1777-1855

1826-1866

Gauss and Riemann
versus
elementary mathematics

Problem at the 1987 International Mathematical Olympiad:

Given that the polynomial $f(x) = x^2 + x + p$ yields primes for $x = 0, 1, 2, \ldots, \left[\sqrt{\frac{p}{3}}\right]$. Show that it yields primes for $x = 0, 1, 2, \ldots, p - 2$.

Example

$p = 41, \left[\sqrt{\frac{41}{3}}\right] = 3.$

$f(0) = 41,\ f(1) = 43,\ f(2) = 47,\ f(3) = 53.$

So $f(4), \ldots, f(39)$, i.e. $61, 71, 83, 97, 113, \ldots, 1601$ are all primes.

(First observed by Euler in 1772.)

Theorem

Let p be prime, and let $f_p(x) = x^2 + x + p$. Then the following conditions are equivalent:

1. $p = 2, 3, 5, 11, 17, 41$
2. $f_p(x)$ *is prime for* $x = 0, 1, 2, \ldots, \left[\sqrt{\frac{p}{3}}\right]$.
3. $f_p(x)$ *is prime for* $x = 0, 1, 2, \ldots, p - 2$.
4. $\mathbb{Q}(\sqrt{1 - 4p})$ *has class number one.*

The prime numbers between 5000 *and* 7498.

$$41 < \sqrt{\frac{p}{3}} < 50$$

5003	5189	5387	5527	5693	5861	6053	6229	6367	6577	6761	6947	7103	7307
5009	5197	5393	5531	5701	5867	6067	6247	6373	6581	6763	6949	7109	7309
5011	5209	5399	5557	5711	5869	6073	6257	6379	6599	6779	6959	7121	7321
5021	5227	5407	5563	5717	5879	6079	6263	6389	6607	6781	6961	7127	7331
5023	5231	5413	5569	5737	5881	6089	6269	6397	6619	6791	6967	7129	7333
5039	5233	5417	5573	5741	5891	6091	6271	6421	6637	6793	6971	7151	7349
5051	5237	5419	5581	5743	5903	6101	6277	6427	6653	6803	6977	7159	7351
5059	5261	5431	5591	5749	5923	6113	6287	6449	6659	6823	6983	7177	7369
5077	5273	5437	5623	5779	5927	6121	6299	6451	6661	6827	6991	7187	7393
5081	5279	5441	5639	5783	5939	6131	6301	6469	6673	6829	6997	7193	7411
5087	5281	5443	5641	5791	5953	6133	6311	6473	6679	6833	7001	7207	7417
5099	5297	5449	5647	5801	5981	6143	6317	6481	6689	6841	7013	7211	7433
5101	5303	5471	5651	5807	5987	6151	6323	6491	6691	6857	7019	7213	7451
5107	5309	5477	5653	5813	6007	6163	6329	6521	6701	6863	7027	7219	7457
5113	5323	5479	5657	5821	6011	6173	6337	6529	6703	6869	7039	7229	7459
5119	5333	5483	5659	5827	6029	6197	6343	6547	6709	6871	7043	7237	7477
5147	5347	5501	5669	5839	6037	6199	6353	6551	6719	6883	7057	7243	7481
5153	5351	5503	5683	5843	6043	6203	6359	6553	6733	6899	7069	7247	7487
5167	5381	5507	5689	5849	6047	6211	6361	6563	6737	6907	7079	7253	7489
5171		5519		5851		6217		6569		6911		7283	
5179		5521		5857		6221		6571		6917		7297	

Gauss ("Disquisitiones Arithmeticae", 1801) stated that $\mathbb{Q}(\sqrt{D})$, where $D < 0$, has class number one if
$D = -1, -2, -3, -7, -11, -19, -43, -67, -163$.

He asked if there were other imaginary quadratic fields that had this property. It was shown independently by Baker and Stark in 1966 that Gauss' list was complete.

p	2	3	5	11	17	41
$1-4p$	-7	-11	-19	-43	-67	-163

Quadratic binary forms

$f(x,y) = ax^2 + bxy + cy^2 \quad ; a,b,c \in \mathbb{Z}$
Discriminant $D = b^2 - 4ac$.

Example

$$f(x,y) = x^2 + y^2 \quad ; D = -4.$$

Equivalent form to $f(x,y)$:

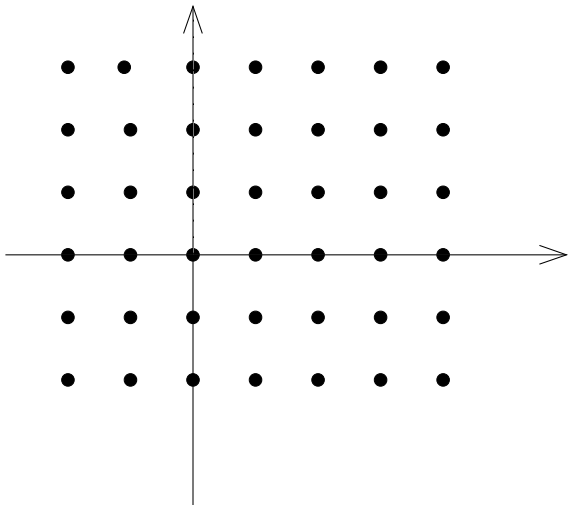
$$\tilde{f}(x,y) = f(\alpha x + \beta y, \gamma x + \delta y),$$

where $\alpha\delta - \beta\gamma = 1$, i.e.

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x + \beta y \\ \gamma x + \delta y \end{bmatrix} \in \mathbb{Z}^2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \delta X - \beta Y \\ -\gamma X + \alpha Y \end{bmatrix} \in \mathbb{Z}^2$$



Example

$\tilde{f}(x, y) = 13x^2 - 42xy + 34y^2 = f(2x - 3y, -3x + 5y)$, where $f(x, y) = x^2 + y^2$.

If $f(x,y)$ and $\tilde{f}(x,y)$ are equivalent forms, they have the same discriminant.

Clearly $f(x,y)$ and $\tilde{f}(x,y)$ have the same set of values as x and y run through the integers $\mathbb{Z}$.

Conversely, let $g(x,y)$ and $h(x,y)$ be two forms with the same discriminant, and assume that $g(x,y)$ and $h(x,y)$ take the same set of values as x and y run through $\mathbb{Z}$. Then $g(x,y)$ is equivalent to $h(x,y)$. (We assume the forms are irreducible and primitive, i.e. the g.c.d. of the coefficients is 1.)
(Schering (1859), Chowla (1966), Perlis (1979))

$$\begin{array}{ccc} ax^2+bxy+cy^2 & \sim & AX^2+BXY+CY^2 \\ \updownarrow & & \updownarrow \\ (a,b,c) & \sim & (A,B,C) \end{array}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

$$D = b^2 - 4ac = B^2 - 4AC < 0.$$

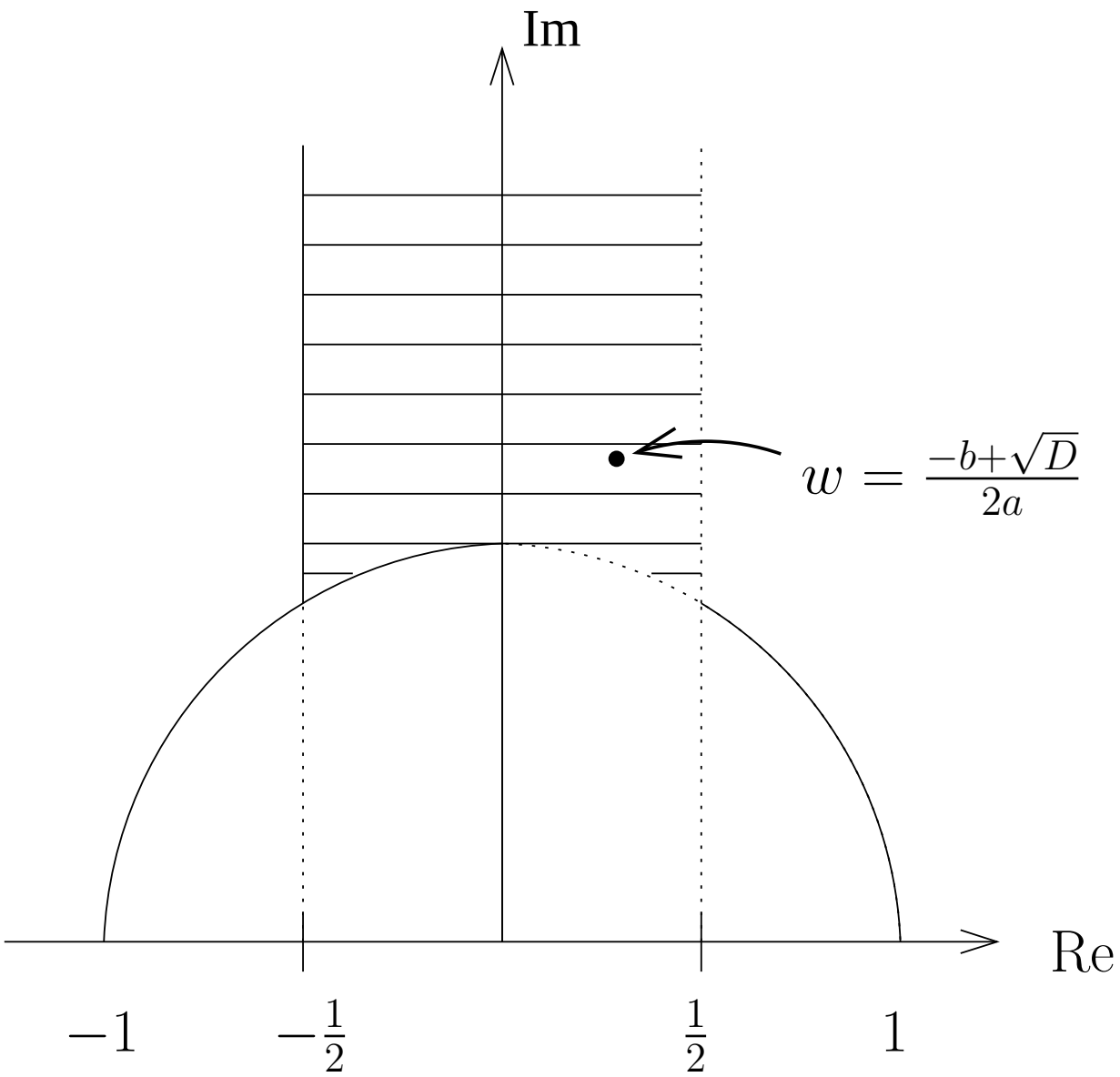
Associate the complex numbers

$$w = \frac{-b + \sqrt{D}}{2a}, \qquad w' = \frac{-B + \sqrt{D}}{2A}.$$

Then

$$w = \frac{\alpha w' + \beta}{\gamma w' + \delta}.$$

A form is called reduced if its associated complex number w lies in the fundamental domain for the modular group $\mathrm{SL}(2, \mathbb{Z})$. A form is reduced if (a, b, c) satisfies $-a < b \leq a < c$ or $0 \leq b \leq a = c$. Every form is equivalent to a unique reduced form.


Im
$w = \frac{-b+\sqrt{D}}{2a}$
Re
-1
$-\frac{1}{2}$
$\frac{1}{2}$
1

Let $h(D)$ denote the number of inequivalent forms $ax^2 + bxy + cy^2$ of discriminant $D = b^2 - 4ac (\equiv 0 \text{ or } 1 (\mod 4))$. Gauss proved that $h(D)$ is finite for all D.

Conjecture

The number of negative discriminants $D < 0$ which have a given class number h is finite. In other words, $h(D) \to \infty$ as $D \to -\infty$.

Fact

$h(D) = 1 \Leftrightarrow \mathbb{Q}(\sqrt{D})$ has class number one,
i.e. the integers A_D in

$$\mathbb{Q}(\sqrt{D}) = \left\{ \alpha + \beta\sqrt{D} \middle| \alpha, \beta \in \mathbb{Q} \right\}$$

is a unique factorization domain $\Leftrightarrow$ A_D is a PID.

In our case, where D is of the form $1 - 4n$,

$$A_D = \left\{ \frac{a + b\sqrt{D}}{2} \middle| a, b \in \mathbb{Z} \text{ with same parity} \right\}.$$

(A_D is the set of solutions to $x^2 + cx + d = 0$; $c, d \in \mathbb{Z}$, that lie in $\mathbb{Q}(\sqrt{D})$.

There are 5 imaginary quadratic fields $\mathbb{Q}(\sqrt{D})$ which are Euclidean w.r.t the norm $N(a+\beta\sqrt{D}) = (a+\beta\sqrt{D})(a-\beta\sqrt{D}) = \alpha^2 - D\beta^2$: $D = -1, -2, -3, -7, -11$.

The units of A_D are ± 1 if $D < 0$, except for

$$D = -1\{\pm 1, \pm \mathrm{i}\} \quad \text{and} \quad D = -3\left\{\pm 1, \pm\rho, \pm\rho^2 \,\middle|\, \rho = \frac{-1+\sqrt{-3}}{2}\right\}.$$

Example

Example of a non-unique factorization domain:

$$A_{-5} = \left\{a + b\sqrt{-5} \,\middle|\, a, b \in \mathbb{Z}\right\} \subset \mathbb{Q}(\sqrt{-5}).$$

$$6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5}).$$

G. Rabinovitch, "Eindeutigkeit der Zerlegung in Primzahlfaktoren in quadratischer Zahlkörpern". Proceedings 5th Congress of Mathematicians, Cambridge 1912.

Theorem

$\mathbb{Q}(\sqrt{1-4p})$ has class number one $\Leftrightarrow$ $f_p(x) = x^2 + x + p$ is a prime for $x = 0, 1, 2, \ldots, p-2$.

We will prove that if $f_p(x)$ is a prime for $x = 0, 1, 2, \ldots, \left[\sqrt{\frac{p}{3}}\right]$, then $\mathbb{Q}(\sqrt{1-4p})$ has class number one.

Proof

We must show that there exists a unique reduced form $ax^2 + bxy + cy^2$, i.e.

$$-a < b \leq a < c \quad \text{or} \quad 0 \leq b \leq a = c \qquad (*)$$

such that the discriminant D equals $1 - 4p$, i.e.

$$D = b^2 - 4ac = 1 - 4p. \qquad (**)$$

Since D is odd, b must be odd, say $b = 2k + 1$. By $(**)$ we get $(2k + 1)^2 - 4ac = 4k^2 + 4k + 1 - 4ac = 1 - 4p$, and so

$$ac = k^2 + k + p = f_p(k). \qquad (***)$$

The inequalities $(*)$, together with $(**)$, yield $-1 - \left[\sqrt{\frac{p}{3}}\right] \leq k \leq \left[\sqrt{\frac{p}{3}}\right]$. Since $f_p(-k) = f_p(k - 1)$, we get by $(***)$ and our assumption that ac is a prime. By $(*)$ we get finally that $a = 1$, $b = -1$ and so by $(**)$, $c = p$. So the unique reduced form is $x^2 - xy + py^2$. Hence the class number of $\mathbb{Q}(\sqrt{1 - 4p})$ is one.

The historical continuity of mathematics (Felix Klein (1894))

Mathematics develops and progresses as old problems are being understood and clarified by means of new methods.
Simultaneously, as a better and deeper understanding of the old questions is thus obtained, new problems naturally arise.

$$\frac{1}{1-\frac{1}{p^s}} = 1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \frac{1}{(p^3)^s} + \cdots \quad (\mathrm{Re}(s) > 1).$$

$$\begin{aligned}\prod_p \frac{1}{1-\frac{1}{p^s}} &= (1 + \frac{1}{2^s} + \frac{1}{(2^2)^s} + \frac{1}{(2^3)^s} + \cdots) \\ &\quad \cdot (1 + \frac{1}{3^s} + \frac{1}{(3^2)^s} + \frac{1}{(3^3)^s} + \cdots) \\ &\quad \cdot (1 + \frac{1}{5^s} + \frac{1}{(5^2)^s} + \frac{1}{(5^3)^s} + \cdots) \cdot \cdots \\ &= \sum_{i_1 < i_2 < \cdots < i_k} \frac{1}{(p_{i_1}^{r_{i_1}} \cdots p_{i_k}^{r_{i_k}})^s} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)\end{aligned}$$

$$\zeta(s) = 2^s \cdot \pi^{s-1} \cdot \sin\frac{\pi s}{2} \cdot \Gamma(1-s) \cdot \zeta(1-s).$$

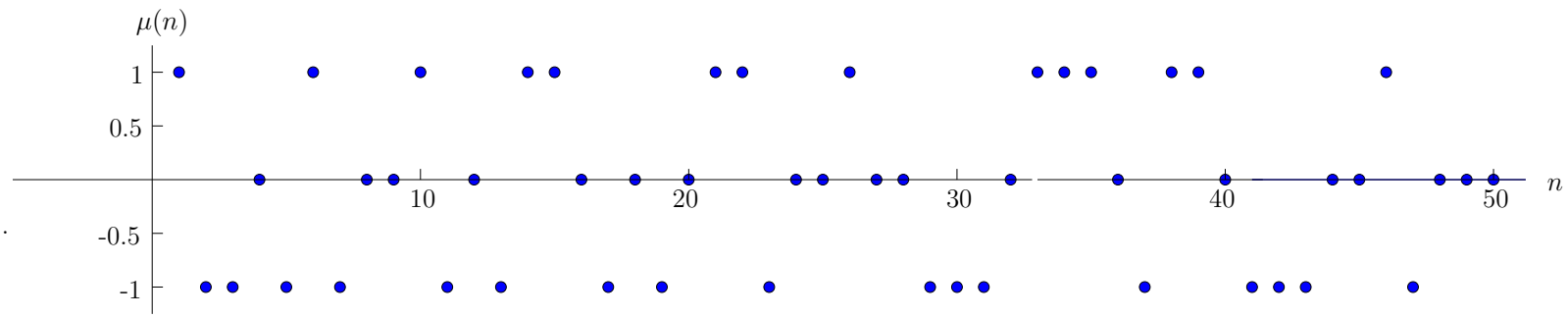
The functional equation for the Riemann zeta-function $\zeta(s)$.

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots$$

$$= 1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{6^s} - \frac{1}{7^s} + \frac{1}{10^s} - \cdots = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

$$= s\int_1^{\infty} M(x)x^{-s-1}\,\mathrm{d}x, \text{ where } M(x) = \sum_{n \le x} \mu(n) \text{ (see next page).}$$

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not square-free} \\ 1 & \text{if } n \text{ is square-free with an even number of distinct prime factors} \\ -1 & \text{if } n \text{ is square-free with an odd number of distinct prime factors} \end{cases}$$

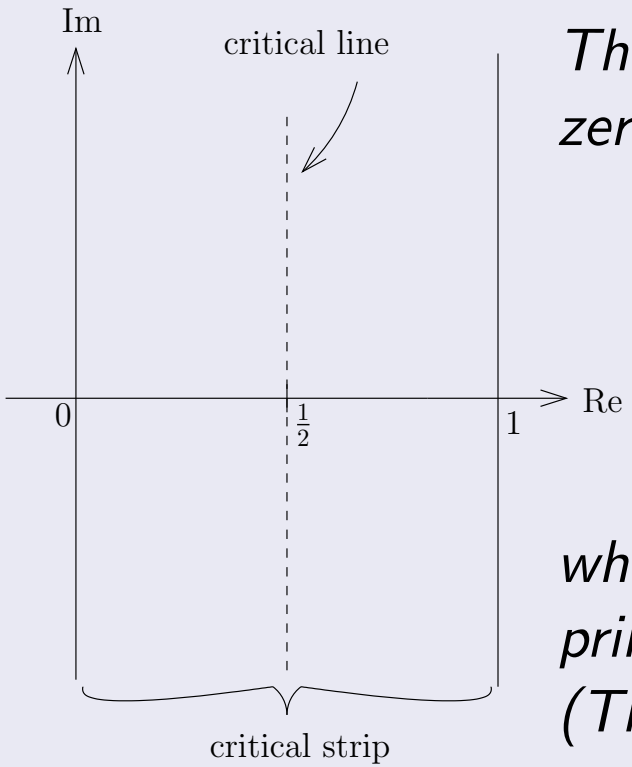


$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\mu(n)}{n^s}.$$

By Abel's partial summation we get:

$$\begin{aligned}
\sum_{n=1}^{N} \frac{\mu(n)}{n^s} &= \sum_{n=1}^{N} \frac{M(n) - M(n-1)}{n^s} \\
&= \frac{M(N)}{N^s} - \sum_{n=1}^{N-1} M(n)\{f(n+1) - f(n)\} \\
&= \frac{M(N)}{N^s} - \sum_{n=1}^{N-1} M(n) \int_n^{n+1} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^s}\right) \mathrm{d}x \\
&= \frac{M(N)}{N^s} + s\int_1^{N+1} M(x)x^{-s-1}\mathrm{d}x \\
&\underset{N\to\infty}{\longrightarrow} s\int_1^{\infty} M(x)x^{-s-1}\mathrm{d}x
\end{aligned}$$

Theorem (Hadamard and de la Vallée Poussin (1896))



The Riemann zeta function $\zeta(s)$ *has no zeros on the line* $\mathrm{Re}(s) = 1$.

$$\Updownarrow$$

$$\frac{\pi(x)}{\frac{x}{\log x}} \longrightarrow 1 \text{ as } x \to \infty,$$

where $\pi(x)$ *denotes the number of primes* $\leq x$.
(The Prime Number Theorem)

The Riemann Hypothesis (Riemann (1859))

The (non-trivial) zeros of the zeta-function $\zeta(s)$ lie on the "critical" line $\mathrm{Re}(s) = \frac{1}{2}$.

Littlewood(1912)

The Riemann hypothesis is equivalent to $M(x) = o(x^{1/2+\epsilon})$ for all $\epsilon > 0$, i.e.

$$\frac{M(x)}{x^{1/2+\epsilon}} \longrightarrow 0 \text{ as } x \to \infty, \text{ where } M(x) = \sum_{n \leq x} \mu(n).$$

Farey fractions $\mathcal{F}_n$ of order n:

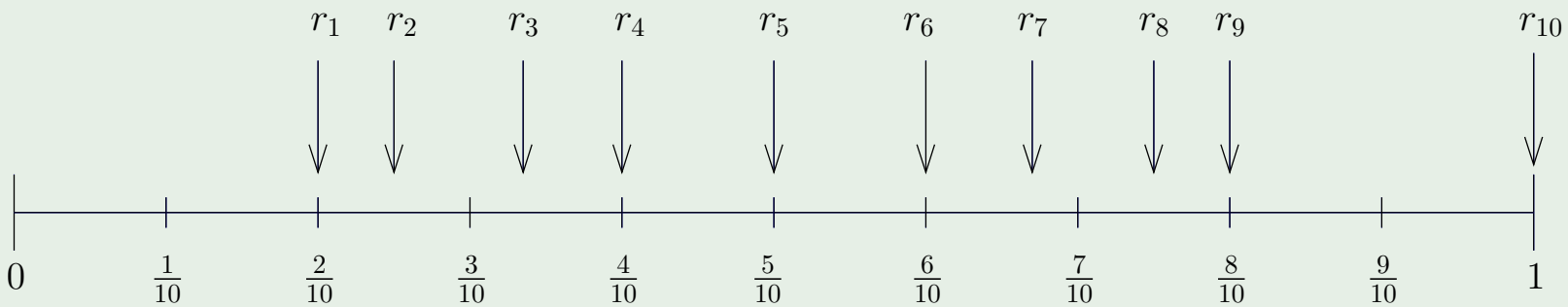
$$\mathcal{F}_n = \left\{ \frac{p}{q} \middle| 0 < \frac{p}{q} \leq 1,\ q \leq n \right\}.$$

Let $A_n = |\mathcal{F}_n|$ and let $\mathcal{F}_n = \left\{ r_1 < r_2 < r_3 < \cdots < r_{A_n} = \frac{1}{1} = 1 \right\}$.
Let

$$\delta_1 = r_1 - \frac{1}{A_n},\ \delta_2 = r_2 - \frac{2}{A_n},\ \delta_3 = r_3 - \frac{3}{A_n}, \ldots, \delta_{A_n} = r_{A_n} - \frac{A_n}{A_n}\ (= 0).$$

Example

$$N = 5,\ \mathcal{F}_5 = \left\{ \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} = 1 \right\},\ A_5 = 10.$$



Theorem (Franel and Landau (1924))

The Riemann hypothesis is true

$$\Updownarrow$$

$$|\delta_1| + |\delta_2| + \cdots + |\delta_{A_n}| = o(n^{1/2+\epsilon}) \text{ for all } \epsilon > 0 \text{ as } n \to \infty.$$

Proof of $\Uparrow$

Let $f : [0,1] \to \mathbb{C}$. Then

$$\sum_{\nu=1}^{A_n} f(r_\nu) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} f\left(\frac{j}{k}\right) M\left(\frac{n}{k}\right) \qquad (*)$$

Comment This is the key identity, where the rather irregular operation of summing f over the Farey fractions can be expressed more regularly using the function M and a double sum (we show this later).

Formula $(*)$ applied to $f(x) = \mathrm{e}^{2\pi \mathrm{i} x}$ gives

$$\sum_{\nu=1}^{A_n} \mathrm{e}^{2\pi \mathrm{i} r_\nu} = \sum_{k=1}^{\infty} \sum_{j=1}^{k} \mathrm{e}^{2\pi \mathrm{i} \frac{j}{k}} M\left(\frac{n}{k}\right) = M(n)$$

since $$\sum_{j=1}^{k} \mathrm{e}^{2\pi \mathrm{i} \frac{j}{k}} = \begin{cases} 0 & \text{if } k \neq 1 \\ 1 & \text{if } k = 1 \end{cases}$$

Proof of $\Uparrow$ (cont.)

Set $A = A_n$. Then

$$\begin{aligned} M(n) &= \sum_{\nu=1}^{A} \mathrm{e}^{2\pi \mathrm{i} r_\nu} \\ &= \sum_{\nu=1}^{A} \mathrm{e}^{2\pi \mathrm{i}\left[\frac{\nu}{A}+\delta_\nu\right]} \\ &= \sum_{\nu=1}^{A} \mathrm{e}^{2\pi \mathrm{i}\frac{\nu}{A}} \left[\mathrm{e}^{2\pi \mathrm{i}\delta_\nu} - 1\right] + \sum_{\nu=1}^{A} \mathrm{e}^{2\pi \mathrm{i}\frac{\nu}{A}} \end{aligned}$$

so

$$|M(n)| \leq \sum_{\nu=1}^{A} \left|\mathrm{e}^{2\pi \mathrm{i}\delta_\nu} - 1\right| + 0 \leq 2 \sum_{\nu=1}^{A} |\sin(\pi\delta_\nu)| \leq 2\pi \sum_{\nu=1}^{A} |\delta_\nu|.$$

Lemma

With f as in the theorem

$$\sum_{\nu=1}^{A_n} f(r_\nu) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} f\left(\frac{j}{k}\right) M\left(\frac{n}{k}\right) \qquad (*)$$

Proof:

Let $0 < p \leq q$, p and q relatively prime. The term

$$f\left(\frac{p}{q}\right) = f\left(\frac{2p}{2q}\right) = f\left(\frac{3p}{3q}\right) = \cdots$$

occurs on the right hand side of $(*)$ with the coefficient

$$M\left(\frac{n}{q}\right) + M\left(\frac{n}{2q}\right) + M\left(\frac{n}{3q}\right) + \cdots = \begin{cases} 1 & \text{if } q \leq n \\ 0 & \text{if } q > n \end{cases} \qquad (**)$$

and so we get $(*)$.

Proof (cont.)

$$M(x) = \sum_{k \leq x} \mu(k) = \sum_{k} \mu(k) D\left(\frac{x}{k}\right)$$

where

$$D(x) = \begin{cases} 1 & \text{for } x \geq 1 \\ 0 & \text{for } x < 1 \end{cases}.$$

By the Möbius inversion (see next page) we get $D(x) = \sum_k M\left(\frac{x}{k}\right)$.
Set $x = \frac{n}{q}$, and we get $(**)$. □

Theorem

Möbius inversion Let $f, g : \mathbb{R}_+ \to \mathbb{C}$.

$$f(x) = \sum_{n \leq x} g\left(\frac{x}{n}\right) \Leftrightarrow g(x) = \sum_{n \leq x} \mu(n) f\left(\frac{x}{n}\right)$$

Proof:

$\Rightarrow$:

$$\begin{aligned}
\sum_{n \leq x} \mu(n) f\left(\frac{x}{n}\right) &= \sum_{n \leq x} \mu(n) \sum_{m \leq \frac{x}{n}} g\left(\frac{x}{mn}\right) \\
&= \sum_{mn \leq x} \mu(n) g\left(\frac{x}{mn}\right) \\
&= \sum_{r \leq x} g\left(\frac{x}{r}\right) \sum_{n|r} \mu(n) \\
&= g\left(\frac{x}{1}\right) = g(x)
\end{aligned}$$

Proof $\Leftarrow$

$\Leftarrow$:

$$\begin{aligned}\sum_{n\leq x} g\left(\frac{x}{n}\right) &= \sum_{n\leq x}\sum_{m\leq \frac{x}{n}} \mu(m) f\left(\frac{x}{mn}\right) \\ &= \sum_{mn\leq x} \mu(m) f\left(\frac{x}{mn}\right) \\ &= \sum_{r\leq x} f\left(\frac{x}{r}\right)\sum_{m|r}\mu(m) \\ &= f\left(\frac{x}{1}\right) = f(x)\end{aligned}$$

$$\left(\text{We have used that } \sum_{d|n}\mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}\right)$$

Dirichlet's L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1,$$

where $\chi : \mathbb{Z}/q\mathbb{Z} \to \{\text{roots of unity}\}$ is a character. $L(s,\chi)$ can be extended to a holomorphic function on $\mathbb{C}$ (if χ is not the trivial character).

The Generalized Riemann Hypothesis (GRH)

The (non-trivial) zeros of $L(s,\chi)$ lie on the line $\mathrm{Re}(s) = 1/2$.

Theorem (Dirichlet (1839))

$L(1,\chi) = 2\pi h(D)/(const.\sqrt{|D|})$, where $D = b^2 - 4ac < 0$ is the discriminant of an imaginary quadratic form $ax^2 + bxy + cy^2$, and $h(D)$ is the class number. (χ is a primitive character mod D.)

Theorem 1 (Hecke (1918))

If GRH is true, then $h(D) \to \infty$ *as* $D \to -\infty$.

Theorem 2 (Heilbronn (1934))

If GRH is false, then $h(D) \to \infty$ *as* $D \to -\infty$.

Corollary

Gauss' conjecture about the class number for imaginary quadratic forms is true.


D. Goldfeld.
Gauss' class number problem for imaginary quadratic fields.
Bulletin AMS 13 (1985), 23-37.


P. Ribenboim.
Euler's famous prime generating polynomial and the class number of imaginary fields.
L'Enseign Math. 34 (1988), 23-42.


J. Oesterlé.
Le problème de Gauss sur le nombre de classes.
L'Enseign Math. 34 (1988), 47-67.


J. Sandbakken.
Et merkverdig problem av Euler og den endelige avklaringen av de tre klassiske problemer stilt av Gauss.
NORMAT 38 (1990), 101-111.


H. M. Edwards.
Riemann's zeta function.
Dover Publications, 1974.


W. Narkiewicz.
The development of prime number theory.
Springer Verlag, 2000.