* holds for (5) if a) holds and

$$\mathbb{P}(\omega \in \Omega : \# C(\Gamma^w b(\omega), x_0) = 1 \text{ for all } x_0 \in \mathbb{R}^d) = 1.$$

Theorem (G.,Harang)

Suppose there exist α, β, λ *with* $\alpha > 1/2$, $\alpha(1+\beta) > 1$, $\beta + \lambda \leq 1$ *such that* $\Gamma^w b(\omega) \in C^\alpha_t C^{1+\beta,\lambda}_x$. *Then path-by-path wellposedness holds.*

Solution theory for perturbed SDEs - II

Theorem

Let $b \in C^s_x$, $s \in \mathbb{R}$, $w \in C^0_t$ be such that $T^w b \in C^{1/2}_t C^{s+\nu}_x$ for

$$s + \nu(2H - 1) > 2;$$

then path-by-path wellposedness holds and eq. (5) admits a random flow of diffeomorphisms. If $s + \nu(2H - 1) > n + 1$, the flow is spatially C^n.

Theorem (G., Harang)

Let $b \in C^s_x$ be compactly supported, w sampled as an fBm of parameter $\delta \in (0, 1)$, w independent of B^H. If

$$s > 2 - \frac{1}{\delta}\left(H - \frac{1}{2}\right)$$

then uniqueness holds for (5), which admits a random flow of diffeom. Similarly higher regularity for $s > n + 1 - (H - 1/2)/\delta$.

In particular: since $H > 1/2$, for any $s \in \mathbb{R}$ we can find $\delta > 0$ small enough such that the conditions are satisfied!

Things I couldn't cover

1. A more detailed explanation of the notion of prevalence: introduced by Hunt, Sauer, York (1992), notion of "Lebesgue full sets" for infinite dim. spaces based on characterization via Fubini theorem.
2. The nonlinear Young theory allows to provide a solution theory for
$$\partial_t u + b \cdot \nabla u + cu + \dot{w} \cdot \nabla u = 0$$
whenever $T^w b, T^w c \in C^\alpha_t C^2_x$ with $\alpha > 1/2$; this includes transport and continuity equations perturbed by w. See [GG20], [G20].
3. The results by Gubinelli,Lejay, Tindel (PA 2020) can be generalised to nonlinear Young parabolic equations of the form
$$\mathrm{d}x_t = -Ax_t \mathrm{d}t + B(\mathrm{d}t, x_t)$$
with $B \in C^\alpha_t C^2_{V,W}$ for suitable V, W, see [G20].
4. Work in progress with Harang, Mayorcas: study DDSDEs of the form
$$X_t = \xi + \int_0^t B(s, X_s, \mathcal{L}(X_s))\mathrm{d}s + W^H_t$$
with distributional B, generalising the results from [CG16].

Essential bibliography

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