Rough path variational principles for fluid equations.

Joint work with

- Dan Crisan
- Darryl Holm
- James-Michael Leahy

1. Variational principles
2. Euler - Arnold
3. Physically relevant noisy decomposition of Lagrangian trajectories
4. Rough path framework
5. Main results

Euler eq.

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p \\ \operatorname{div} u = 0 \end{cases}$$

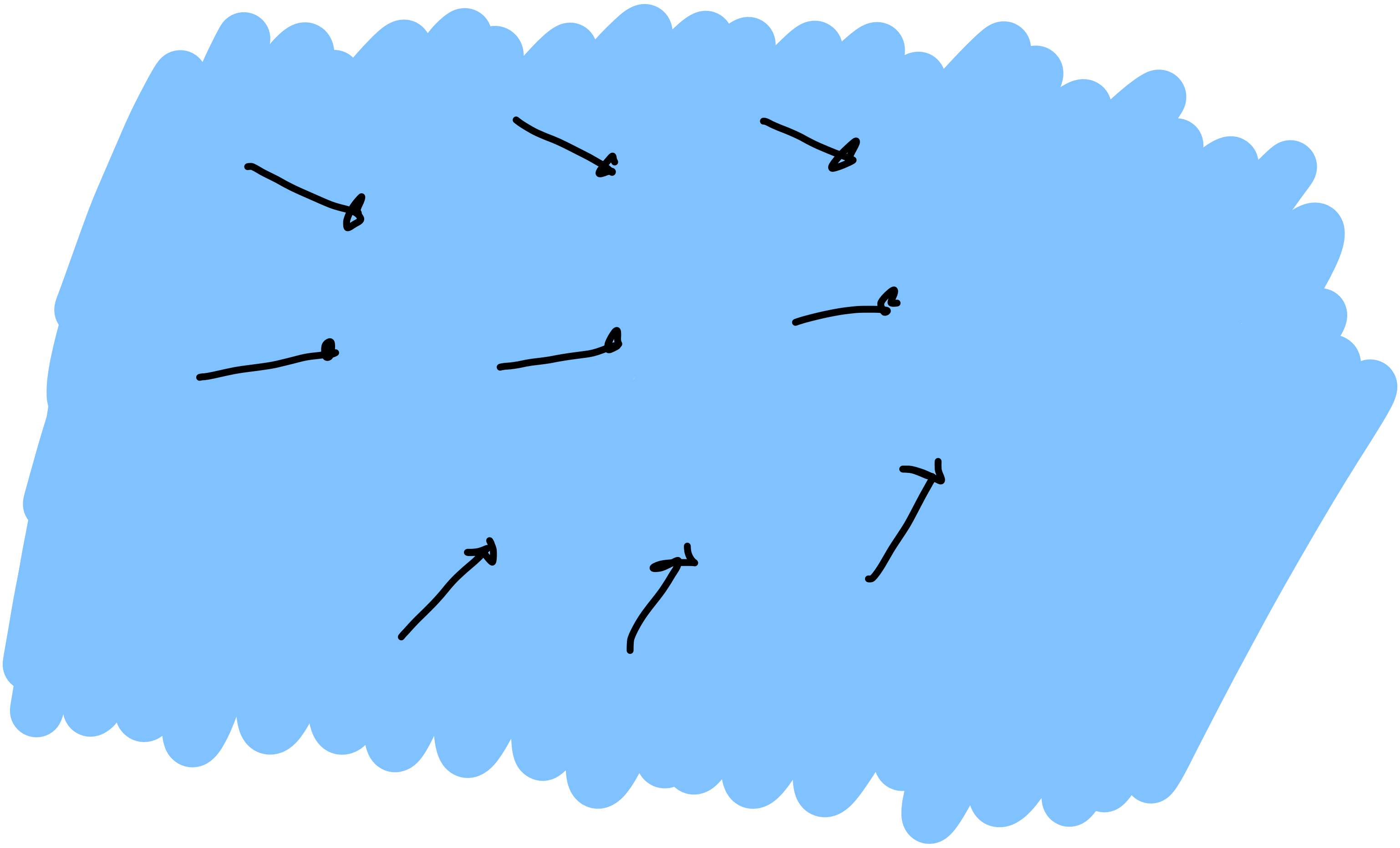
Solution is a pair

u = vector field

p = pressure

In coordinates

$$\partial_t u^i + u^j \frac{\partial u^i}{\partial x^j} = - \frac{\partial p}{\partial x^i} \quad \forall \ i$$



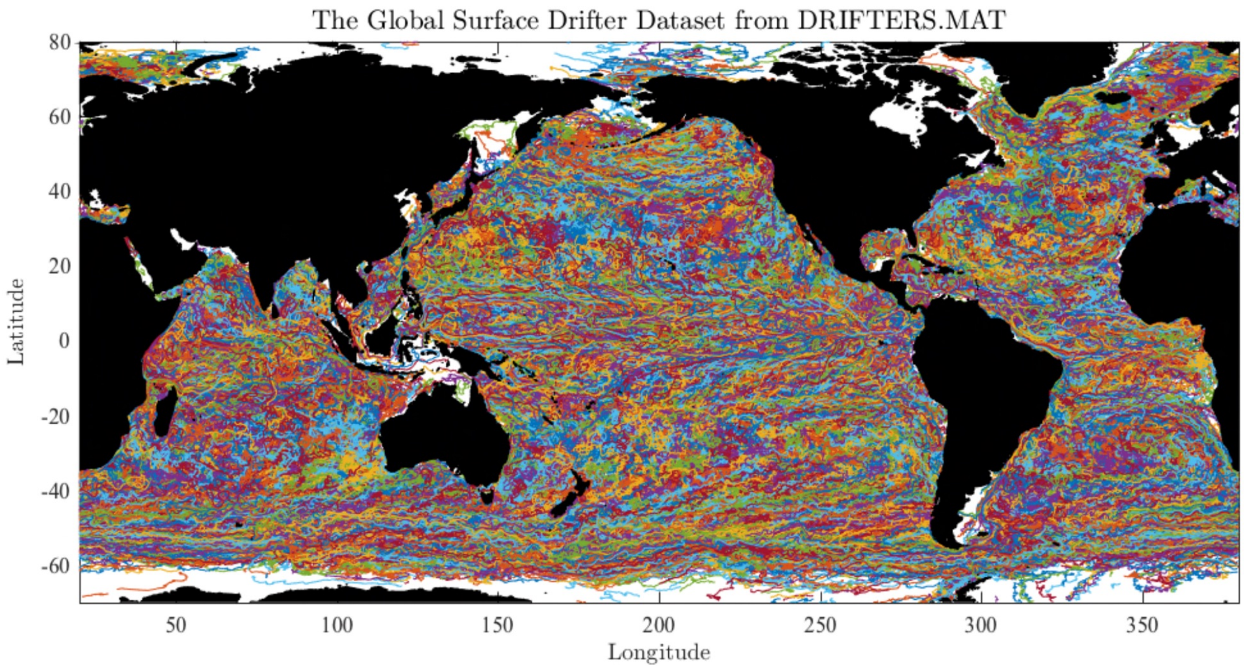
Lagrangian trajectories

$$\dot{\eta}_t = u_t(\eta_t)$$

Solution Properties of a 3D Stochastic Euler Fluid Equation

Dan Crisan, Franco Flandoli & Darryl D. Holm

Journal of Nonlinear Science **29**, 813–870(2019) | Cite this article



$$\dot{\eta}_t = \text{large scale} + \text{small scale}$$

$$\approx u_t(\eta_t) + \text{random term}$$

$$= u_t(\eta_t) + \xi_k(\eta_t)\,\dot{W}_t^k$$

- parametrize uncertainty
- estimate ξ_k from data
- improve predictive power
- preserve geometric structure of Euler eq.

? What is the corresponding PDE ?

1. Variational principles and geodesics

Curve

$$[0,1] \xrightarrow{q} M$$

want to minimize action functional

$$S(q,\dot{q}) = \int_0^1 L(q_t, \dot{q}_t)\, dt$$

for a given Lagrangian

$$L : TM \longrightarrow \mathbb{R}$$

q critical point $\Longleftrightarrow$

$$\frac{\partial L}{\partial q^i}(q_t, \dot{q}_t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}(q_t, \dot{q}_t) = 0$$

Euler-Lagrange eq.

Proof

1. Variations

$$q^{\varepsilon} = q + \varepsilon \delta q$$

$$\delta q_0 = \delta q_1 = 0$$

2. Critical point

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} S(q^{\varepsilon}, \dot{q}^{\varepsilon}) = 0$$

3. integration by parts

$\blacksquare$

Example: N point particles

$$(q^1, \dots, q^N) \in (\mathbb{R}^d)^N$$

w/ masses $(m_1, \dots, m_N) \in \mathbb{R}^N$

Potential $V : (\mathbb{R}^d)^N \longrightarrow \mathbb{R}$

Define the Lagrangian

$$L(q,v) := \sum_n \frac{m_n}{2} |v^n|^2 - V(q)$$

Euler-Lagrange $\iff$ Newton's law

$$m_n \ddot{q}_t^n = -\frac{\partial V}{\partial q^n}(q_t)$$

Example: M Riemannian w/metric $\langle \cdot , \cdot \rangle$.

Geodesic equations:

$$L(q,v) := \langle v, v \rangle_q$$

Euler-Lagrange

$$\Updownarrow$$

$$\ddot{q}_t^i + \Gamma^i_{jk} \dot{q}_t^j \dot{q}_t^k = 0$$ Geodesic equation

where Γ^i_{jk} are the Christoffel symbols

Example: Lie group, $M = G$

Right multiplication $R_q : p \longmapsto pq$

If L is right invariant, i.e.

$$L \circ TR_q = L$$

<u>Def</u> $\quad \ell(u) = L(e, u)$ reduced Lagrangian

$$u \in \mathfrak{g} = T_e G$$

Euler-Poincare reduction

TFAE

- q satisfies Euler-Lagrange

- $u = \dot{q} \circ q^{-1} = R_{q*}^{-1} \dot{q}$ and $\lambda = \frac{\partial l}{\partial u}$ critical points of

$$S(q,u,\lambda) = \int_0^1 l(u_t) + \langle \lambda_t, \dot{q}_t \circ q_t^{-1} - u_t \rangle \, dt$$

- u is a critical point of

$$S(u) = \int_0^1 l(u_t)\, dt$$

when variations are restricted to be on the form

$$\delta u_t = \dot{w}_t + [u_t, w_t]$$

- u satisfies Euler-Poincare eq;

$$\frac{d}{dt} \frac{\partial l}{\partial u} + ad_u^* \frac{\partial l}{\partial u} = 0$$

$$(\, ad_u \xi = - [\xi, u] \,)$$

2. Euler - Arnold

G = volume preserving diffeomorphisms on some manifold M

Tangent space of G:

$c : (-\varepsilon, \varepsilon) \longrightarrow G \qquad c_0 = f \in G$

$c_\cdot(p) : (-\varepsilon, \varepsilon) \longrightarrow M \qquad c_0(p) = f(p) \in M$

$\Rightarrow \quad T_f G \subset \{ X : M \longrightarrow TM \mid X_p \in T_{f(p)} M \}$

Special case:

$f = id = e$ in G

$\mathfrak{g} = T_e G =$ div-free vector fields

G is a Lie - group

w/ composition as group operation

Translations

$R_q p = p \circ q$ $\quad C^\infty$

$L_q p = q \circ p$ $\quad$ Same regularity as q.

C^∞ only if G is Frechet

(Weak) Riemmanian metric

$$\langle X, Y \rangle_{\mathfrak{g}} = \int \langle X(\xi), Y(\xi) \rangle_{\xi} \, \mu(\xi)$$

yields right invariant metric on TG.

Arnold Formally

$$\eta : [0,1] \longrightarrow G$$

is a geodesic iff

$$u = \dot{\eta} \circ \eta^{-1} \in \mathfrak{g} = T_e G$$

solves Euler equation.

Ebin - Marsden

Rigorously,

letting

$G =$ vol.-pres. $W^{k,2}$- Sobolev diffeo's

$k > \frac{d}{2}$

Arnold was right.

+ + +

Gives equivalent formulation of Euler's eq. via critical points of

$$S(u,\eta,\lambda) = \int_0^1 \langle u_t, u_t \rangle_g + \langle \lambda_t, u_t - \dot\eta_t \circ \eta_t^{-1} \rangle dt$$

Taking variations in (u, η, λ) gives that critical points of S satisfies

$$\lambda_t = u_t^\flat = \langle u_t, \cdot \rangle \in \Lambda^1(M)$$

$$u_t = \dot{\eta}_t \circ \eta_t^{-1} \Rightarrow \dot{\eta}_t = u_t(\eta_t)$$

and

$$\partial_t u_t^\flat + \mathcal{L}_{u_t} u_t^\flat = -dp_t$$

In local coordinates

$$\partial_t u_t^i + u_t^j \frac{\partial u_t^i}{\partial x^j} + \underbrace{u_t^j \frac{\partial u_t^j}{\partial x^i}}_{= \frac{1}{2}\frac{\partial}{\partial x^i}\left(u_t^j\right)^2} = -\frac{\partial p_t}{\partial x^i}$$

Convenient to understand u as a 1-form,

$$u^\flat = \langle u, \cdot \rangle$$

- conserved quantities (Kelvin-Noether circulation thm)
- Lie derivatives

More general Lagrangians and underlying Banach manifolds yield a general class of fluid equations, e.g.

- Burgers
- KdV
- Camassa-Holm
- .....

General set up

$$S(u,\eta,\lambda) = \int_0^1 \ell(u_t, \eta_t^* a_0) + \langle \lambda_t, u_t - \dot{\eta}_t \circ \eta_t^{-1} \rangle dt$$

$a_t = \eta_t^* a_0$ - advected quantities

(e.g. heat, density, ...)

3. Use Lagrange multiplier to force decomposition

Wanted

$$\dot{\eta}_t = u_t(\eta_t) + \xi_k(\eta_t)\dot{W}_t^k$$

Use Lagrange multiplier

$$S(u,\eta,\lambda) = \int_0^1 \langle u_t, u_t \rangle_{\mathfrak{g}} + \langle \lambda_t, \dot{\eta}_t \circ \eta_t^{-1} - u_t - \xi_k \dot{W}_t^k \rangle$$

Formally

$$\partial_t u_t^b + \mathcal{L}_{u_t} u_t^b + \mathcal{L}_{\xi_k} u_t^b \, \dot{W}_t^k = -dp_t$$

4. Rough Paths

W irregular, e.g. sample path of Brownian motion.

How to understand

$$\dot{\eta}_t = u_t(\eta_t) + \xi_\alpha(\eta_t)\,\dot{W}^\alpha_t \quad ?$$

Itô - calculus ;

- discont. as a function of sample paths
- restricts to Markov noise
- already studied by Holm

Rough paths

Consider $u \equiv 0$ for simplicity

$$\eta_t - \eta_s = \int_s^t \xi_k(\eta_r)\, dW_r^k$$

$$= \xi_k(\eta_s)(W_t^k - W_s^k)$$

$$+ \nabla \xi_k(\eta_s)\,\xi_\ell(\eta_s) \int_s^t (W_r^\ell - W_s^\ell)\, dW_r^k$$

$$+ O(|t-s|^{1+\varepsilon})$$

The pair

$$(W, \mathbb{W}) = \left(W, \int W\, dW\right)$$

is called a rough path.

Typically need probability to give meaning to

$\int W \circ dW$ (e.g. ~~Ito or~~ Stratonovich)

- Nice continuity properties

$$(W, \mathbb{W}) \longmapsto \eta^{(W, \mathbb{W})}$$

- Can make sense of $\int W dW$ for more general sources of noise

5. Main results

Hamilton Pontryagin

critical points of

$$S(\eta, u, \lambda) = \int_0^1 \ell(u_t, \eta_t^* a_0)\, dt + \int_0^1 \langle \lambda_t, d\eta_t \circ \eta_t^{-1} - u_t\, dt - \xi_k \circ dW_t^k \rangle$$

are exactly solutions of

$$d\eta_t = u_t \circ \eta_t\, dt + \xi_k \circ \eta_t \circ dW_t^k$$

$$d\lambda_t + \mathcal{L}_{u_t} \lambda_t\, dt + \mathcal{L}_{\xi_k} \lambda_t \circ dW_t^k = \lambda_t \diamond a_t\, dt$$

$$\lambda = \frac{\partial \ell}{\partial u}$$

Clebsch

$$S(u,a,\lambda) = \int_0^1 l(u_t, a_t)\,dt + \langle \lambda_t, da_t + \mathcal{L}_{u_t} a_t\,dt + \mathcal{L}_{\xi_k} a_t \circ dW_t^k \rangle$$

Euler - Poincare

$$S(u,a) = \int_0^1 l(u_t, a_t)\,dt$$

variations on the form

$$\delta u\,dt = \partial_t \delta w\,dt - \mathrm{ad}_{u_t} \delta w\,dt - \mathrm{ad}_{\xi_k} \delta w\,dW_t^k$$

$$\delta a = -\mathcal{L}_{\delta w} a$$

Elements of the proof

- setting up the appropriate rough path spaces

- Rough Lie chain rule

$$d\tau_t = \beta_t\,dt + \sigma_\kappa \circ dW_t^\kappa$$

tensor eq.,

$$d\eta_t = u_t \circ \eta_t\,dt + \xi_\kappa \circ \eta_t\,dW_t^\kappa$$

Then

$$d\,\eta_t^* \tau_t = \eta_t^*(\beta_t + \mathcal{L}_{u_t}\tau_t)\,dt$$

$$+\ \eta_t^*(\sigma_\kappa + \mathcal{L}_{\xi_\kappa}\tau_t) \circ dW_t^\kappa$$

Next

- local well-posedness of rough Euler:
  - BKM blow-up criterion
  - well-posedness for $d=2$ w/ smooth initial data
- general theory for rough PDE's
- regularization by noise?

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THANK YOU!