Geometry: Euclid and beyond

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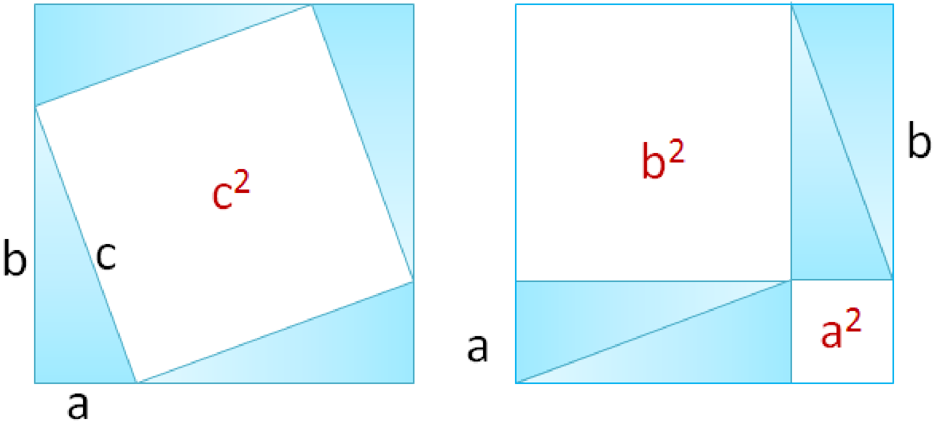
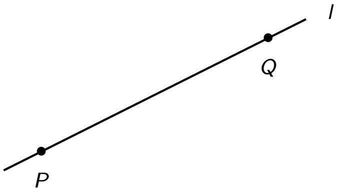


Around 300BC, a little after the time of Plato but before Archimedes, in ancient Greece, a man named Euclid wrote the *Elements* — consisting of 13 books — gathering and improving the work of his predecessors — Pythagoras and Eudoxus, in particular — into one magnificent edifice. These books soon became the standard for geometry in the classical world. With the decline of the great civilizations of Athens and Rome, it moved eastward to the center of the Arabic learning in the court of the caliphs at Baghdad. In the late Middle Ages it was translated from Arabic into Latin, and since the Renaissance it not only has been the most widely used textbook in the world, but has had an influence as a model of scientific thought that extends way beyond the confines of geometry.

Euclid's first postulate:

A straight line may be drawn from any point to any other point.



Notation

$G = (\mathbb{P}, \mathbb{L})$

Incidence Axiom

I1. Given two distinct points P and Q, there exists a unique line l which is incident with P and Q.

I2. Each line l is incident with at least two points.

I3. There exist three points A, B, C which are not collinear.

Theorem

In an incidence geometry two lines l and m satisfy one, and only one, of the following:

(i) $l = m$

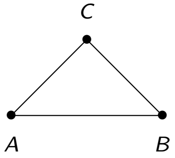
(ii) $l \cap m = \emptyset$

(iii) $l \cap m = \{\text{point}\}$

Example 1

$\mathbb{P} = \{A, B, C\}$

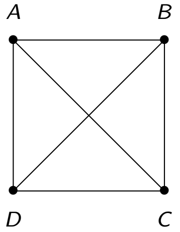
$\mathbb{L} = \{\{A, B\}, \{B, C\}, \{A, C\}\}$



Example 2

$\mathbb{P} = \{A, B, C, D\}$

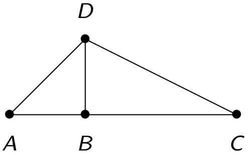
$\mathbb{L} = \{\{A, B\}, \{A, C\}, \{A, D\},$
$\{B, C\}, \{B, D\}, \{C, D\}\}$



Example 3

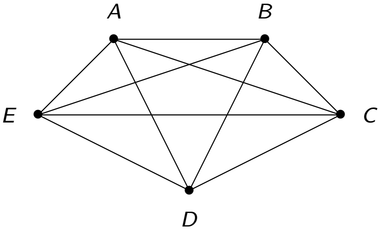
$\mathbb{P} = \{A, B, C, D\}$

$\mathbb{L} = \{\{A, B, C\}, \{A, D\}, \{B, D\},$
$\{C, D\}\}$



Example 4

$\mathbb{P} = \{A, B, C, D, E\}$
$\mathbb{L} = \{\{A, B\}, \dots\}$



Definition

Two lines l and m are *parallel* if $l \cap m = \emptyset$. We write $l || m$.

Euclid's fifth postulate (Euclidean Parallel Postulate):

For every line l and any point P not incident with l there exists a unique line m incident with P and parallel to l.

Hyperbolic Parallel Postulate

For every line l and every point P not incident with l there are at least two distinct lines m and n, both incident with P, and both parallel to l.

Elliptic Parallel Postulate

For every line l and for every point P not incident with l, there is no line m incident with P and parallel to l.

Euclid's second postulate:

A finite straight line ("line segment") may be extended continuously in a straight line.

Euclid's definition:

A line segment is that which lies evenly between its ends.

The following axioms, first introduced by Pasch in 1882, and improved by Hilbert [1899], make precise what Euclid's second postulate hints at.

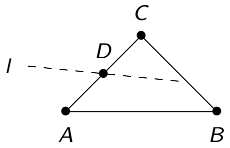
Axioms of "Betweenness":

B1. If B is between A and C (written $A * B * C$), then A, B, C are three distinct points on a line. Furthermore, $C * B * A$.

B2. For any two distinct points A, B there exists a point C such that $A * B * C$.

B3. Given three distinct points on a line, one and only one of them is between the other two.

B4. (Pasch [1882]). Let A, B, C be three non-collinear points, and let l be a line not incident with A, B or C. If l is incident with some D between A and C, it is incident with some point between A and B, or B and C.



Definition

$G = (\mathbb{P}, \mathbb{L})$ is called an *ordered geometry* if it satisfies both the incidence and the betweenness axioms.

In an ordered geometry the *line* segment $\overline{AB}$ is defined by

$$\overline{AB} = \{A, B\} \cup \{P \in \mathbb{P} | A * P * B\} (= \overline{BA}).$$

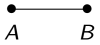
Three non-collinear points A, B, C defines a triangle $\triangle ABC$ by

$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{AC}.$$

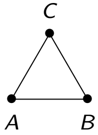
The angle $\angle BAC (= \angle CAB)$ is defined as the union of the *rays* $\overrightarrow{AB}$ and $\overrightarrow{AC}$, where

$$\overrightarrow{AB} = \overline{AB} \cup \{P \in \mathbb{P} | A * B * P\},$$

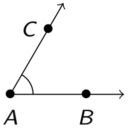
(Similarly for $\overrightarrow{AC}$.)



Line segment



Triangle $\triangle ABC$



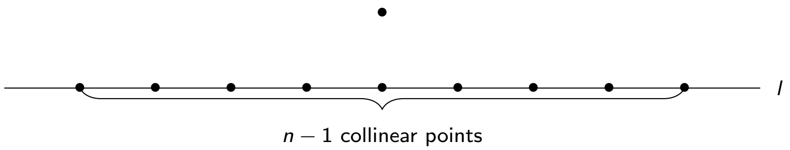
Angle $\angle BAC$

Sylvester's Conjecture [1893]:

If n points ($n \geq 3$) in the Euclidean plane are not collinear, there is at least one line incident with exactly two of these n points. (Such a line will be called an *ordinary line* (with respect to the n points).)

Example 1

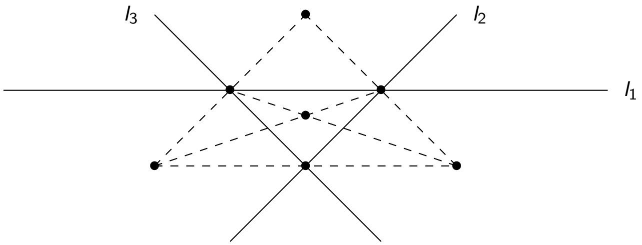
$n - 1$ ordinary lines.



Example 2

$n = 7$,

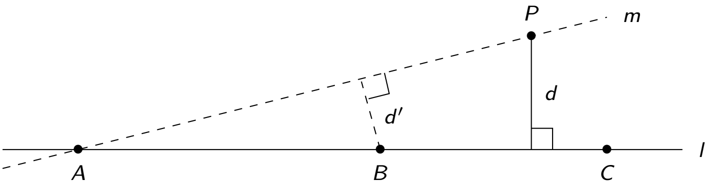
3 ordinary lines l_1, l_2, l_3.



Neither Sylvester nor any of his contemporaries were able to think of a proof. This question was forgotten until 1943, when Erdős revived it, and T. Gallai soon afterwards succeeded in finding a proof using a rather complicated argument.

The following proof is due to L. M. Kelly [1948]

Assume ad absurdum that there exist no ordinary lines. Let l be a line incident with at least three distinct points A, B, C among the n given ones, which has shortest distance to points (among the given ones) not on l. Say P outside l has a shortest distance d to l.



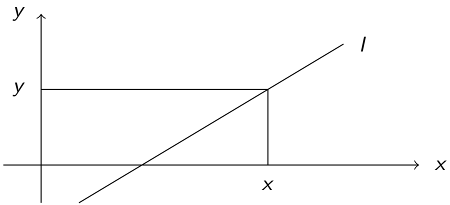
Clearly $d' < d$, contradiction.

However, this is to crack nuts with a sledgehammer! Coxeter proved in 1969 that Sylvester's Conjecture is true in ordered geometry.
Kelly and Moser proved in 1958 that there are at least $3n/7$ ordinary lines (in the Euclidean setting). Greene and Tao proved in 2013 that there are at least $n/2$ ordinary lines for all sufficiently large n.

Recall that Euclidean (plane) geometry can be described in the following way: $G = (\mathbb{P}, \mathbb{L})$ is Euclidean if $\mathbb{P} = \mathbb{R}^2 (= \mathbb{R} \times \mathbb{R})$ and $\mathbb{L}$ consists of all subsets l of $\mathbb{R}^2$ given by

$$l = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} ax + by + c = 0; a, b, c \in \mathbb{R}, \\ \text{not both } a \text{ and } b \text{ equal to zero} \end{array} \right\}$$

We will denote Euclidean geometry by $G_{\mathbb{R}}$.



Let k be a field (not necessarily with commutative multiplication, so we allow what is called a skew field). Let $\mathbb{P} = k \times k$, and let the lines $\mathbb{L}$ be defined by

$$\left\{ (x, y) \in \mathbb{P} \,\middle|\, \begin{array}{l} ax + by + c = 0; a, b, c \in k, \\ \text{not both } a \text{ and } b \text{ equal to zero} \end{array} \right\}$$

The geometry $G = (\mathbb{P}, \mathbb{L})$ thus defined, denoted G_k, satisfies

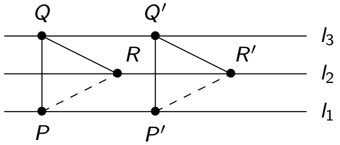
(i) The incidence axioms.

(ii) The Euclidean parallel postulate

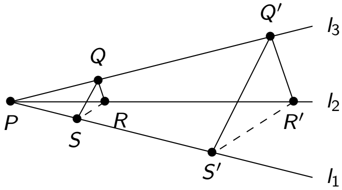
and the following Desargues configurations hold in G_k:

(iii) $l_1 \| l_2 \| l_3$

(a) $(QP \| Q'P') \wedge (QR \| Q'R') \Rightarrow (PR \| P'R')$



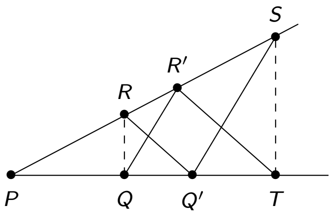
(b) $(QS \| Q'S') \wedge (QR \| Q'R') \Rightarrow (RS \| R'S')$



We call G_k *Desarguian*.

Conversely, if the geometry $G = (\mathbb{P}, \mathbb{L})$ satisfies the incidence axioms and the Euclidean parallel postulate *and* the two Desargues configurations hold in G, then $G = G_k$ for some field k.

Furthermore, k is commutative (hence a field in the ordinary sense of the word "field"), if and only if the Pappus configuration holds in G_k:



$$(QR' \| Q'S) \wedge (Q'R \| TR') \Rightarrow (QR \| ST)$$

Remark

Wedderburn proved that all finite fields k are commutative (and thus equal to $\mathrm{GF}(p^n)$ for some prime p and natural number n). Hence in such fields the Pappus configuration holds.

Definition (Ordered field)

Let k be a (not necessarily commutative) field. We say that k is an *ordered field* if there exits a subset P of k (called the positive cone) such that

(i) $a, b \in P \Rightarrow a + b \in P$ and $ab \in P$

(ii) For any $a \in k$, one and only one of the following holds: $a \in P$, $a = 0$, $-a \in P$.

Facts

Let (k, P) be an ordered field. Then

(a) k has characteristic 0 (hence $\#k = \infty$)

(b) The smallest subfield of k containing 1 is (order) isomorphic to the rational numbers $\mathbb{Q}$

(c) For any $a \in k$, $a \neq 0$, $a^2 \in P$

Theorem

The Desarguian geometry G_k is an ordered geometry if and only if k is an ordered field.

Theorem

The ordered field k is a subfield of the real numbers $\mathbb{R}$ (in its natural ordering) if and only if G_k satisfies the archimedian property, i.e. given two line segments $\overline{AB}$ and $\overline{CD}$, then there is a natural number n such that n copies of $\overline{AB}$ added together will be greater than $\overline{CD}$.

A geometry G that satisfies the four first postulates of Euclid is called *neutral* or *absolute* geometry.

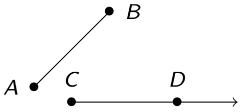
In our set-up, neutral geometry is obtained by adding the so-called congruence axioms to the incidence axioms and the betweenness axioms. (Strictly speaking, we should also add the so-called completeness axiom; also called Dedekind's axiom.) One thus gets a well-defined notion of length of segments, as well as a measurement of angles. In particular, there exists a well-defined notion of perpendicular lines. (If l and m are perpendicular lines, we write $l \perp m$.) For every line l and for every point P, there exists a unique line m incident with P such that $l \perp m$.

Neutral geometry is not *categorical*. It is two geometries in one: Euclidean geometry and hyperbolic geometry.

Congruence axioms for line segments

Undefined notion: congruence of two line segments $\overline{AB}$ and $\overline{CD}$, denoted $\overline{AB} \cong \overline{CD}$.

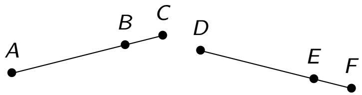
C1. Given a line segment $\overline{AB}$, and given a ray originating at point C, there exists a unique point D on the ray r such that $\overline{AB} \cong \overline{CD}$



C2. If $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{EF}$, then $\overline{CD} \cong \overline{EF}$. Also every line segment is congruent to itself.

Congruence axioms for line segments (cont.)

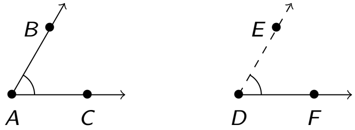
C3. Given three points A, B, C on a line satisfying $A * B * C$, and three other points D, E, F on a line satisfying $D * E * F$. Then if $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, then $\overline{AC} \cong \overline{DF}$



(Undefined notion: Congruence for angles denoted $\cong$)

Congruence axioms for angles

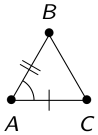
C4. Given an angle $\angle BAC$ and given a ray $\overrightarrow{DF}$, there exists a unique ray $\overrightarrow{DE}$ on a given side of the line $\overleftrightarrow{DF}$, such that $\angle BAC \cong \angle EDF$

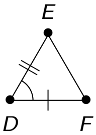


C5. For any three angles α, β, γ, if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Also every angle is congruent to itself.

Congruence axioms for angles (cont.)

C6. (SAS). Given triangles $\triangle ABC$ and $\triangle DEF$, suppose that $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$ and $\angle BAC \cong \angle EDF$. Then the two triangles are congruent, namely $\overline{BC} \cong \overline{EF}$, $\angle ABC \cong \angle DEF$ and $\angle ACB \cong \angle DFE$.





Definition

A *Hilbert plane* is a set (the elements of which are called points) together with certain subsets called *lines*, and undefined notions of *betweenness, congruence for line segments and congruence for angles* that satisfy

(a) the incidence axioms I1, I2, I3,

(b) the betweenness axioms B1, B2, B3, B4, and

(c) the congruence axioms C1, C2, C3, C4, C5, C6.

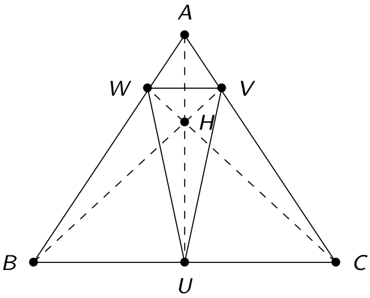
Several mathematicians (e.g. Saccheri, Legendre, Lambert) tried in vain to prove Euclid's fifth postulate (the parallel postulate) from the four first ones.

Independently Gauss (1777-1855) in Germany, János Bolyai (1802-1860) in Hungary, and Nicolai Lobachevsky (1793-1856) in Russia showed that the fifth postulate could not be deduced from the four first ones. Bolyai, in a letter to his father (who happened to be a close friend of Gauss), wrote:

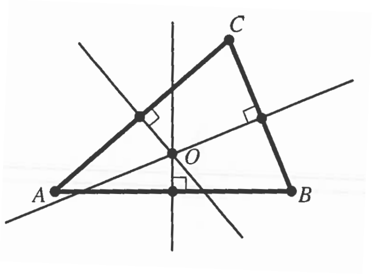
"Out of nothing I have created a strange new universe"

What he referred to was the creation of non-Euclidean geometry, specifically hyperbolic geometry, by adding the hyperbolic parallel postulate to neutral geometry. Later, Beltrami, Klein and Poincaré provided models for the associated axiom system to show its consistency.

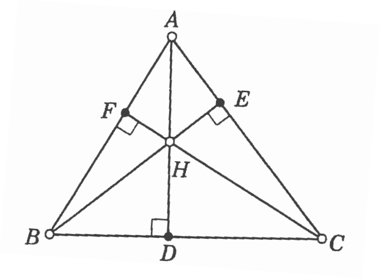
One of the best instances of a genuinely surprising results belonging to neutral geometry is the solution to Fagnano's problem: In a given acute-angled triangle $\triangle ABC$, inscribe a triangle $\triangle UVW$ whose perimeter is as small as possible.



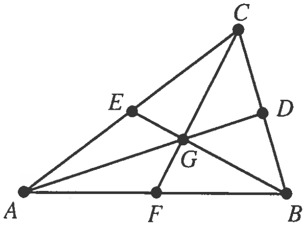
The solution: $\triangle UVW$ is the *orthic triangle* (also called the "pedal triangle"), i.e. U is the point on $\overline{BC}$ meeting the perpendicular line incident with A (called an altitude). Similarly we define V and W. In particular, the altitudes meet in a point H, called the *ortho-centre* of $\triangle ABC$.



Circumcentre O

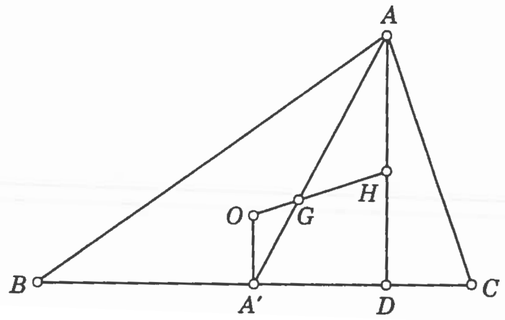


Orthocentre H



Centroid G

The Euler line (in Euclidean geometry):

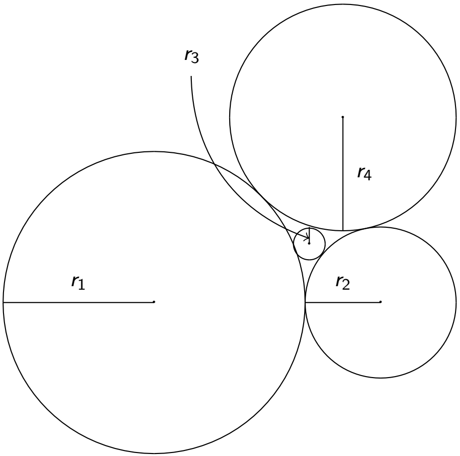


O, G and H are collinear and $|\overline{GH}| = 2|\overline{OG}|$.

This result was proved by Euler (1707-1783), and one can almost imagine the ghost of Euclid saying:

"Why on earth didn't I think of that?"

Descartes circle theorem [1643]:

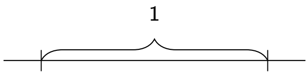


Four circles tangent to one another at six distinct points.

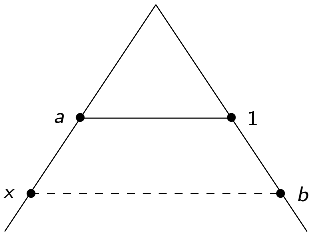
$$\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2$$
$$=2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}\right)$$

Construction by ruler and compass

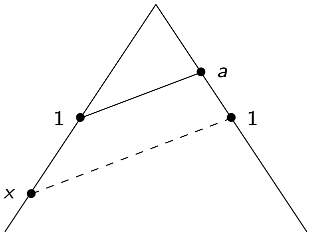
Assume that a segment of unit length



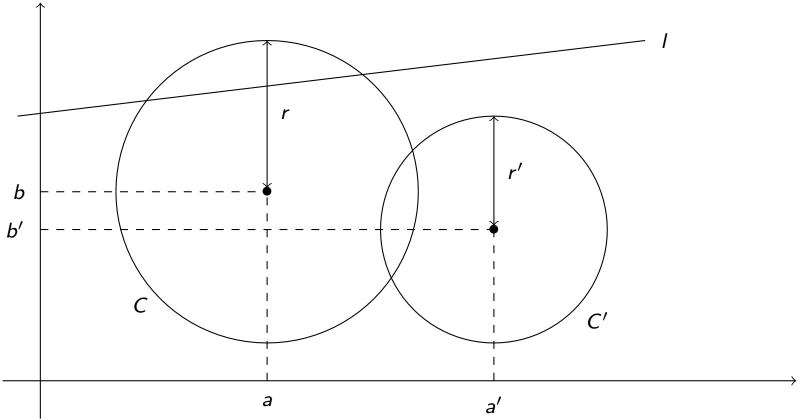
is given. Then one can construct all coordinates $(x, y) \in \mathbb{R}^2$, where x and y are rational numbers.



$$x = ab$$



$$x = 1/a$$

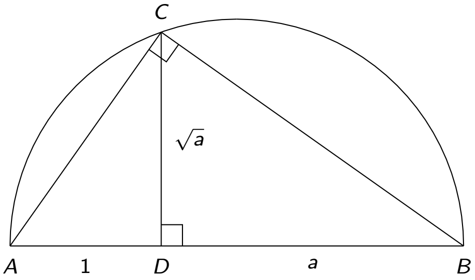


$$C : (x-a)^2 + (y-b)^2 = r^2; \quad C' : (x-a')^2 + (y-b')^2 = r'^2$$
$$l : dx + ey + f = 0$$

Theorem (Descartes [1643])

The coordinates $(x, y) \in \mathbb{R}^2$ can be constructed if and only if x and y are obtained by a finite sequence of successive quadratic extensions, starting with the rational numbers $\mathbb{Q}$.

Construction of $\sqrt{a}$:

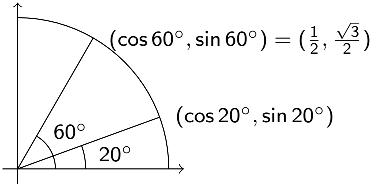


Example (Wantzel [1837]):

The angle 60° can not be trisected by ruler and compass.

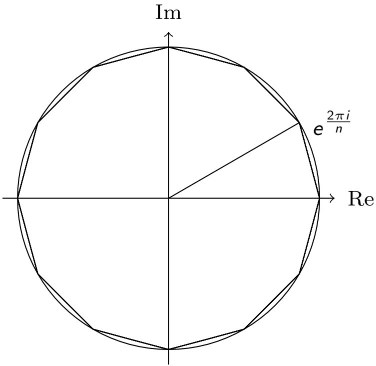
Proof

We must construct $(\cos(20^\circ), \sin(20^\circ))$.



Now we have the trigonometric identity $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$. Let $a = 2\cos(20^\circ)$. Then $a^3 - 3a - 1 = 0$. Solving the cubic equation $x^3 - 3x - 1 = 0$ requires cubic roots, and these can not be obtained by successive quadratic extensions.

Example: Regular n-gons



Must divide the circumference of the unit circle in n equal parts. Algebraically, one must find the solution z of

$$\frac{z^n - 1}{z - 1} = z^{n-1} + z^{n-2} + \cdots + z + 1 = 0.$$

One of the great accomplishments of the Greeks (as proved by Euclid in "The Elements") was to construct the regular 5-gon. (Combining that with the construction of the regular 3-gon, one can construct the regular $3 \cdot 5 = 15$-gon)

Gauss' first entry in his Notizenjournal is dated March 30, 1796, and reads:

"The principles upon which the division of the circle depend, and geometrical divisibility of same into 17 parts."

Theorem (Gauss [1796]):

The regular n-gon can be constructed by ruler and compass if and only if n is a product of distinct Fermat primes. (We assume n is odd.)

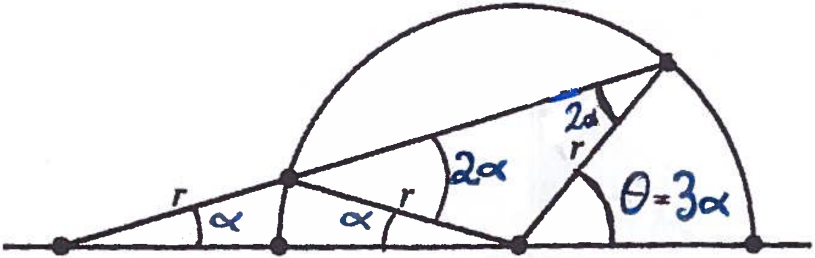
Fermat prime $p_k = 2^{2^k} + 1$; $k = 0, 1, 2, \ldots$

k	0	1	2	3	4	5	6	$\cdots$
p_k	3	5	17	257	65537	?	?	$\cdots$

$p_5 = 4.294.967.297 = 641 \cdot 6700417$ (Euler)

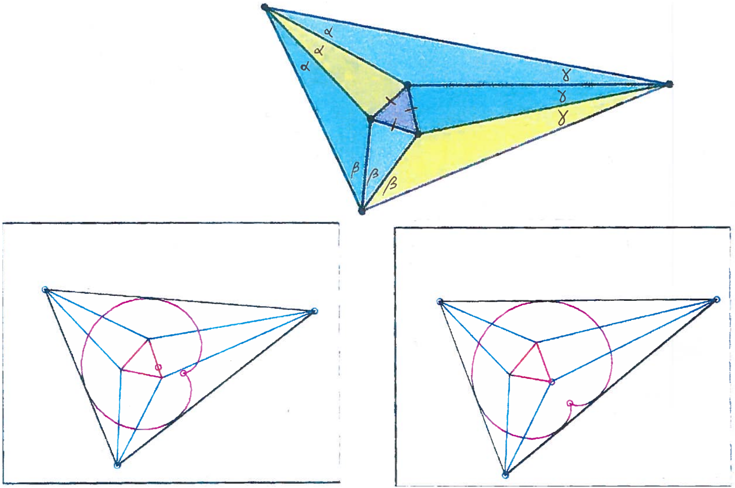
Archimedes' "marked ruler"

Trisecting an angle θ by compass and a "marked ruler":



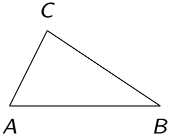
The Greek's "horror trisection" is the reason that they did not discover Morley's Theorem [1899]:

Morley's trisection theorem

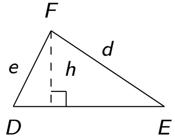


Proof of Morley's theorem

Let $\triangle ABC$ be a triangle, with angles denoted by $A = \angle BAC$, $B = \angle ABC$, $C = \angle ACB$. We start with an equilateral triangle with sides equal to 1.

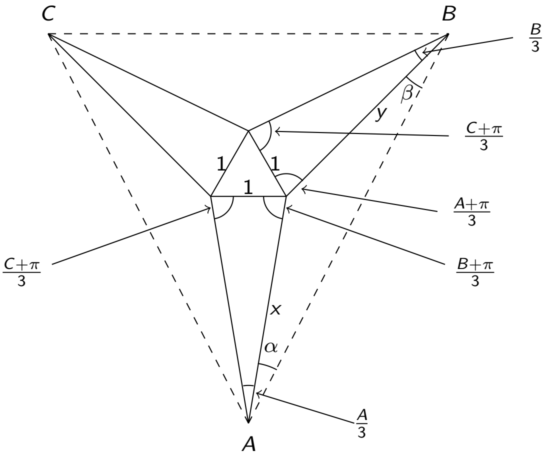


Recall



$$h = e \sin D = d \sin E$$

$$\Rightarrow \frac{\sin D}{d} = \frac{\sin E}{e}$$



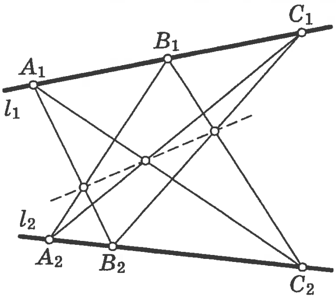
C
B
$\frac{B}{3}$
β
y
$\frac{C+\pi}{3}$
1
1
1
$\frac{A+\pi}{3}$
$\frac{C+\pi}{3}$
$\frac{B+\pi}{3}$
x
α
$\frac{A}{3}$
A

$$\frac{\sin\frac{A}{3}}{1} = \frac{\sin\frac{C+\pi}{3}}{x} \quad \wedge \quad \frac{\sin\frac{B}{3}}{1} = \frac{\sin\frac{C+\pi}{3}}{y} \quad \wedge \quad \frac{\sin\alpha}{y} = \frac{\sin\beta}{x}$$

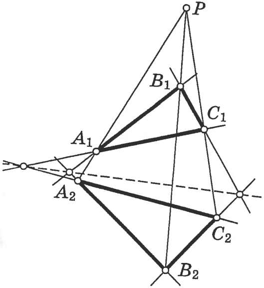
$$\left.\begin{array}{l} \dfrac{\sin\frac{A}{3}}{1} = \dfrac{\sin\frac{C+\pi}{3}}{x} \\ \dfrac{\sin\frac{B}{3}}{1} = \dfrac{\sin\frac{C+\pi}{3}}{y} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dfrac{\sin\frac{A}{3}}{\sin\frac{B}{3}} = \dfrac{y}{x} \\ \dfrac{\sin\alpha}{y} = \dfrac{\sin\beta}{x} \end{array}\right\} \Rightarrow \frac{\sin\frac{A}{3}}{\sin\frac{B}{3}} = \frac{\sin\alpha}{\sin\beta}$$

$$\Rightarrow \alpha = A/3$$

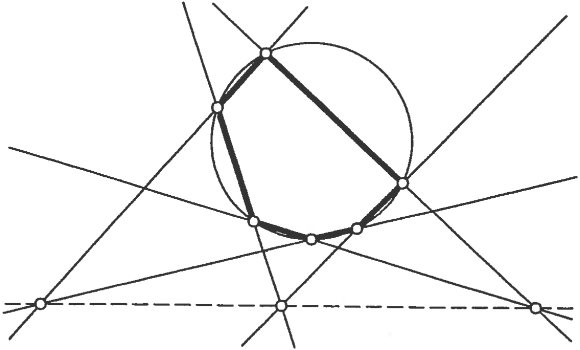
$$\Rightarrow \beta = B/3$$



Pappus' Theorem (c.290-c.350 AD)

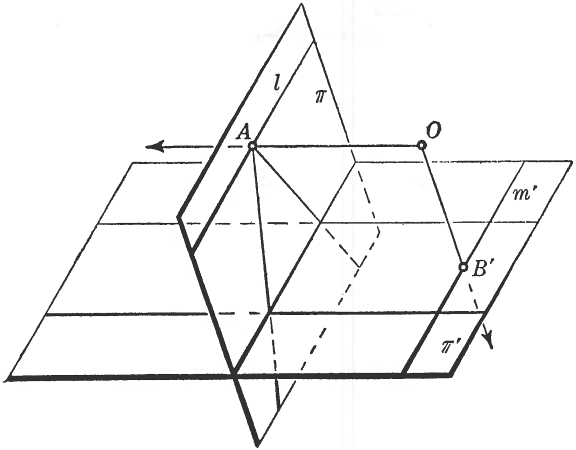


Desargues' Theorem (1593-1662)



Pascal's Theorem
(1623-1662)

"If Desargues, the daring pioneer of the 17th century, could have foreseen what his ingenious method of projection was to lead to, he might well have been astonished. He knew that he had done something good, but he probably had no conception of just how good it proved to be."
(E. T. Bell)


l
π
A
O
m'
B'
π'

Definition

A *projective plane* $\mathbb{P}$ is a set, whose elements are called points, and a set of subsets, called lines, satisfying the following axioms.

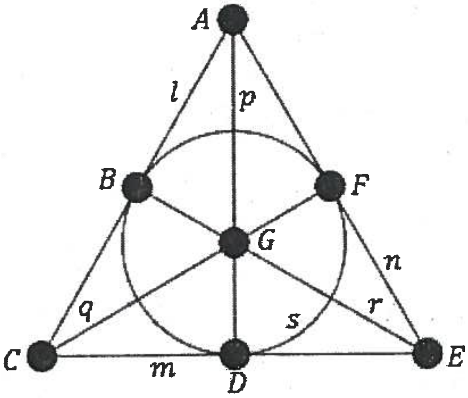
P1. Two distinct points P, Q of $\mathbb{P}$ lie on one and only one line.

P2. Two distinct lines meet in precisely one point.

P3. There exist three non-collinear points.

P4. Every line contains *at least* three points.

Fano's projective plane



Poncelet (1788-1867)
von Staudt (1798-1867)
Steiner (1796-1863)
Möbius (1790-1868)
Plücker (1801-1868)
Laguerre (1834-1886)
Cayley (1821-1895)
Klein (1849-1925)

"Metric geometry is part of projective geometry and projective geometry is <u>all</u> geometry. "
(Cayley [1859])

Klein's Erlangen Program [1872]: "The essential properties of a given geometry can be represented by the group of transformations that preserve those properties."

Cayley's *imaginary absolute* is determined by a quadratic form

$$\Omega(x, y) = ax^2 + 2bxy + cy^2$$

(with $ac - b^2 > 0$) and putting $\Omega = 0$. It is the starting point of a coordinate system on the line, constructed by repeated use of harmonic tetrads. This leads to concepts of distance between points and angles between intersecting lines.