Forum for matematiske perler

Tid:	Fredag 13.11.09, kl. 12.15 – 13.00
Sted:	Lunsjrommet 13. etg., Sentralbygg II, Gløshaugen
Foredragsholder:	Christian Skau
Tittel:	**Gauss and Riemann versus elementary mathematics**

(Som vanlig blir det servert kaffe og kaker som kan nytes under foredraget).

Abstract: At the 1987 International Mathematical Olympiad for high-school students (which was held in Cuba), one of the problems was:

"If the polynomial $f_p(x) = x^2 + x + p$ yields prime numbers for $x = 0, 1, 2, \ldots, \left[\sqrt{p/3}\right]$, then it yields prime numbers for $x = 0, 1, 2, \ldots, p-2$." [Note that this is optimal since $f_p(p-1) = p^2$, and so is not a prime number.]

The students were most likely not aware of the fact that the prime producing property of $f_p(x)$ is intimately connected to a famous problem raised by Gauss in his classic book on number theory, *Disquisitiones Arithmeticae*, from 1801. The problem became known as the class number one problem, and more than 150 years elapsed before it was finally solved.

The Riemann hypothesis (dating from November 3, 1859, almost exactly 150 years ago to the day!) is arguably the most famous open problem in mathematics. Starting with the function $\zeta(s)$, and the Euler product formula defined for $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

one can extend $\zeta(s)$ to an analytic function in the whole complex plane (except at $s = 1$, where it has a simple pole). The location of the zeros of the zeta function $\zeta(s)$ yields highly interesting information about the distribution of the prime numbers. The Riemann hypothesis says simply that all the (non-trivial) zeros of $\zeta(s)$ lie on the so-called critical line $\mathrm{Re}(s) = 1/2$. Very surprisingly the Riemann hypothesis turns out to be equivalent to a seemingly completely elementary question: How do the Farey fractions differ from the regularly spaced points in the interval $[0, 1]$? Specifically, let $\mathcal{F}_n$ be the Farey fractions of order n (i.e. $\mathcal{F}_n$ is the set of fractions $0 < p/q \le 1$, such that the denominator q is less or equal to n), ordered according to size, and let A_n denote the number of terms in $\mathcal{F}_n$. For example,

$$\mathcal{F}_5 = \left\{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} = 1\right\}; \quad A_5 = 10.$$

Since the A_n terms of $\mathcal{F}_n$ are unequally spaced through the interval from 0 to 1, they will differ from the A_n equally spaced points

$$\mathcal{P}_{A_n} = \left\{\frac{1}{A_n}, \frac{2}{A_n}, \ldots, \frac{A_n}{A_n} = 1\right\}$$

The Riemann hypothesis is equivalent to a specific behaviour as n tends to infinity of how $\mathcal{F}_n$ deviates from $\mathcal{P}_{A_n}$. (This will be made precise in the talk.)

Our aim is to give an explanation or, at least, an indication, why the elementary mathematics described above are connected with deep mathematical results associated with Gauss and Riemann, respectively.

Finally $-$ as a unifying feature in this context $-$ we will mention an intriguing connection that exists between the generalized Riemann hypothesis and the general class number problem of Gauss.