Using elementary ideas from dynamics to get striking results in complex analysis

Berit Stensønes

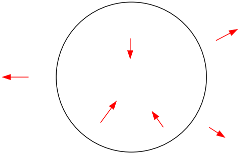
November 16, 2012

$f : C^n \to C^n$
holomorphic (analytic)
$f(\mathbf{0}) = \mathbf{0}$
fixed point

$f^k = f \circ f \circ \cdots \circ f$ k-times
What is the behavior of sequences $\{f^k(z)\}$ for different points z

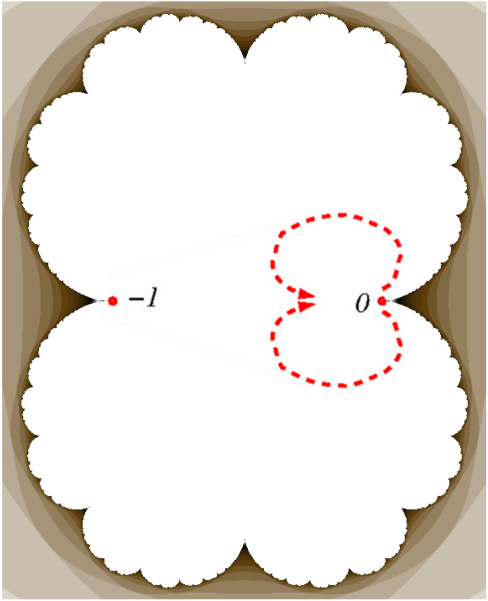
Example

$f(z) = z^2$

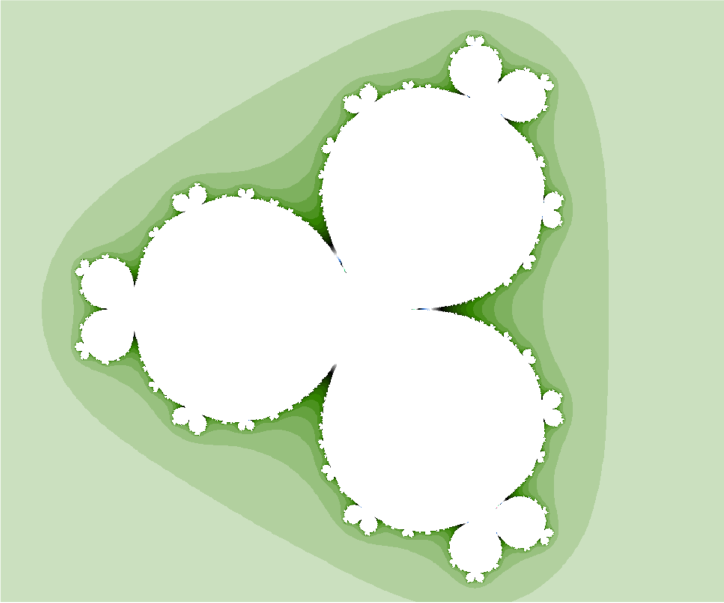


$\lvert z \rvert < 1$	$f^k(z) \to 0$
$\lvert z \rvert > 1$	$f^k(z) \to \infty$
$\lvert z \rvert = 1$	rotation

Example — $f(z) = z(1+z)$



Example — $f(z) = z(1 + z^3)$



Picard's theorem

If f is an analytic function in C (entire),

then $f(C) = \{f(z); z \in C\}$ can avoid at most one point in C.

Automorphisms

$Aut(C^n) = \{F : C^n \to C^n;$ entire, 1-1 and onto$\}$

One variable

$f(z) = az + b,\ a \neq 0.$ If $f(0) = 0$, then $f(z) = az$

The dynamics in this case

$\|a\| < 1$	—	$f^k(z)$ converges to origin at all points
$\|a\| > 1$	—	$f^k(z)$ converges to infinity at all points but the origin
$\|a\| = 1$	—	rotation

Automorphisms in higher dimensions

Two variables

Shears $F(z, w) = (z + f(w), w)$,

Over-shears $F(z, w) = (z, e^{g(z)}w)$

f and g entire functions

Allows us to create much more interesting automorphisms.

Example

Apply the map $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = (w/2, z/2)$

together with the shear $(u, v) \to (u + (2v)^2, v)$

we obtain $F(z, w) = \left(\frac{w}{2} + z^2, \frac{z}{2} \right)$

$F(0, 0) = (0, 0)$

Notice

If $(z, w) \in B((0, 0), 1/4)$, then

$$\|F(z, w)\| \leq \frac{3}{4} \|(z, w)\|$$

so

$$\|F^k(z, w)\| \leq \left(\frac{3}{4}\right)^k \|(z, w)\|$$

and

$$F^k(z, w) \to (0, 0)$$

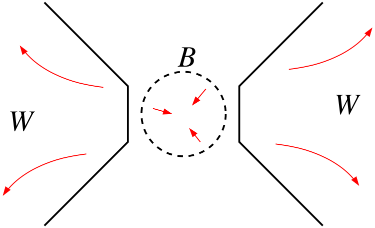
$F(z,w) = \left(\frac{w}{2} + z^2, \frac{z}{2}\right)$ continued

convergence to infinity

If $(z,w) \in W = \{(z,w); |z| > |w|, |z| > 100\}$

then

$|w/2 + z^2| > 99.5|z| > 100$



So $F(W) \subset W$

and also $\|F^k(z,w)\| \to \infty$ as $k \to \infty$ when $(z,w) \in W$.

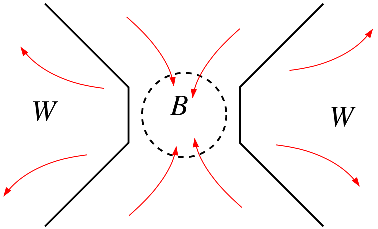
Region of attraction

$$\Omega = \{(z, w); F^k(z, w) \to (0, 0) \text{ as } k \to \infty\}$$

Observe $\Omega = \cup_{k=0}^{\infty} F^{-k}(B((0,0), 1/4)$

NOTE

$W \cap \Omega = \emptyset$



Fatou-Bieberbach domains

A domain $\Omega \subset C^n,\ \Omega \neq C^n$ is called a *Fatou-Bieberbach domain* if there exists a map

$$\Psi^{-1} : C^n \to \Omega$$

Ψ^{-1} is holomorphic and 1-1
or
$\Psi : \Omega \to C^n$ is holomorphic, 1-1 and *onto*

Ψ-factory — Fatou-Bieberbach maps

Start with

$F : C^n \to C^n$ an automorphism

$F(\mathbf{0}) = (\mathbf{0})$

$A = F'(\mathbf{0})$

eigenvalues $|\lambda_1| < \cdots < |\lambda_n| < 1$

Then there exists a small ball B around origin s.t.

$F(B) \subset B$

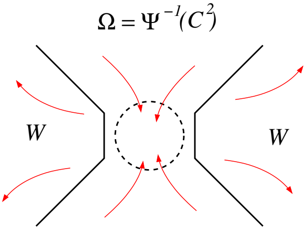
$F^k(z) \to 0$ for $z \in B$

Let $\Omega = \cup_{k=0}^{\infty} F^{-k}(B)$
Then there exists a map $\Psi : \Omega \to C^n$, holomorphic, 1-1, *onto*

No Picard theorem in higher dimension

Example in C^2

$F(z, w) = (w/2 + z^2, z/2)$



$\Omega \cap W = \emptyset$

so Ω is not dense in C^2, but is biholomorphic to C^2

Idea of proof

$A = F'(\mathbf{0})$

$\Psi = \lim_{k\to\infty} A^{-k}F^k$

$\{A^{-k}F^k\}$ converges uniformly om compacts in Ω

so Ψ is 1-1 and holomorphic in Ω

Observe first

$\cup_{k=0}^{\infty} A^{-k}(B) = C^n$

Also

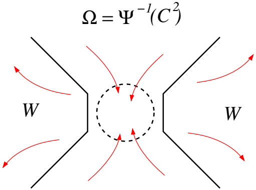
$F(\Omega) = \Omega$

So

$F^k(\Omega) = \Omega$

Hence

$B \subset F^k(\Omega)$ for every k



therefore

$\Psi(\Omega) \supset A^{-k}(B)$ for all k

Hence

$C^n = \cup_{k=0}^{\infty} A^{-k}(B) \subset \Psi(\Omega)$

More general about region of attraction

F an automorphism

Ω a connected component of $\{F^k(p) \to \mathbf{0}\}$

Is Ω biholomorphic to C^n when $\Omega \neq \emptyset$ and open?

Han Peters, Liz Vivas, Erlend Wold:

If $F(\mathbf{0}) = (\mathbf{0})$ is an interior fixed point,

then all eigenvalues of $F'(\mathbf{0})$ have norm less than one.

The previous discussion shows that it is a Fatou-Bieberbach domain.

Fixed point at the boundary of the region of attraction.

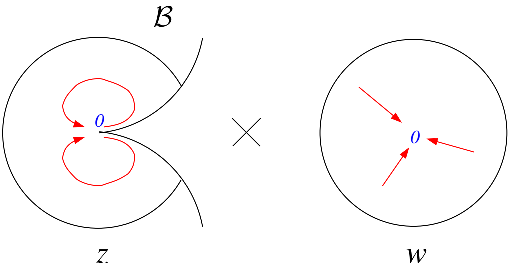
$$F(z,w) = (ze^{ze^{z+w}}, (w + z - ze^{z+w})e^{2ze^{ze^{z+w}}})$$

With the help of good old Taylor, we see that

$$F(z,w) = (z,w)(1+z) + \mathcal{O}(\text{higher order terms})$$

Observe

$$F(0,w) = (0,w)$$
$$F'(\mathbf{0}) = Id$$



$$\Omega_2 = \cup F^{-k}(\mathcal{B})$$

In this example, we can not use $\{A^{-k}F^k\}$ where $A = F'(0)$ to produce the map Ψ.

If we replace A by the second order approximation G of F near $\mathbf{0}$,

then we get Ψ as

$$\Psi = \lim G^{-k}F^k$$

A counterexample in C^3

$$(z, \zeta, w)$$

$$\downarrow$$

$$(z, \zeta, w + z\zeta)$$

$(x,u,y) \to (xe^{y/2}, ue^{y/2}, y)$

$$\downarrow$$

$$\left(ze^{(w+z\zeta)/2}, \zeta e^{(w+z\zeta)/2}, w + z\zeta\right)$$

$(x,u,y) \to (x,u,y-xu)$

$$\downarrow$$

$$\left(ze^{(w+z\zeta)/2}, \zeta e^{(w+z\zeta)/2}, w + z\zeta - e^{w+z\zeta}\right)$$

$(x,u,y) \to (xe^{-y/2}, ue^{-y/2}, y)$

$$\downarrow$$

$$\left(ze^{\frac{1}{2}z\zeta e^{w+z\zeta}}, \zeta e^{\frac{1}{2}z\zeta e^{w+z\zeta}}, w + z\zeta - e^{w+z\zeta}\right)$$

$(x,u,y) \to (x,u,ye^{2xu})$

$$\downarrow$$

$$H(z,\zeta,w) = \left(ze^{\frac{1}{2}z\zeta e^{w+z\zeta}}, \zeta e^{\frac{1}{2}z\zeta e^{w+z\zeta}}, (w + z\zeta - e^{w+z\zeta})e^{2z\zeta e^{z\zeta e^{w+z\zeta}}}\right)$$

$H|_{\{z\zeta=0\}} = Id$ so $\{z\zeta = 0\}$ is not in the region of attraction.
$H'(\mathbf{0}) = Id$

Region of attraction for H:

$$\Omega_3 = \{(z,\zeta,w); (z\zeta, w) \in \Omega_2\}$$

$$(z,\zeta,w) \to (z, z\zeta, w)$$

is a change of coordinates in Ω_3 and

$$(z,\zeta,w) \to (z, \Psi_2(z\zeta, w))$$

sends Ω_3 to $C^* \times C^2$