Theorems of Hardy and Ramanujan and of Erdös and Kac or *"two brains can be better than one"*

14 September 2012

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Then our question is :
Suppose that

$$10^{99} \leq n < 10^{100}.$$

What can be said about $\omega(n)$ and $\Omega(n)$?

Bertrand's postulate

For any $n \geq 2$ there exists a prime number p, $n < p < 2n$.

Stated by Bertrand in 1845 and checked by him for $n \leq 3000000$

Joseph Bertrand (1822-1900)

Proof

The result was proved by Chebyshev in 1850, the proof appeared also in Edmund Landau's *Handbuch der Lehre von der Verteilung der Primzahlen, 1903.* Two-page proof was given by Ramanujan in 1919.

Pafnuty Lvovich Chebyshev (1821-1894)

Proof "From the Book" I

- If $\binom{2n}{n}$ is divisible by p^k where p is a prime number then (i) $p^k \leq 2n$ (ii) if $p > \sqrt{2n}$ then $k = 1$.

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- $\prod_{p \leq m} p < 4^m$ by induction (from m to $2m + 1$)!

Proof "From the Book" II

$$4^n = (1+1)^{2n} \leq (2n+1)\binom{2n}{n} \leq$$

$$(2n+1) \prod_{p \leq \sqrt{2n}} p^{k(p)} \prod_{p \leq 2n/3} p \leq$$

$$(2n+1)(2n)^{\sqrt{2n}} 4^{2n/3}.$$

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For $n \le 1000$ check by taking a sequence of prime numbers $2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631, 1009$.

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Deterministic answer

Clearly,

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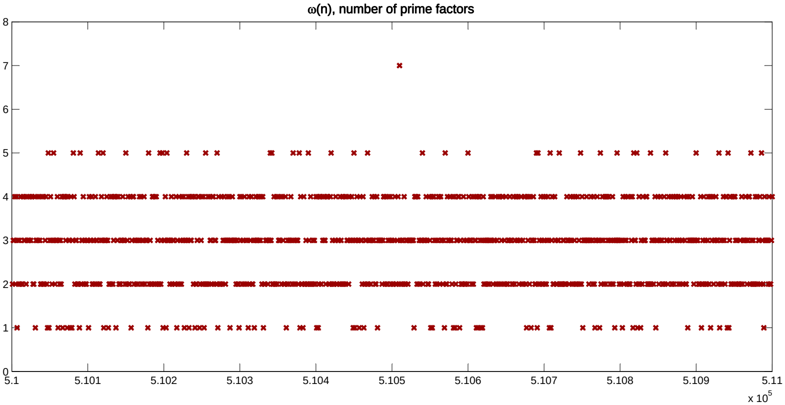
The inequalities are precise but "equalities" occur relatively rare.

What is the typical value of $\omega(n)$ *when* $10^{m-1} \leq n < 10^m$*?*

A picture, thanks to Harald

$\omega(n)$ for n between 510000 and 511000



Average number of prime factors

Let us estimate the average number of prime factors for integers not greater than n,

$$\frac{1}{n}\sum_{k\le n}\omega(k) = \frac{1}{n}\sum_{k\le n}\sum_{p|k}1 =$$

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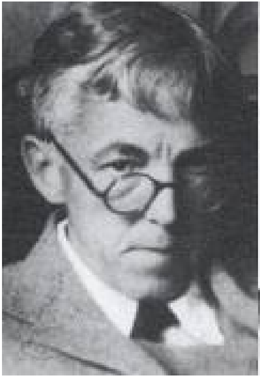
Then

$$E_n(\omega)=\log\log n+O(1).$$

Hardy and Ramanujan




Godfrey H. Hardy,
1877-1947

Srinivasa Ramanujan,
1887-1920

Theorem of 1917: The Law of Large Numbers for $\omega(n)$.

Theorem

Let $g(n) \to \infty$ as $n \to \infty$ and let

$$A_g = \{k : |\omega(k) - \log\log k| \le g(k)\sqrt{\log\log k}\}.$$

Then

$$\lim_{n\to\infty} \frac{|A_g \cap \{1, ..., n\}|}{n} = 1.$$

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So a typical number with 100 digits would have 5.5 **prime divisors.**

Turan's proof from 1934

Idea: estimate the "variance"

$$\sum_{k \le n} (\omega(k) - \log\log n)^2 =$$

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$$\sum_{k\le n}(\omega(k)-\log\log n)^2 =$$

$$\sum_{k\le n}\omega(k)^2-2\log\log n\sum_{k\le n}\omega(k)+$$

$$+n(\log\log n)^2$$

Pál Turán, 1910-1976

Details

$$\sum_{k \le n} \omega(k)^2 = \sum_{k \le n} \left(\sum_{p} \delta_p(k) \right)^2 = \sum_{k \le n} \sum_{p,q} \delta_p(k) \delta_q(k)$$

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$$\le O(n\log\log n)+\sum_{p\ne q}\left[\frac{n}{pq}\right]\le O(n\log\log n)+n\sum_{p,q\le n}\frac{1}{pq}$$

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$$\le O(n\log\log n)+n\left(\sum_{p\le n}\frac{1}{p}\right)^2=n(\log\log n)^2+O(n\log\log n)$$

More details

Finally, $\sum_{k\leq n}(\omega(k) - \log\log n)^2 = O(n \log\log n)$.

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Finally, $\sum_{k\le n}(\omega(k) - \log\log n)^2 = O(n \log\log n)$. Then

$$|\{[n^{1/2}], ..., n\} \setminus A_g| \le \frac{Cn}{\min\{g(m) : \sqrt{n} \le m \le n\}}.$$

And the theorem follows.

Some probability theory: LLN

Weak Law of Large Numbers
Let $X_1, ..., X_j, ...$ *be a sequence of independent random variables.*
For any $\varepsilon > 0$

$$P\left(\frac{1}{N}|\sum_{j=1}^{N}(X_j - E(X_j))| > \varepsilon\right) \to 0.$$

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Idea of the proof: $P\{|Y| > \varepsilon\} < \varepsilon^{-2}E|Y|^2$ plus independence, $Y = (X_1 + ... + X_n)/n$. It gives

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$$P\left(\frac{1}{N}|\sum_{j=1}^{N}(X_j - E(X_j))| > \varepsilon_N\right) \leq \varepsilon_N^{-2}\frac{\sum_{j=1}^{N} Var(X_j)}{N^2}$$

Some probability theory: LLN

Weak Law of Large Numbers (Revised)
Let $X_1, ..., X_j, ...$ be a sequence of independent random variables.

$$P\left(|\sum_{j=1}^{N}(X_j - E(X_j))| > a_N\right) \leq a_N^{-2}\sum_{j=1}^{N} Var(X_j)$$

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Prime factors: $\omega(k) = \sum_p \delta_p(k)$ looks like a sum of independent random variables $X_j = \delta_p$, $1 \le k \le n$,

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$$\sum_{p \leq n} E\delta_p = \sum_{p \leq n} \frac{1}{p} + O(1) = \log\log n + O(1)$$

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Probabilistic reading of the Hardy-Ramanujan theorem

Thinking of δ_p as of independent random variables suggests:

$$P\left(|\sum_{p\le n} \delta_p - \sum_{p\le n} E(\delta_p)| > a(n) \right) \le a(n)^{-2} \log\log n$$

and the Hardy-Ramanujan theorem follows readily.

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Before 1930s the result did not get much attention, but after Turán's work a number of articles followed, including works of Paul Erdös. In the late 1930s Mark Kac noticed the strong resemblance with probability theory and conjectured that $\omega(n)$ enjoys the central limit theorem.

More Probability Theory: CLT

Central Limit Theorem

Let $\{X_j\}$ *be a sequence of independent identically distributed random variables. Let* $\mathcal{E}(X_j) = \mu$ *and* $Var(X_j) = \sigma^2$*. Then*

$$\lim_{n\to\infty} P\left(a < \frac{1}{\sigma\sqrt{n}}\sum_{1}^{n}(X_n - \mu) \le b\right) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-t^2/2}dt.$$

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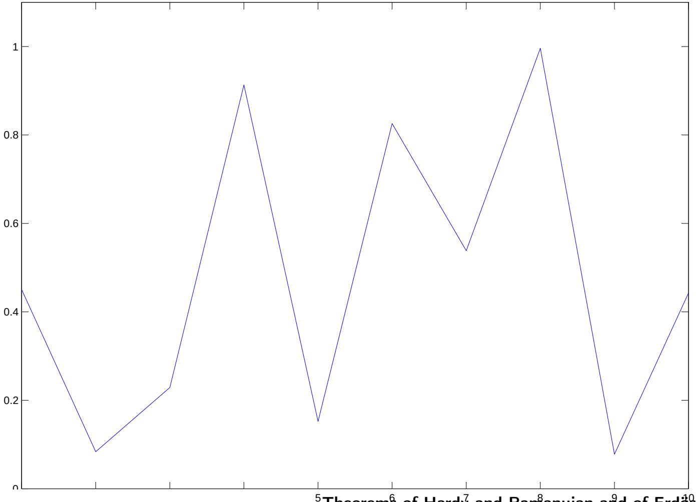
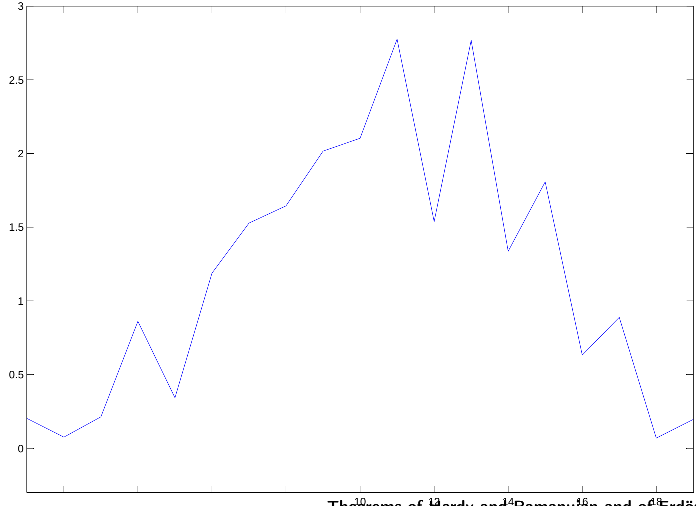
On the level of probability distributions CLT says "for any (positive integrable) function f the sequence of convolutions $f * f * ... * f$ (properly normalized) converges to the bell-shape".

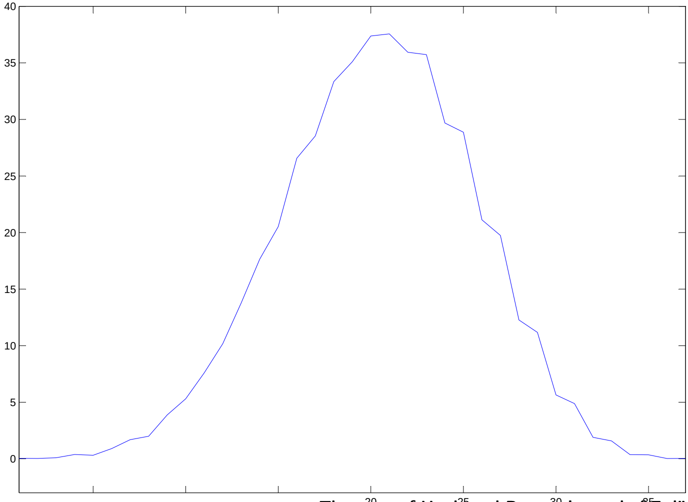
Random peace-wise linear continuous function f_1



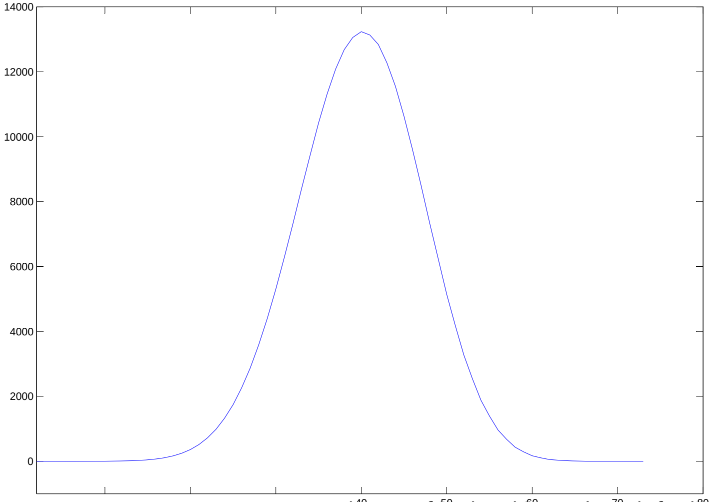
First convolution $f_2 = f_1 * f_1$



Next convolution $f_4 = f_2 * f_2 = f_1 * f_1 * f_1 * f_1$



And yet next convolution $f_8 = f_4 * f_4$



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$$g_n(\tau) = (\hat{f}(0) + \hat{f}'(0)\tau/\sqrt{n} + \hat{f}''(0)\tau^2/(2n) + \ldots)^n = \\ (1 - 2\pi\sigma^2\tau^2/n + \ldots)^n \approx e^{-2\pi\sigma^2\tau^2}$$

CLT for non-identically distributed random variables

Let $\{X_j\}$ be a sequence of independent random variables. Let $\mathcal{E}(X_j) = \mu_j$ and $Var(X_j) = \sigma_j^2$. Then

$$\lim_{n\to\infty} P\left(a < \frac{1}{(\sum_1^n \sigma_j^2)^2} \sum_1^n (X_n - \mu_j) \leq b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-t^2/2} dt.$$

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Khinchin, Kolmogorov, 1933; Feller, Lévy, 1935, 1937

CLT for number of prime factors

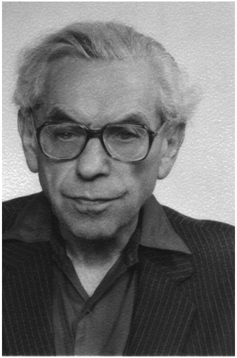
Theorem (Erdös, Kac, 1940)
If $x \leq y$ are real numbers and

$$A_{x,y} = \{m : x \leq \frac{\omega(m) - \log\log m}{(\log\log m)^{1/2}} \leq y\}$$

then

$$D(A_{x,y}) = \lim_{n\to\infty} \frac{|A_{x,y} \cap \{1, \ldots, n\}|}{n} = \frac{1}{\sqrt{2\pi}} \int_x^y e^{-u^2/2} du.$$

Two brains better than one



Paul Erdös, 1913-1996

Mark Kac, 1914-1984

Erdös on the birth of probabilistic number theory

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At this time I could not have formulated even the special case of the Erdös-Kac theorem due to my ignorance in Probability. All these questions were cleared up when Kac and I met in 1939 in Baltimore and Princeton.

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After his lecture we immediately got together. Neither of us completely understood what the other was doing, but we realized that our joint work will give the theorem and to be a little impudent and conceited, Probabilistic Number Theory was born.

Sketch of a proof, I

Statement:

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- 2nd reduction from the original proof of Erdös and Kac. Let $\omega_n(m) = \sum_{p\le a(n)} \delta_p(m)$, where $a(n) = \exp(\log n / \log\log n)$. Then it suffices to show

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  $\log\log a(n) = \log\log n + o((\log\log n)^{1/2})$ but $a(n) = o(n^q)$ for any $q > 0$.

Sketch of a proof, II

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- Moment method, suggested to be used in the proof by Kac in 1949, it is enough to show that for each $s = 1, 2, \ldots$

$$\lim_{n\to\infty} EY_n^s = \frac{1}{\sqrt{2\pi}} \int u^s e^{-u^2/2} du$$

where $Y_n = (\log\log n)^{-1/2}(\omega - \log\log n)$.

Sketch of a proof, III

- One more reduction, Billingsley, 1969. Introduce **independent** random variables X_p such that $X_p = 1$ with probability $1/p$ and $X_p = 0$ with probability $1 - 1/p$. Let $\tilde{Y}_n = (\log\log n)^{-1/2}(\sum_{p \leq a(n)} X_p - \log\log n)$, moments of Y_n converge to those of Φ by CLT, one needs to estimate $E(Y_n^s - \tilde{Y}_n^s)$.

Just compare!

$$\left(\sum_{p\le a(n)} \delta_p\right)^s = \sum_I C_I \prod_{p\in I} \delta_p, \quad \sum C_I = \pi(a(n))^s \le a(n)^s.$$

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And by one more binomial formula

$$E_n(Y_n^s - \tilde{Y}_n^s) \le \frac{(a(n)+\log\log n)^s}{n(\log\log n)^{s/2}}$$

Further results

- Everything above holds for $\Omega(n)$

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Book tips

- P. Billigsley, Probability and Measure
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- G. Tenenbaum, Introduction to Analytic and Probabilistic Number Theory
- N. Alon, J. H. Spencer, The Probabilistic Method