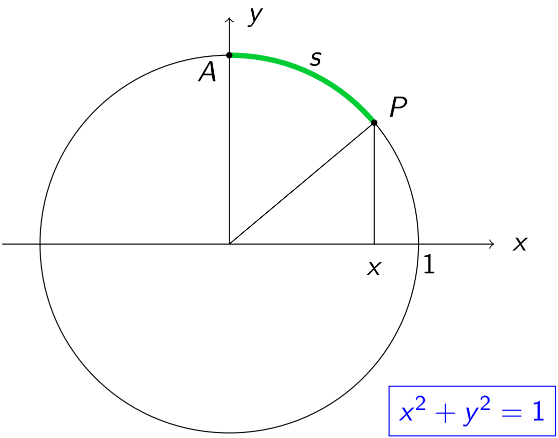
The lemniscate and Abel's discovery of complex multiplication for elliptic curves.

Christian Skau

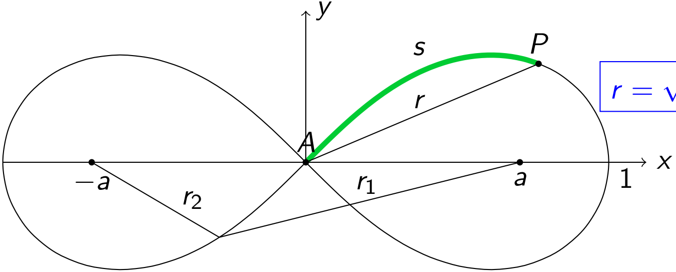
"It appears to me that if one wants to make progress in mathematics, one should study the masters and not the pupils."
(Remark by Abel in his mathematical notebook)

The (unit) circle



$$s = \mathrm{arc}(AP) = \int_0^x \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x \qquad x = \sin s$$

$$s = \operatorname{arc}(AP) = \int_0^r \frac{\mathrm{d}r}{\sqrt{1-r^4}} \qquad\qquad r = \phi(s)$$



$$\boxed{r = \sqrt{\cos 2\theta}}$$

$$\boxed{(x^2+y^2)^2 = x^2 - y^2}$$

$$r_1 \cdot r_2 = a^2 \quad (2a^2 = 1)$$

David Hilbert (1932)

"The theory of complex multiplication of elliptic curves is not only the most beautiful part of mathematics, but of all science."

Born Aug. 5 — 1802

1815

1816

1817

1818 — New math teacher (Bernt Michael Holmboe)

1819

Pupil at the Cathedral School in Christiania (later Oslo)

1820 — Death of Abel's father

1821 — "Proof" of solvability (sic!) of the quintic

Examen Artium

1822

Letter from Degen to Hansteen; May 21 1821

"It is difficult for me to suppress a wish that the time and mental exertion which Herr Abel expends on such a topic – to my view somewhat sterile subject– could better be applied to a topic whose development would have the greatest consequences for analysis and mechanics. I refer to the elliptic transcendentals. A serious investigator with suitable qualifications for research of this kind would by no means be restricted to the many beautiful properties of these most remarkable functions, but could discover a Strait of Magellan leading into wide expanses of a tremendous analytic ocean."

1822

Entrance exam to the university

1823 — Visits professor Degen in Copenhagen

Preparation for the travel to Göttingen and Paris – in isolation in Christiania

1824

"Anni mirabiles" (The miraculous years)

Discoveries:
1) Abel's integral equation
2) Unsolvability of the quintic
3) Elliptic functions
4) The addition theorem

1825

Berlin and Crelle: "Journal für die reine und Angewandte Mathematik."

1826

Abel's foreign trip to Berlin and Paris

Paris and the disappearance of the Paris Memoir.

1827 — Avoids travelling to Göttingen and Gauss.

Hectic work period in Christiania

1828

Development of the theory of elliptic functions.
(Abel-Jacobi "competition".)
Theory of equations.

1829

January 6: Last manuscript – proof of the addition theorem in its most general form.

†April 6

... "You remember the little paper which treated the inverse functions of the elliptic transcendentals; I asked Degen to read it; but he could not discover any erroneous conclusions or where the mistake may be hidden. God knows how I can pull out of it! “

Letter from Abel to Degen, Christiania den 2den Marts 1824, 1

Siden jeg havde den Ære at nyde Hr. Professorens lærerige Omgang, har jeg især beskjæftiget mig med Integralregningen, og jeg tør maaskee sige ikke uden alt Held. Jeg har gjennemgaaet paa nye den første Deel af Integralregningen, som jeg havde udarbejdet førend jeg rejste til Kjøbenhavn og været sa heldig at give dem en som jeg troer, temmerlig systematisk Form. Jeg havde Haab om at faae den trykt her i Christiania paa Universitetets Bekostning, men da jeg paa samme Tid ble foreslaaet til at erholde et Rejsestipendium, og man desuden holdt for at min Afhandling vilde finde et mere passende Sted i et Videnskabers Selskabs Skrifter, saa blev der ikke noget av den Sag.

Algebraic functions

Definition (Algebraic function (or curve))

$v = v(x)$ is an <u>algebraic function</u> of x if v satisfies an equation

$$\Theta(x, v) = v^n + R_{n-1}(x)v^{n-1} + \cdots + R_1(x)v + R_0(x) = 0$$

where $R_i(x)$ is a rational function of x ($i = 0, \ldots, n-1$).

Example

$$v = \frac{\sqrt{1+x} - \sqrt[3]{1-x}}{\sqrt{1+x} + \sqrt[3]{1-x}}$$

$$\Theta(x, v) = v^4 - 4xv^2 - 4xv - x = 0$$

Example

$\Theta(x, v) = v^5 - v - x = 0$ has no algebraic solution.

Two results by Abel

1. Let y be an algebraic function of x, and assume that $\int y \mathrm{d}x = u$ is an algebraic function of x. Then u is a rational function of x and y.

Example

$$\int \frac{2-x^3}{(1+x^3)\sqrt{1+x^3}}\,\mathrm{d}x = \frac{2x}{\sqrt{1+x^3}}$$

2. $\int \frac{\log x}{x-a}\mathrm{d}x \quad (a \neq 0)$ is not an elementary function.

Preface to 2nd edition of Hardy's book "The integration of functions of a single variable".

…The new edition differs from the first in one important point only. <u>In the first edition I reproduced a proof of Abel's which Mr J. E. Littlewood afterwards discovered to be invalid. The correction of this error has led me to rewrite a few sections (pp. 36-41 of the present edition) completely.</u> The proof which I give now is due to Mr H. T. J. Norton. I am also indebted to Mr Norton and to Mr S. Pollard, for many other criticisms of a less important character.

G. H. H.

January 1916.

Letter from Abel to Degen, Christiania den 2den Marts 1824, 2

Hvad Franskmanden kalder Comparaison des transcendantes har jeg beskjæftiget mig meget med, og da denne staaer i nøie Forbindelse med den *complete algebraiske* Opløsning af Differentialligningen $\phi(y)\mathrm{d}y + \phi(x)\mathrm{d}x = 0$, saa har jeg søgt og fundet en almindelig Methode, for at bestemme den Form som Funktionen ϕ må have, for at det complete Integral af denne Ligning kan udtrykkes under en algebraisk Form. Jeg har forudsat at ϕ betegner en algebraisk Function. Jeg har anvendt den almindelige Methode paa den ved Eulers og Lagranges Undersøgelser bekjendte separerede Differentialligning

$$\frac{\mathrm{d}y}{\sqrt{a + a_1 y + \cdots + a_m y^m}} + \frac{\mathrm{d}x}{\sqrt{a + a_1 x + \cdots + a_m x^m}} = 0.$$

Letter from Abel to Degen, Christiania den 2den Marts 1824, 3

Saavidt mig er bekjendt ere alle de transcendendte Functioner hvis Egenskaber man hidentil har kunnet uttrykke ved en endelig Ligning mellem *to variable* indeholdte under Formen

$$\int \frac{p\mathrm{d}x}{\sqrt{a + a_1x + a_2x^2 + a_3x^3 + a_4x^4}};$$

eller kunne ved en bequem Substitusjon bringes dertil. Jeg er af en Hændelse kommen dertil, at jeg kan udtrykke en Egenskab af alle transcendente Functioner af Formen $\int \phi(z)\mathrm{d}z$, hvor $\phi(z)$ betegner en hvilkensomhelst algebraisk irrational Function af z, ved en saadan Ligning, og det mellem saa mange variable Størrelser som man vil; nemlig dersom man betegner $\int \phi(z)\mathrm{d}z = \psi(z)$ saa kan man altid finde en Ligning af Formen

$$\psi(z_1) + \psi(z_2) + \psi(z_3) + \cdots + \psi(z_n) = \psi(\alpha_1) + \psi(\alpha_2) + \cdots + \psi(\alpha_n) + p$$

hvor z_1, z_2 etc. ere algebraiske Functioner af et hvilketsomhelst Antal variable Størrelser (n er afhængig av dette Antal og i Alm. meget større; α_1, α_2 etc. ere constante Størrelser og p en algebraisk og logarithmisk Function; den er i mange Tilfælde liig Nul). Dette Theorem og en Afhandling derom har jeg tænkt at sende til det franske Institut, da jeg synes det vil udbrede Lys over de transcendente Functioner i det Hele.

Letter from Abel to Degen, Christiania den 2den Marts 1824, 4

Af alle transcendente Functioner har jeg dog især gjort mig Umag med de elliptiske Transcendenter, da jeg alltid har havt for Øje Hr. Professorens Raad til mig i et Brev til Hr. Professor Hansteen. Jeg har derfor bestræbt mig for at udarbejde en "Theorie des transcendantes elliptiques", i hvilket jeg saa vidt jeg har kunnet har søgt at vise den Methode man bør følge for at gjøre denne interessante og overmaade nyttige Theorie saa fuldkommen, som det, efter Analysens nuværende Tilstand, er muligt; men det er som sagt kun et Forsøg og det være langt fra at jeg har den Mening at denne Theorie er fuldkommen, hvilket den vel neppe i Aarhundreder vil blive.

Letter from Abel to Degen, Christiania den 2den Marts 1824, 5

Idet jeg taler om de elliptiske Transcendenter kan jeg ikke afholde mig fra at yttre at Legendre maa besidde en almindeligere Methode end den han i sine Undersøgelser legger for Dagen. Der findes virkelig Stæder hos ham som ere hvad Hr. Krejdal kalder himmelfaldne, og som det neppe synes muelig at komme til uden en almindelig Methode. Jeg troer Hr. Professoren er med mig af den Mening at neppe noget er mere skadelig for Videnskabens Fremme end saaledes at skjule Methoden.[1]

[1] Sammenlign til denne, for opfattelsen av Abels karakter eiendommelige uttalelse, hvad *Gauss* skriver til *Weber* om sine arbeider med de elliptiske funktioner: "Nu er teorien færdig, kun bevisene mangler"!

December 23, 1751: Birth date of elliptic functions

$$\frac{\mathrm{d}x}{\sqrt{1-x^4}} = \frac{\mathrm{d}y}{\sqrt{1-y^4}}$$

Particular solution: $y = -\sqrt{\dfrac{1-x^2}{1+x^2}}$ (Fagnano)

General solution: $x^2 + y^2 + c^2y^2x^2 = c^2 + 2xy\sqrt{1-c^4}$ (Euler)

Euler (1754)

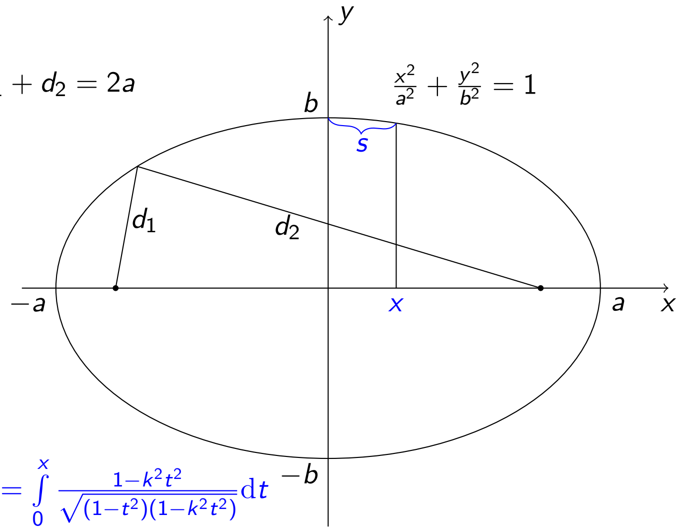
$$\frac{\mathrm{d}x}{\sqrt{A+2Bx+Cx^2+2Dx^3+Ex^4}}+\frac{\mathrm{d}y}{\sqrt{A+2By+Cy^2+2Dy^3+Ey^4}}=0$$

$$\left(\frac{\sqrt{A+2Bx+\cdots}-\sqrt{A+2By+\cdots}}{x-y}\right)^2=2D(x+y)+E(x+y)^2+F$$

$$\int_0^x \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}+\int_0^y \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_0^z \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$z=\frac{x\sqrt{(1-y^2)(1-k^2y^2)}+y\sqrt{(1-x^2)(1-k^2x^2)}}{1-k^2x^2y^2}$$

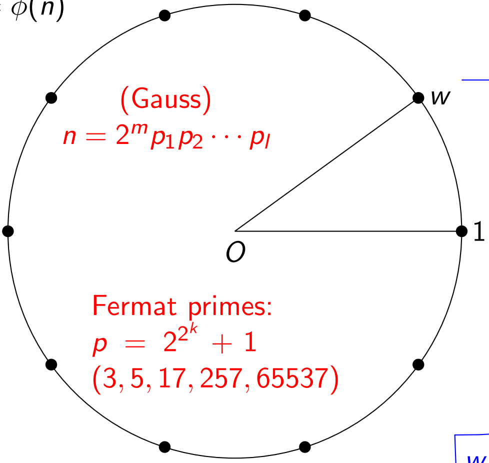
$$d_1 + d_2 = 2a \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$s = \int_0^x \frac{1-k^2t^2}{\sqrt{(1-t^2)(1-k^2t^2)}}\,\mathrm{d}t$$

$(a = 1) \quad k^2 = 1 - b^2,\ 0 \leq k < 1$

$$G = G(\mathbb{Q}(w)/\mathbb{Q}) = (\mathbb{Z}/n\mathbb{Z})^*$$
$$|G| = \phi(n)$$



(Gauss)

$$n = 2^m p_1 p_2 \cdots p_l$$

Fermat primes:

$$p = 2^{2^k} + 1$$

$(3, 5, 17, 257, 65537)$

n–division of the period 2π.

$k = 0$ (the circle)

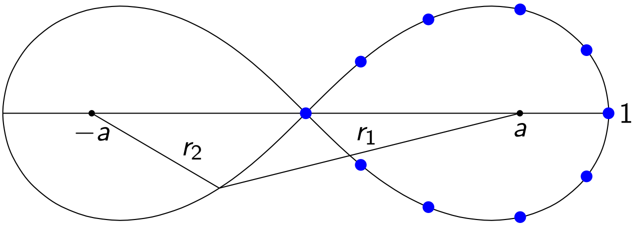
$$\frac{\pi}{2} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)}}$$

$$\frac{2\pi}{n} = \int_0^y \frac{\mathrm{d}t}{\sqrt{(1-t^2)}}$$
$$\Updownarrow$$
$$\sin\frac{2\pi}{n} = y$$
$$\rightsquigarrow$$
$$\mathrm{e}^{\frac{2\pi \mathrm{i}}{n}} = w$$
$$1 = w^n$$
$$\boxed{w^{n-1} + \cdots + w + 1 = 0}$$

(cyclotomic equation)

The lemniscate

$$\underline{r_1 \cdot r_2 = a^2 \quad (2a^2 = 1)}$$



$$\frac{\omega}{2} = \int_0^1 \frac{\mathrm{d}r}{\sqrt{1-r^4}}$$

$$s = \int_0^r \frac{\mathrm{d}r}{\sqrt{1-r^4}}$$

$$2\omega = \frac{1}{M(1,\sqrt{2})} = 5,2411\ldots \ \text{(Gauss)}$$

$$r = \phi(s)$$

n–division of the period ω:

$$\frac{\omega}{n} = \int_0^r \frac{\mathrm{d}r}{\sqrt{1-r^4}} \qquad\qquad r = \phi\left(\frac{\omega}{n}\right)$$

$$\alpha = \int_0^x \frac{\mathrm{d}x}{\sqrt{(1-c^2x^2)(1+e^2x^2)}}$$

$$x = \phi(\alpha)$$

$$f(\alpha) = \sqrt{1-c^2\phi^2(\alpha)}, \qquad F(\alpha) = \sqrt{1+e^2\phi^2(\alpha)}.$$

$$\phi(\alpha+\beta) = \frac{\phi(\alpha)f(\beta)F(\beta)+\phi(\beta)f(\alpha)F(\alpha)}{1+e^2c^2\phi^2(\alpha)\phi^2(\beta)},$$

$$f(\alpha+\beta) = \frac{f(\alpha)f(\beta)-c^2\phi(\alpha)F(\beta)F(\alpha)}{1+e^2c^2\phi^2(\alpha)\phi^2(\beta)},$$

$$F(\alpha+\beta) = \frac{F(\alpha)F(\beta)+e^2\phi(\alpha)f(\beta)f(\alpha)}{1+e^2c^2\phi^2(\alpha)\phi^2(\beta)}.$$

$$\frac{\omega}{2} = \int_0^{\frac{1}{c}} \frac{\mathrm{d}x}{\sqrt{(1-c^2x^2)(1+e^2x^2)}}, \quad \frac{\tilde{\omega}}{2} = \int_0^{\frac{1}{e}} \frac{\mathrm{d}x}{\sqrt{(1-e^2x^2)(1+c^2x^2)}}.$$

$$\phi(x) = \phi(\alpha)$$
$$x = \pm\alpha + m\omega + n\tilde{\omega}\mathrm{i}, \quad (-1)^{m+n} = 1$$

$$f(x) = f(\alpha) : x = \pm\alpha + 2m\omega + n\tilde{\omega}\mathrm{i};$$
$$F(x) = f(\alpha) : x = \pm\alpha + m\omega + 2n\tilde{\omega}\mathrm{i}$$

$$\phi(-\alpha) = -\phi(\alpha)$$

$$\phi(\beta \mathrm{i}) = \mathrm{i}x, \text{ where } \beta = \int_0^x \frac{\mathrm{d}x}{\sqrt{(1+c^2x^2)(1-e^2x^2)}}$$

$$\phi(0) = 0, \quad \phi(\tfrac{\omega}{2}) = \tfrac{1}{c}, \quad \phi(\tfrac{\tilde{\omega}}{2}\mathrm{i}) = \tfrac{1}{e}\mathrm{i}$$

$$f(0) = 1, \quad f(\pm\tfrac{\omega}{2}) = 0,$$

$$F(0) = 1, \qquad\qquad , \quad F(\pm\tfrac{\tilde{\omega}}{2}\mathrm{i}) = 0$$

$$F, f, \phi\left(\frac{\omega}{2} + \frac{\tilde{\omega}}{2}\mathrm{i}\right) = \infty$$

Special case: $c = e = 1$

$$\alpha = \int_0^x \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1+x^2)}} = \int_0^x \frac{\mathrm{d}x}{\sqrt{1-x^4}}\,; \quad x = \phi(\alpha)$$

Then $\underline{\phi(\alpha \mathrm{i}) = \mathrm{i}\phi(\alpha)}$ $\qquad f(\alpha \mathrm{i}) = F(\alpha)$ $\qquad F(\alpha \mathrm{i}) = f(\alpha)$.

$$\frac{\tilde{\omega}}{2} = \frac{\omega}{2} = \int_0^x \frac{\mathrm{d}x}{\sqrt{1-x^4}}$$

Abel ("Recherche ..." (1827/1828))

"The values $\phi(\frac{m\omega}{n})$ can always be expressed algebraically, i.e. by radicals. The values can be expressed by square roots in all cases where n is an integer of the form 2^k, or a prime of the form $2^k + 1$, or a product of any number of factors of these two forms."

Modern formulation:

The associated elliptic curve E has the Weierstrass equation

$$y^2 = 4x^3 - \frac{1}{4}x.$$

The coordinates of the n–division points on E generate an abelian extension K_n of $\mathbb{Q}(i)$, with Galois group

$$G = G(K_n/\mathbb{Q}(i)) \cong (\mathbb{Z}[i]/n\mathbb{Z}[i])^* .$$

It is an easy exercise to show that $(\mathbb{Z}[i]/n\mathbb{Z}[i])^*$ is of order 2^l if and only if n is a power of 2 times a product of distinct Fermat primes.

Bessel to Gauss, Nov. 30, 1827

"As much as the beautiful results of Jacobi on the elliptic transcendentals have pleased me, I regret that he and Abel, who presumably is stronger than Jacobi, have deprived you of many results which have previously been in your possession. Jacobi does not possess the ability to present this theory in as elegant a form as undoubtedly can be done. Here, as well as in every other direction, your work will again be the classical model. Would you not be so good, since your work has long been ripe, to prepare at least some application of it, and thus secure the priority for yourself? "

Gauss to Bessel

"I shall most likely not soon prepare my investigations on the transcendental functions which I have had for many years – since 1798 – because I have many other matters which must be cleared up. Herr Abel has now, as I see, anticipated me and relieved me of the burden in regard to one third of these matters, particularly since he has executed all developments with great stringency and elegance. He has followed exactly the same road which I traveled in 1798; it is no wonder that our results are so similar. To my surprise this extended also to the form and even, in part, to the choice of notations, so several of his formulas appeared as if they were copied from mine. But to avoid every misunderstanding, I must observe that I cannot recall ever having communicated any of these investigations to others."

Sir Michael Atiyah (Oslo, 2004) Abel Prize Recipient 2004 (shared with Isodore Singer)

"I first read the biography of Niels Henrik Abel 50 years ago. Last month I read it again and was better able to appreciate how Abel was really the first modern mathematician. His whole approach, with its generality, its insight and its elegance set the tone for the next two centuries. His early death was a terrible loss - imagine if Mozart had died at a similar age. It has been said that, had Abel lived longer, he would have been the natural successor to the great Gauss: a statement with which I fully concur except for the qualification that Abel was a much nicer man, modest, friendly and likeable. "

Abel's addition theorem

$*$: limits are algebraic functions of $(x_1, y_1) \ldots, (x_m, y_m)$

$$\sum_{i=1}^{m} \int_0^{(x_i, y_i)} R(x,y)\mathrm{d}x = \sum_{i=1}^{g} \int_0^{*} R(x,y)\mathrm{d}x + \text{algebraic/logarithmic terms}$$

$$\Theta(x,y) = y^n + p_{n-1}(x)y^{n-1} + \cdots + p_1(x)y + p_0(x) = 0$$

$$\begin{aligned} p_i(x) &= \text{rational.} \\ R(x,y) &= \text{rational function of } x \text{ and } y. \\ g &= \text{genus of the algebraic function (or curve).} \end{aligned}$$

Example

$$\int_0^{x_1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} + \int_0^{x_2} \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int_0^{x_1\sqrt{1-x_2^2}+x_2\sqrt{1-x_1^2}} \frac{\mathrm{d}x}{\sqrt{1-x^2}}$$

$$\Theta(x,y) = y^2 - (1-x^2)$$

Example

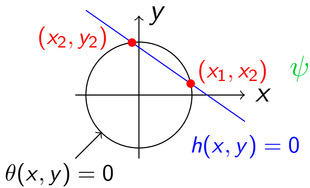
$$\int_0^{x_1} \frac{\mathrm{d}x}{\sqrt{1-x^4}} + \int_0^{x_2} \frac{\mathrm{d}x}{\sqrt{1-x^4}} = \int_0^{\frac{x_1\sqrt{1-x_2^4}+x_2\sqrt{1-x_1^4}}{1+x_1^2x_2^2}} \frac{\mathrm{d}x}{\sqrt{1-x^4}}$$

$$\Theta(x,y) = y^2 - (1-x^4)$$

$$\theta(x,y) = y^2 + (x^2 - 1) = 0 \qquad \Big(\,\Big) : y = \sqrt{1-x^2}\Big)$$

$$h(x,y) = y - ax - b = 0$$

$$\psi(x) = \int_0^x \frac{\mathrm{d}x}{\sqrt{1-x^2}} \Big(= \arcsin x \Big)$$

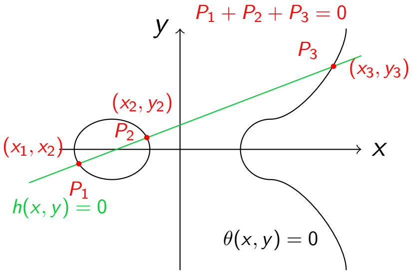


$$\psi(x_1) + \psi(x_2) = \mathrm{i}\log(1+\mathrm{i}a) - \mathrm{i}\log(1-\mathrm{i}a)$$
$$\Big(= \arcsin\Big(\frac{-2a}{a+a^2}\Big)\Big)$$

$$\frac{-2a}{1+a^2} = x_1y_1 + x_2y_2 = x_1\sqrt{1-x_2^2} + x_2\sqrt{1-x_1^2}$$

$$\int_0^{x_1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} + \int_0^{x_2} \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int_0^{x_1\sqrt{1-x_2^2}+x_2\sqrt{1-x_1^2}} \frac{\mathrm{d}x}{\sqrt{1-x^2}}$$

$$\arcsin x_1 + \arcsin x_2 = \arcsin\left(x_1\sqrt{1-x_2^2} + x_2\sqrt{1-x_1^2}\right)$$

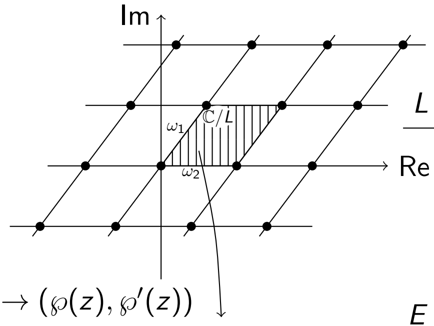


$$\theta(x,y) = y^2 - x(1-x)(1-k^2x) = 0$$
$$h(x,y) = y - ax - b = 0$$
$$\psi(x) = \int_0^x \frac{\mathrm{d}x}{\sqrt{x(1-x)(1-k^2x)}}$$

$$\psi(x_1) + \psi(x_2) + \psi(x_3) = 0$$

$$\int_0^{x_1} \frac{\mathrm{d}x}{\sqrt{x(1-x)(1-k^2x)}} + \int_0^{x_2} \frac{\mathrm{d}x}{\sqrt{x(1-x)(1-k^2x)}} = \int_0^{\frac{\left(\frac{x_1y_2+x_2y_1}{1-k^2x_1x_2}\right)^2}{x_1x_2}} \frac{\mathrm{d}x}{\sqrt{x(1-x)(1-k^2x)}}$$

$$\left(\text{Substitution } : x = z^2 ; \int \frac{\mathrm{d}x}{\sqrt{x(1-x)(1-k^2x)}} = \int \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-k^2z^2)}}\right)$$



$z \to (\wp(z), \wp'(z))$

$$\wp(z+\omega_1) = \wp(z)$$
$$\wp(z+\omega_2) = \wp(z)$$

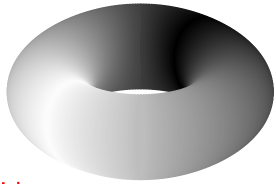
(period lattice)

$$L = \left\{ \omega = m\omega_1 + n\omega_2 \,\middle|\, m, n \in \mathbb{Z} \right\}$$

$$z = \int_0^x \frac{\mathrm{d}t}{\sqrt{4t^3 - g_2 t - g_3}}$$
$$x = \wp(z),\ y = \wp'(z)$$

$$E : \left\{ (x,y) \,\middle|\, y^2 = 4x^3 - g_2 x - g_3 \right\}$$

$$g_2 = 60 \sum_{\omega \neq 0} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \neq 0} \frac{1}{\omega^6}$$

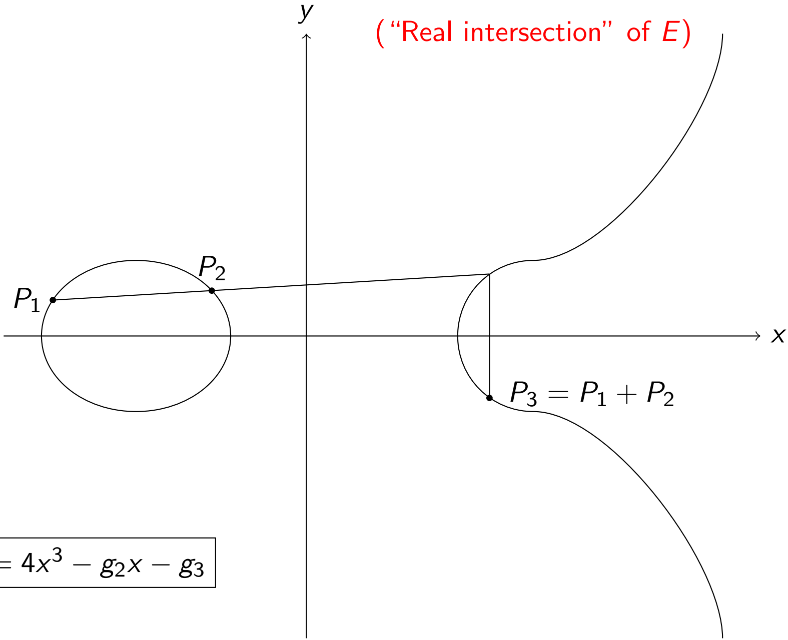


E

(Euclidean parametrization)

In particular, the lemniscate function is associated to $y^2 = 4x^3 - \frac{1}{4}x$.

(“Real intersection” of E)



$$\boxed{y^2 = 4x^3 - g_2 x - g_3}$$

$$\boxed{u = \int_0^x \frac{\mathrm{d}x}{\sqrt{1-x^2}}} \qquad \frac{\pi}{2} = \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}}$$

$$\sin u \overset{\text{def}}{=} x$$

$$\cos u \overset{\text{def}}{=} \sqrt{1-x^2} = \sqrt{1-\sin^2 u} \quad \left(\Rightarrow \sin u = \sqrt{1-\cos^2 u}\right)$$

$$\sin 0 = 0,\ \sin\frac{\pi}{2} = 1,\ \cos 0 = 1,\ \cos\frac{\pi}{2} = 0$$

$$\frac{\mathrm{d}(\sin u)}{\mathrm{d}u} = \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{\frac{\mathrm{d}u}{\mathrm{d}x}} = \sqrt{1-x^2} = \cos u$$

$$\frac{\mathrm{d}(\cos u)}{\mathrm{d}u} = \frac{\mathrm{d}(\cos u)}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{x}{\sqrt{1-x^2}}\cdot\sqrt{1-x^2} = -x = -\sin u$$

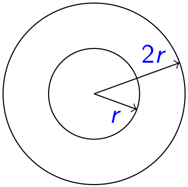
$$(*) \quad \begin{cases} \sin(u+v) & = \sin u\cos v + \cos u \sin v \\ \cos(u+v) & = \cos u \cos v - \sin u \sin v \end{cases}$$

Set $u = v$:

$$\sin 2u = 2\sin u \cos u$$

$$\cos 2u = \cos^2 u - \sin^2 u$$

Assume $\sin u$ and $\cos u$ have power series that converge for $|u| < r$. By



$$\sin 2u = 2 \sin u \cos u$$

$$\cos 2u = \cos^2 u - \sin^2 u$$

one can extend $\sin u$ and $\cos u$ to $|u| < 2r$, and this yield power series for $\sin u$ and $\cos u$ that converge for $|u| < 2r$. Iterating this we get two entire analytic functions $\sin u$ and $\cos u$. Furthermore, $(*)$ still holds.

$$\sin\left(u + \frac{\pi}{2}\right) = \sin u \cos \frac{\pi}{2} + \cos u \sin \frac{\pi}{2} = \cos u$$

$$\sin(u + \pi) = \cos\left(u + \frac{\pi}{2}\right) = \cos u \cos \frac{\pi}{2} - \sin u \sin \frac{\pi}{2} = -\sin u$$

$$\sin(u + 2\pi) = -\sin(u + \pi) = \sin u$$

Similarly,

$$\underline{\cos(u + 2\pi) = \cos u}.$$

$$\frac{\mathrm{d}y}{\sqrt{(1-y^2)(1+e^2y^2)}} = a\frac{\mathrm{d}x}{\sqrt{(1-x^2)(1+e^2x^2)}} \qquad (*)$$

Abel: The separable differential equation $(*)$ has an algebraic solution $y = y(x)$ iff

(i) $a \in \mathbb{Q}$. Then the modulus e^2 can be arbitrary.

(ii) $a = m \pm \sqrt{-n}$, $m, n \in \mathbb{Q}$, $n > 0$. Then e^2 can not be arbitrary but will have to be a root of a specific polynomial, in fact, a *solvable* polynomial.

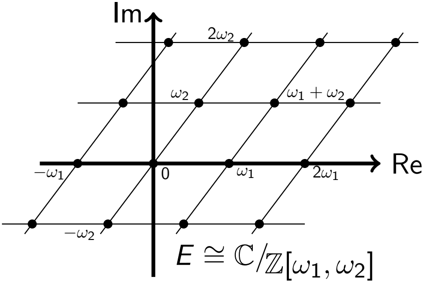
In both cases (i) and (ii), y will actually be a *rational* function of x.

Example

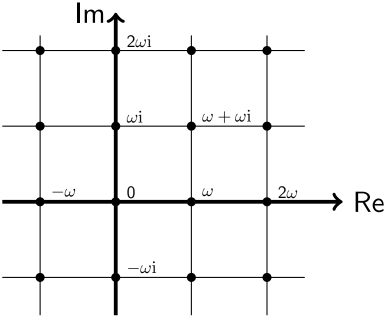
$$a = \sqrt{-1}, \quad e = 1, \quad \boxed{y = \mathrm{i}x}$$

Example

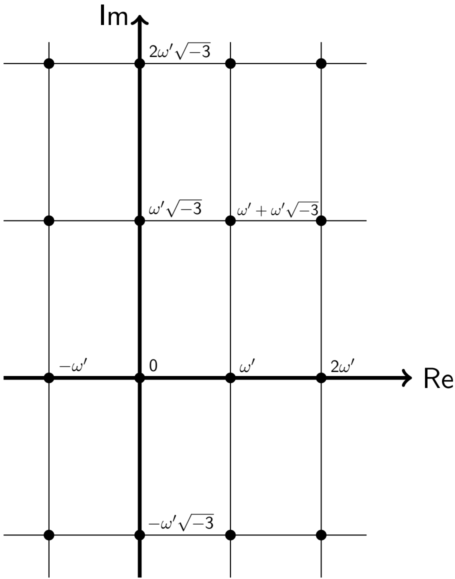
$$a = \sqrt{-3}, \quad e = \sqrt{3} + 2, \quad \boxed{y = \mathrm{i}x\frac{\sqrt{3}-(2+\sqrt{3})x^2}{1+\sqrt{3}(2+\sqrt{3})x^2}}$$



$E \cong \mathbb{C}/\mathbb{Z}[\omega_1, \omega_2]$



$E \cong \mathbb{C}/\mathbb{Z}[\mathrm{i}]\omega \cong \mathbb{C}/\mathbb{Z}[\mathrm{i}]$



$\tilde{E} \cong \mathbb{C}/\mathbb{Z}[\sqrt{-3}]\omega' \cong \mathbb{C}/\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$

Modern definition of complex multiplication

Let E be an elliptic curve, and let $m \in \mathbb{Z}$. The multiplication-by-m map

$$\Theta_m : E \to E$$

defined by $x \to mx$

is an element of $\mathrm{End}(E)$. (Note that $\ker(\Theta_m) = m-$torsion points of E.) So $\mathbb{Z} \subseteq \mathrm{End}(E)$

Definition

E has complex multiplication provided $\mathbb{Z} \subsetneqq \mathrm{End}(E)$.

m–division of the periods of an elliptic function $\phi(z)$ (or the associated elliptic curve E)

Abel connected the solvability of the m–division equation with complex multiplication. He showed that if E has complex multiplication the associated Galois group is abelian (i.e. commutative), and so the equation is solvable. He conjectured that otherwise the equation is not solvable. (The last was only completely clarified 150 years later, mainly in the works of Jean Pierre Serre – the first Abel Prize recipient.)

“Moonshine”, Kronecker’s “Jugendtraum” and complex multiplication

$\mathrm{e}^{\pi\sqrt{163}} = 640320^3 + 743,99999999999925007\cdots$
$\mathbb{Z}\left[\frac{1+\sqrt{-163}}{2}\right]$ is a UFD (“unique factorization domain”).

If τ is any element of $\mathbb{Q}(m+\sqrt{-n})$, then $j(\tau)$ is an algebraic integer, and

$$G\left(\mathbb{Q}(j(\tau),\tau)/_{\mathbb{Q}(\tau)}\right)$$

is abelian.

$$j(\tau) = 1728\frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{1}{q} + 744 + 196884q\cdots; \quad q = \mathrm{e}^{2\pi\mathrm{i}\tau}$$

is the modular function.