Transport and continuity equations with (very) rough noise

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FG6

Introduction

We consider the equation

$$\begin{cases} du_t(x) = \sum_{i=1}^{d} f_i(x) \cdot \nabla u_t(x) \, dW_t^i & t \in [0,T], x \in \mathbb{R}^n \\ u_T(x) = g(x) \end{cases}$$

where the vector fields $f_1, \ldots, f_d$ and the function g are "nice" and $W \in \mathscr{C}^\gamma$ for some $\gamma \in (0,1)$.

We also consider the associated continuity equation

$$\begin{cases} \partial_t \mu_t = \sum_{i=1}^{d} \operatorname{div}(f_i(x)\mu_t) \, dW_t^i \\ \mu_0 = \nu \end{cases} .$$

Definition

We define the first order differential operators $\Gamma_i := f_i \cdot \nabla$, i.e. if ϕ is C^1 then $\Gamma_i \phi(x) := f_i(x) \cdot \nabla \phi(x)$.

Method of characteristics

If the driver W is smooth we can easily get a solution:

Proposition

Suppose $f_i \in C_b^3$. Take $X^{s,x}$ to be the unique solution of

$$\dot{X}_t^{s,x} = \sum_{i=1}^{d} f_i(X_t^{s,x})\, \dot{W}_t^i, \quad X_s^{s,x} = x,$$

and set $u_t(x) := g(X_T^{t,x})$. Then u is a solution to the transport equation.

Proposition (Diehl–Friz–Stannat 2017)

The induced flow $(t,x) \mapsto X^{t,x}$ is a C^1 diffeomorphism in space and

$$\frac{\mathrm{d}}{\mathrm{d}t} X_T^{t,x} = -DX_T^{t,x} f(x) \dot{W}_t, \qquad \frac{\mathrm{d}}{\mathrm{d}t} D_t^{s,x} = Df(X_t^{s,x}) DX_t^{s,x} \dot{W}_t$$

$$\frac{\mathrm{d}}{\mathrm{d}t} DX_T^{t,x} h = -D^2 X_T^{t,x}(f(x), h)\dot{W}_t - DX_T^{t,x} Df(x) h \dot{W}_t$$

Method of characteristics

Proof.

We compute

$$\begin{aligned}\partial_t u_t(x) &= \nabla g(X_T^{t,x}) \cdot \frac{\mathrm{d}}{\mathrm{d}t} X_t^{s,x} \\ &= -f(x) \cdot \nabla g(X_T^{t,x}) \dot{W}_t + \int_t^T Dg(X_T^{t,x}) Df(X_u^{t,x}) \frac{\mathrm{d}}{\mathrm{d}t} X_u^{t,x} \,\mathrm{d}u \\ &= -f(x) \cdot \nabla g(X_T^{t,x}) \dot{W}_t - \int_t^T Dg(X_T^{t,x}) Df(X_u^{t,x}) DX_u^{t,x} f(x) \dot{W}_u \,\mathrm{d}u\end{aligned}$$

and

$$\begin{aligned}f(x) \cdot \nabla u_t(x) \dot{W}_t &= Dg(X_T^{t,x}) DX_T^{t,x} f(x) \dot{W}_t \\ &= f(x) \cdot \nabla g(X_T^{t,x}) \dot{W}_t + \int_t^T Dg(X_T^{t,x}) Df(X_u^{t,x}) DX_u^{t,x} f(x) \dot{W}_u \,\mathrm{d}u.\end{aligned}$$

□

Solving ODEs

Signatures are connected to solutions to *controlled* ODEs.

Proposition (Fliess 1981)

If

$$\dot{X}_t^{s,x} = \sum_{i=1}^{d} f_i(X_t^{s,x})\dot{W}_t^i, \quad X_s^{s,x} = x$$

then, for any $N \geq 1$

$$X_t^{s,x} = \sum_{i=1}^{d} f_i(x) \int_s^t \dot{W}_\tau^i \,\mathrm{d}\tau + \sum_{i,j=1}^{d} Df_j(x) f_i(x) \int_s^t \int_s^{\tau_2} \dot{W}_{\tau_1}^i \dot{W}_{\tau_2}^j \,\mathrm{d}\tau_1 \mathrm{d}\tau_2$$
$$+ \sum_{i,j,k=1}^{d} \Big(D^2 f_k(x)(f_j(x), f_i(x)) + Df_k(x) Df_j(x) f_i(x) \Big) \int_s^t \int_s^{\tau_3} \int_s^{\tau_2} \dot{W}_{\tau_1}^i \dot{W}_{\tau_2}^j \dot{W}_{\tau_3}^k \,\mathrm{d}\tau_1 \mathrm{d}\tau_2 \mathrm{d}\tau_3 + \cdots + \mathrm{O}(|t-s|^{N+1})$$

Rough integration

Let G be the character group of the shuffle algebra. For any truncation level N, there is a metric ρ_N on G^N homogeneous wrt the grading.

Definition (Lyons 1998, Hairer–Kelly 2015)

A *weakly geometric rough path* of roughness $\gamma \in (0,1)$ is a γ-Hölder path $\mathbf{W}\colon [0,T] \to (G^N, \rho_N)$, where $N \coloneqq \lfloor \gamma^{-1} \rfloor$.

Let's return to $\mathrm{d}X_t = \sum f_i(X_t)\,\mathrm{d}W_t^i$.

Reinterpret in the integral sense:

$$X_t - X_s = \sum_{i=1}^{d} \int_s^t f_i(X_\tau)\,\mathrm{d}W_\tau^i.$$

Definition (Gubinelli 2003)

Fix $\frac{1}{3} < \gamma < \frac{1}{2}$ and $\mathbf{W}$ a weakly geometric rough path. A path $X \in \mathscr{C}^\gamma$ is controlled by $\mathbf{W}$ if there exists $X' \in \mathscr{C}^\gamma(\mathbb{R}^d)$ such that

$$X_{st} = \sum_{j=1}^{d} \langle j, X_s' \rangle \langle \mathbf{W}_{st}, j \rangle + \mathrm{O}(|t-s|^{2\gamma}).$$

Proposition (Gubinelli 2003)

Let (X, X') be controlled by $\mathbf{W}$. Then

$$\int_s^t X_\tau\,\mathrm{d}\mathbf{W}_\tau^i \coloneqq \lim_{|\pi|\to 0} \sum_{[a,b]\in\pi} \left(X_a \langle \mathbf{W}_{ab}, i \rangle + \sum_{j=1}^{d} \langle j, X_a' \rangle \langle \mathbf{W}_{ab}, ji \rangle \right)$$

exists, and we call it the rough integral of X wrt W.

Rough integration

Remark

Chen's relation can be reformulated as

$$\delta\langle \mathbf{W}_{sut}, i_1\cdots i_n\rangle = \sum_{j=1}^{n-1}\langle \mathbf{W}_{su}, i_1\cdots i_j\rangle\langle \mathbf{W}_{ut}, i_{j+1}\cdots i_j\rangle.$$

Proof.

Let $\Xi_{st} = X_s\langle \mathbf{W}_{st}, i\rangle + \sum\langle j, X_s'\rangle\langle \mathbf{W}_{st}, ji\rangle$. Then

$$\delta\Xi_{sut} = -X_{su}\langle \mathbf{W}_{ut}, i\rangle + \sum_{j=1}^{d}(\langle j, X's\rangle\langle \mathbf{W}_{su}, j\rangle\langle \mathbf{W}_{ut}, i\rangle - \langle j, X_{su}'\rangle\langle \mathbf{W}_{ut}, ji\rangle)$$

$$= -\left(X_{su} - \sum_{j=1}^{d}\langle j, X_s'\rangle\langle \mathbf{W}_{su}, j\rangle\right)\langle \mathbf{W}_{ut}, i\rangle - \sum_{j=1}^{d}\langle j, X_{su}'\rangle\langle \mathbf{W}_{ut}, ji\rangle$$

□

Rough integration

Proposition (Gubinelli 2003)

Let $\mathbf{X} = (X, X')$ be a controlled rough path and $\phi \in C^2$. Then $\phi(X)$ is also controlled and $\phi(X)'_t = \phi'(X_t)X'_t$.

Proof.

$$\begin{aligned} \phi(X_t) - \phi(X_s) &= \phi'(X_s)(X_t - X_s) + \mathrm{O}(|t-s|^{2\gamma}) \\ &= \sum_{j=1}^{d} \phi'(X_s)\langle j, X'_s\rangle\langle \mathbf{W}_{st}, j\rangle + \mathrm{O}(|t-s|^{2\gamma}) \end{aligned}$$

□

Definition (Rough differential equation)

We say that X is a solution to $\mathrm{d}X_t = \sum f_i(X_t)\,\mathrm{d}W^i_t$, $X_s = x$ if

$$X_{st} = x + \sum_{i=1}^{d} \int_s^t f_i(X_\tau)\,\mathrm{d}\mathbf{W}^i_\tau .$$

Rough integration

Theorem (Davie's expansion; Davie 2008)

X is a solution to $\mathrm{d}X_t = \sum f_i(X_t)\,\mathrm{d}W_t^i$, $X_s = x$ if and only if X is controlled with $\langle j, X_t' \rangle = f_j(X_t)$ and

$$X_t - x = \sum_{i=1}^{d} f_i(x)\langle \mathbf{W}_{st}, i\rangle + \sum_{i,j=1}^{d} Df_j(x)f_i(x)\langle \mathbf{W}_{st}, ij\rangle + \mathrm{o}(|t-s|)$$

Theorem (see e.g. Friz–Victoir 2010)

Suppose $f_i \in C_b^3$. Then the Rough IVP

$$\mathrm{d}X_t^{s,x} = \sum_{i=1}^{d} f_i(X_t)\,\mathrm{d}W_t^i, \quad X_s^{s,x} = x$$

has a unique global solution for any x. Moreover, if $f_i \in C_b^{3+k}$ then the solution flow is in C^k and

$$\mathrm{d}DX_t^{s,x} = \sum_{i=1}^{d} Df_i(X_t^{s,x})DX_t^{s,x}\,\mathrm{d}W_t^i, \quad DX_s^{s,x} = I.$$

Solving the rough transport eq.

Let $g \in C^3$ and let $X^{s,x}$ be the solution flow to the Rough IVP with C_b^6 vector fields.

Define $u_t(x) := g(X_T^{t,x})$. Then $u_t \in C^3$ uniformly in t and moreover, for $s < t$ the formula $u_s(x) = u_t(X_t^{s,x})$ holds.

So

$$u_s(x) - u_t(x) = u_t(X_t^{s,x}) - u_t(x)$$

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So

$$\begin{aligned} u_s(x) - u_t(x) &= u_t(X_t^{s,x}) - u_t(x) \\ &= Du_t(x)(X_t^{s,x} - x) + \frac{1}{2}D^2u_t(x)(X_t^{s,x} - x)^2 + o(|t-s|) \end{aligned}$$

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So

$$\begin{aligned} u_s(x) - u_t(x) &= u_t(X_t^{s,x}) - u_t(x) \\ &= Du_t(x)(X_t^{s,x} - x) + \frac{1}{2} D^2 u_t(x)(X_t^{s,x} - x)^2 + o(|t-s|) \\ &= \sum_{i=1}^d Du_t(x) f_i(x) \langle \mathbf{W}_{st}, i \rangle + \sum_{i,j}^d Du_t(x) Df_j(x) f_i(x) \langle \mathbf{W}_{st}, ij \rangle \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d D^2 u_t(x)(f_i(x), f_j(x)) \langle \mathbf{W}_{st}, i \rangle \langle \mathbf{W}_{st}, j \rangle + o(|t-s|) \\ &= \sum_{i=1}^d \Gamma_i u_t(x) \langle \mathbf{W}_{st}, i \rangle + \sum_{i,j=1}^d \Gamma_{ij} u_t(x) \langle \mathbf{W}_{st}, ij \rangle + o(|t-s|). \end{aligned}$$

Solving the rough transport eq.

Also, since $\Gamma_i u_s(x) = Du_t(X_t^{s,x})DX_t^{s,x} f_i(x)$ we have

$$\Gamma_i u_s(x) - \Gamma_i u_t(x) = Du_t(X_t^{s,x})DX_t^{s,x} f_i(x) - Du_t(x) f_i(x)$$

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$$= Du_t(X_t^{s,x}) f_i(x) - Du_t(x) f_i(x) + \sum_{j=1}^{d} Du_t(X_t^{s,x}) Df_j(X_t^{s,x}) f_i(x) \langle \mathbf{W}_{st}, j \rangle + O(|t-s|^{2\gamma})$$

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&= Du_t(X_t^{s,x}) f_i(x) - Du_t(x) f_i(x) + \sum_{j=1}^{d} Du_t(X_t^{s,x}) Df_j(X_t^{s,x}) f_i(x) \langle \mathbf{W}_{st}, j \rangle + O(|t-s|^{2\gamma}) \\
&= D^2 u_t(x)(X_t^{s,x} - x, f_i(x)) + \sum_{j=1}^{d} Du_t(X_t^{s,x}) Df_j(X_t^{s,x}) f_i(x) \langle \mathbf{W}_{st}, j \rangle + O(|t-s|^{2\gamma})
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&= Du_t(X_t^{s,x}) f_i(x) - Du_t(x) f_i(x) + \sum_{j=1}^{d} Du_t(X_t^{s,x}) Df_j(X_t^{s,x}) f_i(x) \langle \mathbf{W}_{st}, j\rangle + O(|t-s|^{2\gamma}) \\
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&= \sum_{j=1}^{d} D^2 u_t(x)(f_j(x), f_i(x)) \langle \mathbf{W}_{st}, j\rangle + \sum_{j=1}^{d} Du_t(X_t^{s,x}) Df_j(X_t^{s,x}) f_i(x) \langle \mathbf{W}_{st}, j\rangle + O(|t-s|^{2\gamma})
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&= Du_t(X_t^{s,x})f_i(x) - Du_t(x)f_i(x) + \sum_{j=1}^{d} Du_t(X_t^{s,x})Df_j(X_t^{s,x})f_i(x)\langle \mathbf{W}_{st}, j\rangle + O(|t-s|^{2\gamma}) \\
&= D^2 u_t(x)(X_t^{s,x} - x, f_i(x)) + \sum_{j=1}^{d} Du_t(X_t^{s,x})Df_j(X_t^{s,x})f_i(x)\langle \mathbf{W}_{st}, j\rangle + O(|t-s|^{2\gamma}) \\
&= \sum_{j=1}^{d} D^2 u_t(x)(f_j(x), f_i(x))\langle \mathbf{W}_{st}, j\rangle + \sum_{j=1}^{d} Du_t(X_t^{s,x})Df_j(X_t^{s,x})f_i(x)\langle \mathbf{W}_{st}, j\rangle + O(|t-s|^{2\gamma}) \\
&= \sum_{j=1}^{d} \Gamma_{ij} u_t(x)\langle \mathbf{W}_{st}, j\rangle + O(|t-s|^{2\gamma}).
\end{aligned}$$

Solving the rough transport eq.

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\Gamma_i u_s(x) - \Gamma_i u_t(x) &= Du_t(X_t^{s,x})DX_t^{s,x}f_i(x) - Du_t(x)f_i(x) \\
&= Du_t(X_t^{s,x})f_i(x) - Du_t(x)f_i(x) + \sum_{j=1}^{d} Du_t(X_t^{s,x})Df_j(X_t^{s,x})f_i(x)\langle \mathbf{W}_{st}, j\rangle + O(|t-s|^{2\gamma}) \\
&= D^2u_t(x)(X_t^{s,x} - x, f_i(x)) + \sum_{j=1}^{d} Du_t(X_t^{s,x})Df_j(X_t^{s,x})f_i(x)\langle \mathbf{W}_{st}, j\rangle + O(|t-s|^{2\gamma}) \\
&= \sum_{j=1}^{d} D^2u_t(x)(f_j(x), f_i(x))\langle \mathbf{W}_{st}, j\rangle + \sum_{j=1}^{d} Du_t(X_t^{s,x})Df_j(X_t^{s,x})f_i(x)\langle \mathbf{W}_{st}, j\rangle + O(|t-s|^{2\gamma}) \\
&= \sum_{j=1}^{d} \Gamma_{ij}u_t(x)\langle \mathbf{W}_{st}, j\rangle + O(|t-s|^{2\gamma}).
\end{aligned}$$

Naturally, $t \mapsto \Gamma_{ij}u_t(x)$ is in $\mathscr{C}^\gamma$.

Solving the rough transport eq.

Definition (Friz–Hairer 2020)

Let $\gamma \in (\frac{1}{3}, \frac{1}{2})$. A function u is a solution to the rough transport equation if

$$u_s(x) - u_t(x) = \sum_{i=1}^{d} \Gamma_i u_t(x) \langle \mathbf{W}_{st}, i \rangle + \sum_{i,j=1}^{d} \Gamma_{ij} u_t(x) \langle \mathbf{W}_{st}, ij \rangle + o(|t-s|),$$

$$\Gamma_i u_s(x) - \Gamma_i u_t(x) = \sum_{j=1}^{d} \Gamma_{ij} u_t(x) \langle \mathbf{W}_{st}, j \rangle + O(|t-s|^{2\gamma}),$$

$$\Gamma_{ij} u_s(x) - \Gamma_{ij} u_t(x) = O(|t-s|)$$

Theorem (Friz–Hairer 2020)

Suppose $f_i \in C_b^6$, $g \in C^3$. There exists a unique solution to the rough transport equation, in the sense of the previous definition.

General rough paths

Fix $\gamma \in (0,1)$ and let $N := \lfloor \gamma^{-1} \rfloor$.

Definition (Gubinelli 2010, Hairer–Kelly 2015)

A *controlled rough path* is a path $\mathbf{X}\colon [0,T] \to \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2} \oplus \cdots \oplus (\mathbb{R}^d)^{\otimes (N-1)}$ such that

$$\langle v, \mathbf{X}_t \rangle = \sum_{|v'| \le N-|v|} \langle vv', \mathbf{X}_s \rangle \langle \mathbf{W}_{st}, v' \rangle + \mathrm{O}(|t-s|^{(N+1-|v|)\gamma}).$$

Theorem (Hairer–Kelly 2015; Bellingeri 2019; BDFT 2020)

Let $\mathbf{X}$ be controlled by $\mathbf{W}$. The limit

$$\int_s^t X_\tau \, \mathrm{d}\mathbf{W}^i_\tau := \lim_{|\pi| \to 0} \sum_{[a,b] \in \pi} \sum_{|v| \le N-1} \langle v, \mathbf{X}_a \rangle \langle \mathbf{W}_{ab}, vi \rangle$$

exists. We call it the rough integral of X wrt to $\mathbf{W}$.

General rough paths

Proposition (BDFT 2020)

Let $\phi \in C^N$ and $\mathbf{X}$ controlled by $\mathbf{W}$. Then $\phi(X)$ is also controlled by $\mathbf{W}$ and

$$\langle v, \phi(\mathbf{X}_t)\rangle = \sum_{k=1}^{|v|} \frac{1}{k!} \sum_{v=u_1 \sqcup \cdots \sqcup u_k} D^k\phi(X_t)(\langle u_1, \mathbf{X}_s\rangle, \dots, \langle u_k, \mathbf{X}_s\rangle).$$

Theorem (Gubinelli 2010)

Suppose $f_i \in C_b^{N+1}$. Then there exists a unique solution to the Rough IVP.

Theorem (Hairer–Kelly 2015; BDFT 2020)

X is a solution to the Rough IVP if and only if it is controlled with $\langle v, \mathbf{X}_t^{s,x}\rangle = F_w(X_t^{s,x})$ and

$$X_t^{s,x} = \sum_{|v|\le N} F_v(x)\langle \mathbf{W}_{st}, v\rangle + o(|t-s|)$$

where $F_\varnothing = \mathrm{id}$ and $F_{iv}(x) = DF_v(x)f_i(x)$.

Theorem (BDFT 2020)

Suppose $f_i \in C_b^{N+k+1}$. Then the solution flow of the Rough IVP is of class C^k and

$$\mathrm{d}D^m X_t^{s,x}(h_1,\ldots,h_m) = \sum_{i=1}^{d} \sum_{P \in \mathcal{P}(m)} D^{|P|} f_i(X_t^{s,x})(D^{|B_1|} X_t^{s,x} h_{B_1}, \ldots, D^{|B_p|} X_t^{s,x} h_{B_p})\, \mathrm{d}W_t^i$$
$$= \sum_{i=1}^{d} Df(X_t^{s,x}) D^m X_t^{s,x}(h_1,\ldots,h_m) + \gamma_t^m$$

with initial conditions $X_s^{s,x} = x$, $DX_s^{s,x} = I$ and $D^m X_s^{s,x} = 0$ for $m \geq 2$.

Corollary (BDFT 2020)

The partial derivatives $\partial^\alpha X_t^{s,x}$ admit the following Davie-type expansion:

$$\partial^\alpha X_t^{s,x} = \sum_{|v| \leq N} \partial^\alpha F_v(x) \langle \mathbf{W}_{st}, v \rangle + \mathrm{o}(|t-s|).$$

General transport eq.

Proposition (BDFT 2020)

For a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$,

$$\Gamma_v \phi(x) = \sum_{k=1}^{|v|} \frac{1}{k!} \sum_{v = u_1 \sqcup \cdots \sqcup u_k} D^k \phi(x)(F_{u_1}(x), \ldots, F_{u_k}(x)).$$

Definition (BDFT 2020)

A function u is a solution to the rough transport equation driven by $\mathbf{W}$ if for all words with $|v| \leq N$,

$$\Gamma_v u_s(x) = \sum_{|v'| \leq N - |v|} \Gamma_{vv'} u_t(x) \langle \mathbf{W}_{st}, v' \rangle + O(|t-s|^{(N+1-|v|)\gamma}).$$

Theorem (BDFT 2020)

Let $f \in C^{2N+2}$ *and* $g \in C^{N+1}$. *There exists a unique solution of the rough transport equation in the sense of the previous definition.*

Continuity eq.

Definition (BDFT 2020)

By a solution of the *rough continuity equation* we mean a path of measures μ_t such that for every $s < t$ we have

$$\mu_t(\Gamma_v \phi) = \sum_{|v'| \leq N - |v|} \mu_s(\Gamma_{vv'} \phi) \langle \mathbf{W}_{st}, v' \rangle + \mathrm{O}(|t-s|^{(N+1-|v|)\gamma}),$$

uniformly in $\phi \in C_b^{N+1}$.

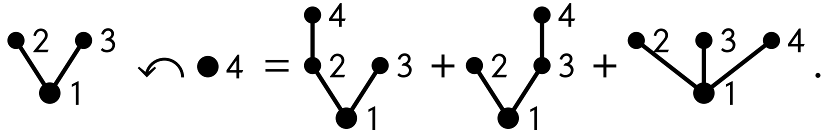
Theorem (BDFT 2020)

Let $f_i \in C^{2N+2}$ and ν a measure on $\mathbb{R}^n$. There is a unique solution to the continuity equation, in the sense of the previous definition. It is explicitly given by

$$\mu_t(\phi) = \int_0^t \phi(X_t^{0,x}) \, \nu(\mathrm{d}x).$$

Brace algebras

Let $\mathfrak{T}$ denote the linear span of decorated rooted trees.
Consider the grafting operator $\curvearrowleft : \mathfrak{T} \otimes \mathfrak{T} \to \mathfrak{T}$. For example

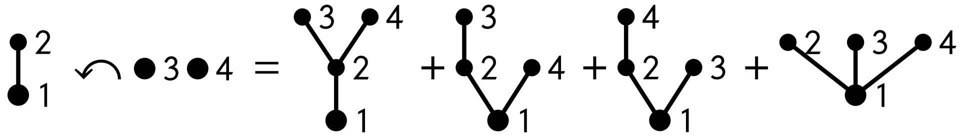


It is well known that $(\mathfrak{T}, \curvearrowleft)$ is a (right) pre-Lie algebra.

The symmetric algebra $S(\mathfrak{T})$ corresponds to forests. Grafting is extended (Guin–Oudom, 2004) to $\mathfrak{T} \otimes S(\mathfrak{T})$ recursively:

$$\tau \curvearrowleft \sigma\tau' := (\tau \curvearrowleft \sigma) \curvearrowleft \tau' - \tau \curvearrowleft (\sigma \curvearrowleft \tau').$$

Example:



Theorem (Guin–Oudom 2004)

This extension turns $\mathfrak{T}$ into a symmetric brace algebra.

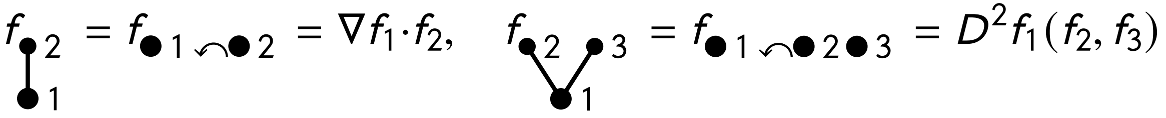
Let $f, g : \mathbb{R}^d \to \mathbb{R}^d$ be vector fields. Define $f \triangleright g := \nabla f \cdot g$.
Then $(\mathfrak{X}, \triangleright)$ is a right pre-Lie algebra. Indeed,

$$f \triangleright (g \triangleright h) - (f \triangleright g) \triangleright h = -D^2 f(g, h).$$

Proposition (Chapoton–Livernet 2001)

There is a unique pre-Lie morphism $\mathfrak{T} \to \mathfrak{X}$ such that $\bullet_i \mapsto f_i$.

Example:



Lemma (Bayer–Friz–T. 2020)

Let $\tau_1, \ldots, \tau_n \in \mathfrak{T}$ and $\sigma = \sigma_1 \cdots \sigma_k \in S(\mathfrak{T})$. Then

$$f_{B_i^+(\tau_1 \cdots \tau_n) \curvearrowleft \sigma} = \sum_{r=n}^{n+k} \frac{1}{(r-n)!} \sum_{\rho\gamma=\sigma} D^r f_i(f_{\tau_1 \curvearrowleft \rho_1}, \ldots, f_{\gamma_1}, \ldots)$$

References

[1] C. Bellingeri. *Quasi-geometric rough paths and rough change of variable formula*. arXiv: 2009.00903 [math.PR]

[2] C. Bellingeri, A. Djurdjevac, P. K. Friz, N. Tapia. *Transport and continuity equations with (very) rough noise*. arXiv: 2002.10432 [math.AP]

[3] K.-T. Chen. *Iterated integrals and exponential homomorphisms*. Proc. London Math. Soc. (3) **s3–4** (1954), pp. 502–512.

[4] A. M. Davie. *Differential Equations Driven by Rough Paths: An Approach via Discrete Approximation*. Appl. Math. Res. eXpress (AMRX) (2008).

[5] J. Diehl, P. K. Friz, W. Stannat. *Stochastic partial differential equations: a rough paths view on weak solutions via Feynman–Kac*. Ann. Fac. Sci. Toulous Math. (6) **26(4)** (2017), pp. 911–947.

[6] D. Feyel, A. de La Pradelle. *Curvilinear integrals along enriched paths*. Electron. J. Probab. **11** (2006), paper no. 34, pp. 860–892.

[7] M. Fliess. *Fonctionelles causales non linéaires et indéterminées non commutatives*. Bull. Soc. Math. France **109** (1981), pp. 3–40.

[8] P. K. Friz, M. Hairer. A course on Rough Paths. 2nd. ed. (2020). Universitext, Springer, Cham.

[9] P. K. Friz, N. Victoir. Multidimensional stochastic processes as Rough Paths. Cambridge Studies in Advanced Mathematics 120, 1st. ed. (2010). CUP.

[10] M. Gubinelli. *Controlling Rough Paths*. J. Funct. Anal. **216(1)** (2003), pp. 86–140.

[11] M. Gubinelli. *Ramification of Rough Paths*. J. Differential Equations **248(4)** (2010), pp. 693–271.

[12] J.-M. Oudom, D. Guin. *On the Lie envelopping algebra of a pre-Lie algebra*. arXiv:math/0404457 [math.QA]

[13] M. Hairer, D. Kelly. *Geometric versus non-geometric Rough Paths*. Ann. Inst. Henri Poincaré Probab. Stat. **51(1)** (2015), pp. 207–251.

[14] T. Lyons. *Differential equations driven by rough signals*. Rev. Mat. Iberoam. **14** (1998), pp. 215–310.

[15] T. Lyons, N. Victoir. *An extension theorem to Rough Paths*. Ann. Inst. Henri Poincaré Anal. Non Linéaire **24(5)** (2007), pp. 835–847.

[16] D. Manchon. Hopf algebras in renormalization. In *Handbook of Algebra* vol. 5, 2008, pp. 365–427.

[17] R. Ree. *Lie elements and an algebra associated with shuffles.* Ann. of Math. **68** (1958), pp. 211–220.