THE GENERALISED RIEMANN PROBLEM: THE BASIS FOR VERY-HIGH ORDER SCHEMES

E F Toro

Laboratory of Applied Mathematics
Faculty of Engineering
University of trento, Italy
E-mail: toro@ing.unitn.it
Website: `http//www.ing.unitn.it/oro`

ABSTRACT

Advection-reaction type partial differential equations model a wide variety of phenomena in several disciplines in physics, chemistry, environmental sciences, geometry, financial mathematics and many others. Generally, these non-linear inhomogeneous equations must be solved in complicated multi-dimensional domains and thus analytical solutions are only available under very special circumstances. Situations in which exact solutions are available include Riemann problems. Conventionally, the Riemann problem for a system of conservation laws in two independent variables x and t is the initial value problem for the system with initial conditions consisting of two *constant* states separated by a discontinuity at the origin $x = 0$; for background see [2] and [5], for example. Ben-Artzi and Falcovitz [1] and others have generalised the concept of Riemann problem by admiting initial conditions that are *linear* functions in x, separated by a discontinuity at $x = 0$. We introduce the notation GRP_1 to denote this generalisation of the conventional Riemann problem, denoted by GRP_0.

Here we extend further the concept of generalised Riemann problem in two respects, the first concerns initial conditions and the second concerns the type of governing equations. As to initial conditions, we admit k-th order polynomial functions of x and denote the corresponding generalised Riemann problem by GRP_k. Concerning the governing differential equations, in addition to pure non-linear advection, we include here, non-stiff, reaction-like terms.

We propose a semi-analytical method of solution of the generalised Riemann problem GRP_∞ for non-linear advection-reaction partial differential equations [7]. The method gives the solution at $x = 0$ at a time τ, assumed to be sufficiently small, in terms of a time Taylor series expansion at $x = 0$ about $t = 0$. The leading term in this expansion is the exact solution of a conventional non-linear Riemann problem, GRP_0, with piece-wise constant initial conditions. All remaining terms in the expansion have coefficients that are time derivatives of the solution; these time derivatives are replaced by spatial derivatives by repeated use of the differential equations, a technique known as the Cauchy-Kowalewski procedure, or the Lax-Wendroff procedure [3]. It is then shown that all spatial derivatives in the expansion obey inhomogeneous advection equations. As derivative values are required at time $t = 0^+$, all source terms in these evolution equations for all-order spatial derivatives vanish and one only requires the solution of the conventional Riemann problems with piece-wise constant initial conditions for each of these advection equations for the derivatives. In summary, we reduce the solution of the generalised advection-reaction Riemann problem GRP_k to that of solving a sequence of $k+1$ conventional Riemann problems GRP_0 for homogeneous advection equations, the solutions of which are pieced together to determine the complete solution.

The approach can be extended further to solve the *general initial value problem* for advection-reaction partial differential equations with piece-wise smooth initial conditions for sufficiently small times. At any position x where the initial condition is smooth the GRP_k solution procedure applies trivially. In the case in which the initial condition includes more than one discontinuity, one places local origins at these positions to define local problems GRP_k and applies the solution method as described. In this manner, for a sufficiently short time, one obtains the solution everywhere at any point x that is sufficiently far away from points of initial discontinuities. This restriction is necessary to preserve time smoothness in the local Riemann problems. The solution technique presented here can be used for assessing the performance of

numerical methods intended for solving complicated problems involving both non-linear advection and reaction. Given its local character, our solution procedure has also the potential for providing sub-cell resolution in numerical computation procedures for the general initial-boundary value problem.

Finally, our solution procedure can be used to construct numerical methods of very high order of accuracy in both space and time to solve the general initial-boundary value problem. The solution procedure for GRP_k has already been applied to the ADER method [6] using ENO and WENO spatial reconstruction procedures; see for example [4]. The methodology of the GRP_k can also be applied to the Discontinuous Galerkin Finite Element Method, and this is the subject of current investigations. In this talk we shall present high-order numerical results obtained from GRP_k and the ADER approach.

References

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