2 Results on singular SDEs

NICOLAS PERKOWSKI (FU BERLIN)

WITH HELENA KREMP AND

WILLEM VAN ZUIJLEN

Singular Diffusions

distribution

here: α- stable process, $\alpha \in (1,2]$

($\alpha = 2$: Brownian motion)

$$dX_t = b(t, X_t)\,dt + d\text{"Noise"}$$

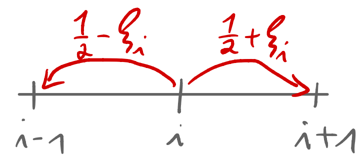
Questions:

- Why are we interested?
- How to solve them?
- Qualitative aspects?

Singular Diffusions as Processes in Random Media

Sinai's random walk in random environment:

$(\xi_j)_{j\in\mathbb{Z}}$ iid, centered, values in $\left(-\frac{1}{2},\frac{1}{2}\right)$



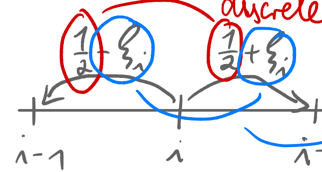
$$\mathbb{P}\left(S_{n+1}=i+1 \mid S_n=i,(\xi_j)\right)=\frac{1}{2}+\xi_i, \qquad \mathbb{P}\left(S_{n+1}=i-1 \mid S_n=i,(\xi_j)\right)=\frac{1}{2}-\xi_i$$

Singular Diffusions as Processes in Random Media

Sinai's random walk in random environment:

$(\xi_j)_{j\in\mathbb{Z}}$ i.i.d, centered, values in $\left(-\frac{1}{2}, \frac{1}{2}\right)$

discrete r.w. $\to W$



random perturbation, $\to$ white noise

$\mathbb{P}\left(S_{n+1}=i+1 \mid S_n=i, (\xi_j)\right) = \frac{1}{2}+\xi_i$, $\quad \mathbb{P}\left(S_{n+1}=i-1 \mid S_n=i, (\xi_j)\right) = \frac{1}{2}-\xi_i$

Continuous analogue: Brox diffusion

$$\boxed{dX_t = \xi(X_t)\,dt + dW_t}$$

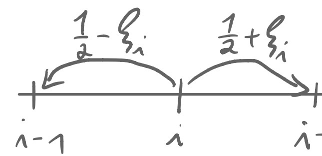
white noise $= B'$

1d B.m.

Singular Diffusions as Processes in Random Media

Sinai's random walk in random environment:

$(\xi_j)_{j\in\mathbb{Z}}$ iid, centred, values in $(-\frac{1}{2},\frac{1}{2})$



$\mathbb{P}(S_{n+1}=i+1 \mid S_n=i,(\xi_j)) = \frac{1}{2}+\xi_i$ $\quad$ $\mathbb{P}(S_{n+1}=i-1 \mid S_n=i,(\xi_j)) = \frac{1}{2}-\xi_i$

Continuous analogue: Brox diffusion

$$\boxed{dX_t = \xi(X_t)dt + dW_t}$$

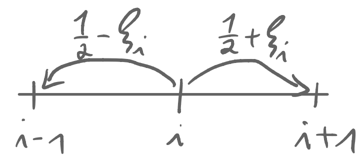
white noise $= B'$ $\quad$ 1d B.m.

- scaling limit if ξ_i "small"
- easier to analyze because of scaling invariance
- $\alpha^{-2}X_{e^\alpha}$ converges, $\alpha\to\infty$

Singular Diffusions as Processes in Random Media

Sinai's random walk in random environment:

$(\xi_j)_{j\in\mathbb{Z}}$ iid, centered, values in $(-\frac{1}{2},\frac{1}{2})$



$\mathbb{P}(S_{n+1}=i+1 \mid S_n=i,(\xi_j)) = \frac{1}{2}+\xi_i,$ $\quad\mathbb{P}(S_{n+1}=i-1 \mid S_n=i,(\xi_j)) = \frac{1}{2}-\xi_i$

Continuous analogue: Brox diffusion

$$\boxed{dX_t = \xi(X_t)\,dt + dW_t}$$

white noise $= B'$

1d B.m.

faster for $W \to$ Lévy noise?
(compare Imkeller – Pavlyukevich '06)

- scaling limit if ξ_i "small"
- easier to analyze because of scaling invariance
- $\alpha^{-2} X_{e^\alpha}$ converges, $\alpha\to\infty$

very slow!

Singular diffusions as random polymer measures

Random directed polymer measure

Alberts - Khanin - Quastel '14, Delarue - Diel '16

$$d\mathbb{P} = \frac{1}{Z} \exp\left(\int_0^T \xi(s, B_s)\, ds \right) dW$$

space-time white noise

Wiener measure

$\Rightarrow$ under $\mathbb{P}$ the coordinate process solves

$$dX_t = \partial_x h(t, X_t)\, dt + dW_t$$

solution to KPZ, non-differentiable

Singular diffusions as random polymer measures

Random directed polymer measure

Alberts - Khanin - Quastel '14, Delarue - Diel '16

$$d\mathbb{P} = \frac{1}{Z} \exp\left(\int_0^T \xi(s, B_s)\, ds \right) dW$$

space-time white noise

Wiener measure

$\Rightarrow$ under $\mathbb{P}$ the coordinate process solves

$$dX_t = \partial_x h(t, X_t)\, dt + dW_t$$

solution to KPZ, non-differentiable

1d

Cannizzaro - Chouk '18: Similar story for space white noise, $d = 1, 2, 3$.

Singular diffusions as stochastic characteristics I

KPZ equ.: $\partial_t h = -\left(\frac{1}{2}\partial_{xx} h + \frac{1}{2}(\partial_x h)^2 + \xi\right), \quad h(T) = \bar{h}$

Gubinelli-P.'17 $\Rightarrow$ $h(t,x) = \sup_v \mathbb{E}_{t,x}\left[\int_t^T \xi(s, X_s^v)\,ds + \bar{h}(X_T^v)\right]$

$$dX_t^v = v_t\,dt + dW_t$$

Optimal $v_t = \partial_x h(t, X_t^v)$

distributional drift

$\longrightarrow$ leads to good L^∞ bounds for h

Singular diffusions as stochastic characteristics I

KPZ equ.: $\partial_t h = -\left(\frac{1}{2}\partial_{xx} h + \frac{1}{2}(\partial_x h)^2 + \xi\right), \quad h(T) = \bar{h}$

Gubinelli-P.'17 $\Rightarrow$ $$h(t,x) = \sup_v \mathbb{E}_{t,x}\left[\int_t^T \xi(s, X_s^v)\,ds + \bar{h}(X_T^v)\right]$$

$$dX_t^v = v_t\,dt + dW_t$$

Optimal $v_t = \partial_x h(t, X_t^v)$

distributional drift

$\longrightarrow$ leads to good L^∞ bounds for h

Expect:

- Feynman-Kac type representations for general linear singular SPDEs (parabolic)
- Nonlinear equ. via BSDEs, McKean-Vlasov, stoch. control

What's new here?

- Extension of Cannizzaro - Chouk '18 to Lévy noise
- "Quantitative" heat kernel bounds for singular diffusions in the "Young regime"

Solution theories for singular diffusions

- 1d / gradient drift: Itô-McKean construction / Dirichlet forms

  Brox '86 / Mathieu '94

- 1d via martingale problem / Zvonkin transform

  Flandoli-Russo-Wolf '03 / Bass-Chen '01

$$dX_t = b(X_t)dt + dW_t$$

$\Rightarrow$ formally: $f(X_t) - f(X_0) - \int_0^t \mathcal{L} f(X_s)\,ds = \text{Mart.}$

$\mathcal{L} f = b\,\partial_x f + \frac{1}{2}\partial_{xx} f$

f smooth $\Rightarrow \mathcal{L} f$ distribution

Solution theories for singular diffusions

- 1d / gradient drift: Itô-McKean construction / Dirichlet forms

  Brox '86 / Mathieu '94

- 1d via martingale problem / Zvonkin transform

  Flandoli-Russo-Wolf '03 / Bass-Chen '01

$$dX_t = b(\cancel{t,}X_t)dt + dW_t$$

$\Rightarrow$ formally: $f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds = \text{Mart.}$

$$\hookrightarrow \mathcal{L}f = b\partial_x f + \frac{1}{2}\partial_{xx} f$$

Idea: find non-smooth f s.t. $\mathcal{L}f \in C_b$.

f smooth $\Rightarrow \mathcal{L}f$ distribution

E.g. solve $\mathcal{L}f = g$ (linear ODE in 1d)

- $d \geq 1$ and $b \in C(\mathbb{R}_+, C^{-\alpha})$, $\alpha < \frac{1}{2}$ (Young regime):

  Flandoli - Issoglio - Russo '17 via regularity analysis for

  $$\partial_t u = \mathcal{L}u + f, \quad u(0) = \varphi$$

- $d = 1$ and $b \in C(\mathbb{R}_+, C^{-\alpha})$, $\alpha < \frac{2}{3}$ (rough path regime):

  Delarue - Diel '16 via rough path analysis of $\partial_t u = \mathcal{L}u + f$

  (similar to Hairer's rough stochastic PDEs)

- $d \geq 1$ and $b \in C(\mathbb{R}_+, C^{-\alpha})$, $\alpha < \frac{2}{3}$ (rough regime) → $\alpha < 1$ would be subcritical (RS?)

  Cannizzaro - Chouk '18 via paracontrolled distrib.

Extension of Cannizzaro - Chouk to Lévy noise

$$dX_t = b(t, X_t)dt + dL_t$$

α- stable Lévy process, generator $\simeq -(-\Delta)^{\frac{\alpha}{2}}$

Formal martingale problem:

$$f(t, X_t) - f(0, X_0) - \int_0^t \underbrace{(\partial_s + \mathcal{L})}_{= (\partial_s + b(s,x)\cdot\nabla - (-\Delta)^{\alpha/2})} f(s, X_s)ds = \text{Mart.}$$

Extension of Cannizzaro - Chouk to Lévy noise

$$dX_t = b(t, X_t)dt + dL_t$$

$\uparrow$ α-stable Lévy process, generator $\simeq -(-\Delta)^{\frac{\alpha}{2}}$

Formal martingale problem:

$$f(t, X_t) - f(0, X_0) - \int_0^t \underbrace{(\partial_s + \mathcal{L})}_{= \left(\partial_s + b(s,x)\cdot\nabla - (-\Delta)^{\alpha/2}\right) f} f(s, X_s)\,ds = \text{Mart.}$$

$\rightsquigarrow$ Kolmogorov backward equation: $\in C^{-\gamma} \Rightarrow$ guess $u \in C^{\alpha-\gamma} \Rightarrow \nabla u \in C^{\alpha-\gamma-1}$

gains C^{α}

$$\left(\partial_t - (-\Delta)^{\alpha/2}\right) u = -b \cdot \nabla u + f$$

$\Rightarrow b \cdot \nabla u$ ok if

$-\gamma + (\alpha - \gamma - 1) > 0 \Leftrightarrow \gamma < \frac{\alpha - 1}{2}$

Extension of Cannizzaro - Chouk to Lévy noise

$$dX_t = b(t, X_t)\,dt + dL_t$$

α-stable Lévy process, generator $\simeq -(-\Delta)^{\frac{\alpha}{2}}$

Young regime, Athreya - Butkovsky - Mytnik '20, Ling - Zhao '19, de Raynal - Menozzi '19

Lévy Brox would be $\xi \in C^{-\frac{1}{2}-}$, $\frac{1}{2} > \frac{\alpha - 1}{2}$

Formal martingale problem:

$$f(t, X_t) - f(0, X_0) - \int_0^t \underbrace{(\partial_s + \mathcal{L})}_{= \left(\partial_s + b(s,x)\cdot\nabla - (-\Delta)^{\alpha/2}\right)} f(s, X_s)\,ds = \text{Mart.}$$

$\rightsquigarrow$ Kolmogorov backward equation:

$$\left(\partial_t - (-\Delta)^{\alpha/2}\right) u = -b \cdot \nabla u + f$$

gains C^α

$b \in C^{-\gamma} \Rightarrow$ guess $u \in C^{\alpha-\gamma} \Rightarrow \nabla u \in C^{\alpha-\gamma-1}$

$\Rightarrow b \cdot \nabla u$ ok if $-\gamma + (\alpha - \gamma - 1) > 0 \Leftrightarrow \gamma < \frac{\alpha - 1}{2}$

Paracontrolled ansatz

"paraproducts"

$$\left(\partial_t - (-\Delta)^{\alpha/2}\right) u = -\nabla u \cdot b + f = \underbrace{-\nabla u \varogreaterthan b + f}_{\substack{\in C^{\alpha-2\gamma-1} \\ \text{(better)}}} + \underbrace{-\nabla u \varolessthan b}_{\in C^{-\gamma}}$$

isolates singular part of the product

Thm. (Kremp-P. '20)

For $\gamma < \frac{2\alpha-2}{3}$, $b \in C_T C^{-\gamma}$, $b \odot \nabla\left(\partial_t - (-\Delta)^{\alpha/2}\right)^{-1} b \in C_T C^{\alpha-2\gamma-1}$:

$\exists!$ paracontrolled solution u. Similar to Cannizzaro-Chouk with $\Delta \to -(-\Delta)^{\alpha/2}$

Paracontrolled ansatz

"paraproducts"

$$(\partial_t - (-\Delta)^{\alpha/2}) u = -\nabla u \cdot b + f = \underbrace{-\nabla u \circledcirc b + f}_{\in C^{\alpha-2\gamma-1}} + \underbrace{-\nabla u \odot b}_{\in C^{-\gamma}}$$

(better)

isolates singular part of the product

white noise: $\gamma = \frac{1}{2}+$, ok if $\alpha > \frac{7}{4}$. Critical regularity: $\alpha = \frac{3}{2}$ (via RS?)

Thm. (Kremp-P. '20)

For $\gamma < \frac{2\alpha-2}{3}$, $b \in C_T C^{-\gamma}$, $b \circ \nabla (\partial_t - (-\Delta)^{\alpha/2})^{-1} b \in C_T C^{\alpha-2\gamma-1}$:

$\exists!$ paracontrolled solution u. Similar to Cannizzaro-Chouk with $\Delta \to -(-\Delta)^{\alpha/2}$

Construction of the process

$dX_t^n = b_n(t, X_t^n)\,dt + dL_t$ $b_n \to b$ in "rough path norm"

Need: • Tightness of (X^n).

• Any limit point X is s.t. $\forall u$ solving $(\partial_t - (-\Delta)^{\alpha/2})u = -b \cdot \nabla u + f$:

$$u(t, X_t) - u(0, X_0) - \int_0^t f(s, X_s)\,ds = \text{Mart.}$$

Construction of the process

$$dX_t^n = b_n(t, X_t^n)\,dt + dL_t \qquad b_n \to b \text{ in "rough path norm"}$$

Need: • Tightness of (X^n).

• Any limit point X is s.t. $\forall u$ solving $(\partial_t - (-\Delta)^{\alpha/2})u = -b\cdot\nabla u + f$:

$$u(t, X_t) - u(0, X_0) - \int_0^t f(s, X_s)\,ds = \text{Mart.}$$

Kremp - P. '20: • Tightness via stoch. calculus for Poisson random measures (Campbell's formula), get $\mathbb{E}\left[\left|\int_s^t b_n(r, X_r^n)\,dr\right|^p\right] \lesssim |t-s|^{p(1-\frac{\gamma}{\alpha})}$

• Martingale property easy once we know $\mathbb{P}(\Delta X_t \neq 0) = 0 \ \forall t \geq 0$.

Brox diffusion with Lévy noise

$$dX_t = \xi(X_t)\,dt + dL_t\,, \quad d=1$$

take periodic white noise (avoid weighted spaces)

- Clear: $\xi \in C^{-\frac{1}{2}-}$

- Kremp–P. '20: • can construct $\xi \odot \nabla(\partial_t - (-\Delta)^{\alpha/2})^{-1}\xi \in C^{\alpha-2-}$

  no renormalization

  if $\alpha > \frac{3}{2}$

  $\alpha = \frac{3}{2}$: critical regularity

  • get periodic Brox diffusion for $\alpha > \frac{7}{4}$.

Next: heat kernel estimates in the Brownian case

Motivation: Mass asymptotics for 2d PAM

$$\partial_t u = \Delta u + u\xi \quad , \qquad \xi \text{ 2d white noise}$$

$$U(t) := \int_{\mathbb{R}^d} u(t,x)\,dx$$

Goal: study $U(t)$ for $t \to \infty$.
(a.s. limit)

Known results

- ξ smooth Gaussian potential $\Rightarrow$ $\frac{1}{t(\log t)^{\frac{1}{2}}} \log u(t) \longrightarrow \sqrt{2d\,\mathbb{E}[\xi(0)^2]}$

  Carmona - Molchanov '95, Gärtner - König - Molchanov '00 : 2nd order asymptotics

- $\mathbb{E}[\xi(x)\xi(y)] = |x-y|^{-\alpha}$, $0<\alpha<\min\{2,d\}$

  $\Rightarrow$ $\frac{1}{t(\log t)^{\frac{2}{4-\alpha}}} \log u(t) \longrightarrow c_\alpha$ Chen '14

- ξ 1d white noise $\Rightarrow$ $\frac{1}{t(\log t)^{2/3}} \log u(t) \longrightarrow c$ Chen '14

- $\xi = \partial_t B^H$ $\Rightarrow$ $\frac{1}{t(\log t)^{\frac{1}{1+H}}} \log u(t) \longrightarrow c$ Chakraborty - Chen - Gao - Tindel '18

  fractional B.m., Hurst index $H \in (0,1)$

$$\frac{1}{t(\log t)^{\frac{1}{2}}} \qquad \frac{1}{t(\log t)^{\frac{2}{4-\alpha}}} \qquad \frac{1}{t(\log t)^{2/3}} \qquad \frac{1}{t(\log t)^{\frac{1}{1+H}}}$$

$\alpha \in (0, 2 \wedge d)$ $\qquad$ $H \in (0,1)$

<u>Note</u>: Always $\frac{1}{t(\log t)^{\gamma}} \log U(t) \to c$, $\gamma \in [\frac{1}{2}, 1)$

$\gamma = 1$ is natural barrier, $\gamma \geq 1$ needs renormalization.

<u>Expect</u>: $\xi \in C^{-\kappa-} \Rightarrow \gamma = \frac{1}{2-\kappa} \vee \frac{1}{2}$

$\rightsquigarrow \quad \gamma < 1 \iff \kappa < 1$

$$\frac{1}{t(\log t)^{\frac{1}{2}}} \qquad \frac{1}{t(\log t)^{\frac{2}{4-\alpha}}} \qquad \frac{1}{t(\log t)^{2/3}} \qquad \frac{1}{t(\log t)^{\frac{1}{1+H}}}$$

$\alpha \in (0, 2 \wedge d)$ $\qquad$ $H \in (0,1)$

<u>Note</u>: Always $\frac{1}{t(\log t)^{\gamma}} \log U(t) \to c$, $\gamma \in [\frac{1}{2}, 1)$

$\gamma = 1$ is natural barrier, $\gamma \geq 1$ needs renormalization.

<u>Expect</u>: $\xi \in C^{-\kappa-} \Rightarrow \gamma = \frac{1}{2-\kappa} \vee \frac{1}{2}$

$\rightsquigarrow \quad \gamma < 1 \Leftrightarrow \kappa < 1$

2d white noise:

$\kappa = 1$

What breaks at $\kappa = 1$?

$$\partial_t u = \Delta u + u\xi, \quad u(0) = \delta_0$$

All proofs based on Feynman-Kac: $u(t) = \mathbb{E}\left[\exp\left(\int_0^t \xi(W_s)\,ds\right)\right]$

But: $\xi \in C^{-1-} \Rightarrow$ cannot construct $\cancel{\int_0^t \xi(W_s)\,ds}$ ↯

What breaks at $\kappa = 1$?

All proofs based on Feynman-Kac: $U(t) = \mathbb{E}\left[\exp\left(\int_0^t \xi(W_s)\,ds\right)\right]$

But: $\xi \in C^{-1-} \Rightarrow$ cannot construct $\cancel{\int_0^t \xi(W_s)\,ds}$ ↯

König - van Zuijlen - P. '20: Still can show

$$\lim_{t\to\infty} \frac{1}{t\log t} \log U(t) \to \chi \quad \text{for 2d white noise}$$

via "transformed Feynman-Kac formula"

Making the Feynman-Kac formula work

Let $-\Delta Z = \xi$

"$\frac{1}{2} = 1$"

$$\Rightarrow Z(B_t) = Z(B_0) + \int_0^t \nabla Z(B_s)\cdot dB_s + \underbrace{\int_0^t \Delta Z(B_s)\,ds}_{= -\int_0^t \xi(B_s)\,ds}$$

"Itô trick"

$$\Rightarrow \int_0^t \xi(B_s)\,ds = Z(B_0) - Z(B_t) + \int_0^t \nabla Z(B_s)\cdot dB_s$$

ok if $\xi \in C^{-1+\varepsilon}$ $(\Rightarrow Z \in C^{1+\varepsilon})$

here: just fails

Making Feynman-Kac work

$$\int_0^t \xi(B_s)\,ds = Z(B_0) - Z(B_t) + \int_0^t \nabla Z(B_s)\cdot dB_s$$

$$\Rightarrow U(t) = \mathbb{E}\left[\exp\left(Z(B_0) - Z(B_t) + \int_0^t \nabla Z(B_s)\cdot dB_s\right)\right]$$

$$= \mathbb{E}\left[\exp\left(Z(0) - Z(B_t) + \left(\int_0^t \nabla Z(B_s)\cdot dB_s - \frac{1}{2}\int_0^t |\nabla Z(B_s)|^2 ds\right)\right.\right.$$

$$\left.\left. + \frac{1}{2}\int_0^t |\nabla Z(B_s)|^2 ds\right)\right]$$

Making Feynman-Kac work

$$\int_0^t \xi(B_s)\,ds = Z(B_0) - Z(B_t) + \int_0^t \nabla Z(B_s)\cdot dB_s$$

$$\Rightarrow U(t) = \mathbb{E}\left[\exp\left(Z(B_0) - Z(B_t) + \int_0^t \nabla Z(B_s)\cdot dB_s\right)\right]$$

$$= \mathbb{E}\left[\exp\left(Z(0) - Z(B_t) + \left(\int_0^t \nabla Z(B_s)\cdot dB_s - \frac{1}{2}\int_0^t |\nabla Z(B_s)|^2 ds\right) + \frac{1}{2}\int_0^t |\nabla Z(B_s)|^2 ds\right)\right]$$

$$= \mathbb{E}\left[\exp\left(Z(0) - Z(X_t) + \frac{1}{2}\int_0^t |\nabla Z(X_s)|^2 ds\right)\right]$$

$$X_t = \int_0^t \nabla Z(X_s)\,ds + B_t \quad \text{(weakly)}$$

Making Feynman-Kac work

$$\int_0^t \xi(B_s)\,ds = Z(B_0) - Z(B_t) + \int_0^t \nabla Z(B_s)\cdot dB_s$$

$$\Rightarrow U(t) = \mathbb{E}\Big[\exp\Big(Z(B_0) - Z(B_t) + \int_0^t \nabla Z(B_s)\cdot dB_s\Big)\Big]$$

$$= \mathbb{E}\Big[\exp\Big(Z(0) - Z(B_t) + \Big(\int_0^t \nabla Z(B_s)\cdot dB_s - \frac{1}{2}\int_0^t |\nabla Z(B_s)|^2 ds\Big) + \frac{1}{2}\int_0^t |\nabla Z(B_s)|^2 ds\Big)\Big]$$

$$= \mathbb{E}\Big[\exp\Big(Z(0) - Z(X_t) + \frac{1}{2}\int_0^t |\nabla Z(X_s)|^2 ds\Big)\Big]$$

$|\nabla Z|^2$ ill-defined ($Z \in C^{+1-\varepsilon}$)

$$X_t = \int_0^t \nabla Z(X_s)\,ds + B_t \quad \text{(weakly)}$$

Probabilistic argument: construct $|\nabla Z|^2$ as (random) distribution in $C^{-\varepsilon}$

Rigorous Feynman-Kac for 2d PAM

Itô trick once more:

$$-\Delta Z = \xi, \qquad (-\Delta - \nabla Z \cdot \nabla) Y = -|\nabla Z|^2, \qquad W = Z + Y$$

$\hookrightarrow \in C^{-\varepsilon} \Rightarrow Y \in C^{2-\varepsilon}$

$\Rightarrow |\nabla Y|^2$ ok

$$dX_t = \nabla W(X_t)\,dt + dB_t$$

$$\Rightarrow U(t) = \mathbb{E}\left[\exp\left(W(0) - W(X_t) - \frac{1}{2}\int_0^t |\nabla Y|^2(X_s)\,ds\right)\right]$$

$\rightarrow$ Get asymptotics of $U(t)$ <u>if</u> we understand X well.

(upper + lower tail bounds)

Heat kernel estimates for singular diffusions

$$dX_t = b(t, X_t)dt + dB_t \qquad b \in C(\mathbb{R}_+, C^{-\alpha}), \quad \alpha < \frac{1}{2}$$

$\rightsquigarrow$ not hard to show: $\exists$ transition density

$$\Gamma_{s,t}(x,y)\,dy = \mathbb{P}(X_t \in dy \mid X_s = x)$$

Zhang - Zhao '17: $\exists\, \mu > 1$, $\forall\, T > 0$ $\exists\, C > 0$ depending only on T and $\|b\|_{C([0,T], C^{-\alpha})}$: (if $\alpha < \frac{1}{2}$)

based on "Zvonkin transform"

$$\frac{1}{C}\, p\left(\frac{t}{\mu}, x-y\right) \leq \Gamma_t(x,y) \leq C\, p(\mu t, x-y)$$

standard Gaussian kernel

Heat kernel estimates for singular diffusions

$$dX_t = b(t, X_t)dt + dB_t \qquad b \in C(\mathbb{R}_+, C^{-\alpha}), \quad \alpha < \frac{1}{2}$$

$\rightsquigarrow$ not hard to show: $\exists$ transition density

$$\Gamma_t(x,y)dy = \mathbb{P}_x(X_t \in dy)$$

We need explicit dependence on T, $\|b\|$.

Zhang-Zhao '17: $\exists\, M > 1$, $\forall\, T > 0$ $\exists\, C > 0$ depending only on T and $\|b\|_{C([0,T], C^{-\alpha})}$:

based on "Zvonkin transform"

$$\frac{1}{C}\, p\left(\frac{t}{M}, x-y\right) \leq \Gamma_t(x,y) \leq C\, p(Mt, x-y)$$

(if $\alpha < \frac{1}{2}$)

standard Gaussian kernel

"Quantitative" heat kernel bounds for singular diffusions

$$dX_t = b(t, X_t)\,dt + dB_t \quad , \quad \Gamma_{s,t}(x,y)\,dy = \mathbb{P}(X_t \in dy \mid X_s = x)$$

young regime

Thm. (van Zuijlen - P. '20) $\forall \alpha \in (0, \tfrac{1}{2})$, $c > 1$ $\exists C > 0, \mu > 1$:

$$\frac{1}{C} \exp\left(-C(t-s)\|b\|^{2/1-\alpha}_{C_t C^{-\alpha}}\right) p\left(\frac{t-s}{\mu}, x-y\right) \leq \Gamma_{s,t}(x,y)$$

$$\leq C \exp\left(C(t-s)\|b\|^{\frac{2}{1-\alpha}}_{C_t C^{-\alpha}}\right) p\left(c(t-s), x-y\right)$$

$$\forall b \in C(\mathbb{R}_+, C^{-\alpha}), \ \forall\, 0 \leq s < t$$

Idea of the proof

Fix $s=0$, $\Gamma_t := \Gamma_{0,t}$

$$\partial_t \Gamma_t = \underbrace{\tfrac{1}{2}\Delta\Gamma_t + b\cdot\nabla\Gamma_t}_{\mathcal{L}\Gamma_t}, \quad \Gamma_t = \delta_y$$

Parametrix method: $\partial_t p_t = \frac{1}{2}\Delta p_t$, $p_0 = \delta_y$

compare Friedman's book

$\Longrightarrow r_t := \Gamma_t - p_t$ solves $(\partial_t - \mathcal{L}) r_t = \underbrace{-b\cdot\nabla r_t}_{=:\Phi_t}$, $r_0 = 0$

space-time "convolution"

$\rightsquigarrow$ $r = \Gamma * \Phi$

Idea of the proof

Fix $s=0$, $\Gamma_t := \Gamma_{0,t}$

$$\partial_t \Gamma_t = \underbrace{\tfrac{1}{2}\Delta \Gamma_t + b\cdot\nabla\Gamma_t}_{\mathcal{L}\Gamma_t}, \quad \Gamma_t = \delta_y$$

Parametrix method: $\partial_t p_t = \frac{1}{2}\Delta p_t$, $p_0 = \delta_y$

$$\Rightarrow r_t := \Gamma_t - p_t \text{ solves } (\partial_t - \mathcal{L}) r_t = \underbrace{-b\cdot\nabla r_t}_{=:\Phi_t}, \quad r_0 = 0$$

space-time "convolution"

$$\rightsquigarrow r = \Gamma * \Phi$$

$$\Gamma = p + r \rightsquigarrow = p * \Phi + \Gamma * \Phi^{*2} \quad \dots \Rightarrow \Gamma = \sum_{n=0}^{\infty} p * \Phi^{*n}$$

$\hookrightarrow$ control each term inductively

$$\Gamma = \sum_{n=0}^{\infty} \eta * \Phi^{*n}$$

control each term inductively

$\approx$ 8 pages of computations:

get upper bound for $|\Gamma - \eta|$

$$\Gamma = \sum_{n=0}^{\infty} p * \Phi^{*n}$$

control each term inductively

$\approx$ 8 pages of computations:

get upper bound for $|\Gamma - p|$

Chapman-Kolmogorov

+ "chaining"

$\Rightarrow$ lower bound (cf. Stroock's PDE book)

Summary

- Singular diffusions arise as processes in random media
  + as stochastic characteristics for singular SPDEs.
- Weak solution theory via martingale problem.
- Here 2 new results:
  - Brox diffusion with Lévy noise
  - quant. heat kernel estimates in Brownian Young regime

Thank You