Analysis is almost algebra

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[1] Nothing of what follows is due to me. See the list of references at the end. Thanks to whoever brought this pearl to my attention

Linear algebra meets coding theory

1. **Algebra:** A linear function $f\colon \mathbf{Z}\to\mathbf{Z}$ satisfies $f(a+b)=f(a)+f(b)$ **!**
2. **Coding theory:** Really? On the nose *all* the time **?**[2]
3. **Compromise:** allow some slack.

[2] "C'mon! these functions grow without bounds. You can't expect to keep track of the very last of billions of decimal places!"

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Compromise: allow some slack

Definition

A function $f\colon \mathbf{Z}\to\mathbf{Z}$ has *cross effect*

$$\mathrm{cr}^f\colon \mathbf{Z}\times\mathbf{Z}\to\mathbf{Z},\qquad \mathrm{cr}^f(a,b)=f(a+b)-f(a)-f(b)$$

- f is *almost linear* (ali) if cr^f is bounded.
- Two ali f and g are *almost the same* ($f\approx g$) if $f-g$ is bounded.

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Linear algebra meets coding theory

Examples

- $f(x)=7x$ is linear, and so ali.
- $g(x)=7x-6$ is ali and $g\approx f$
- $h(x)=7x+(-1)^x$ is ali and $h\approx f$.

This is something we shouldn't be naïve about: there are significantly more exotic alis[3] than even ...[4] can imagine.
Reality[5] *is* quite exotic: also some of the alis are irrational.
This talk is about them.

[3]
- $f\colon \mathbf{Z}\to\mathbf{Z}$ is *almost linear* if cr^f is bounded. $\quad \mathrm{cr}^f(a,b)=f(a+b)-f(a)-f(b)$
- "almost the same" $f\approx g$: bounded difference.

[4] name of politician deleted
[5] pun intended

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Ali defined[6]

Definition

$$\mathrm{Ali}(\mathbf{Z},\mathbf{Z})=\{\text{ ali } f\colon \mathbf{Z}\to\mathbf{Z}\}/\approx$$

A true generalization

$$\mathrm{Hom}(\mathbf{Z},\mathbf{Z})\hookrightarrow \mathrm{Ali}(\mathbf{Z},\mathbf{Z}).$$

Linear functions are easy to understand:

$$\mathbf{Z}\cong\mathrm{Hom}(\mathbf{Z},\mathbf{Z}),\qquad n\mapsto\{x\mapsto nx\};$$

what about the almost linear?

[6]
- f is *almost linear* if cr^f is bounded. $\quad \mathrm{cr}^f(a,b)=f(a+b)-f(a)-f(b)$
- "almost the same" $f\approx g$: bounded difference.

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An estimate

Two useful facts

If f ali and n a natural number, then
- $f(nx)$ (coarser sampling) and $nf(x)$ (scaled output) are ali,
- and are almost the same : $f(nx)\approx nf(x)$

Proof: There is an M such that for all a,b we have $|\mathrm{cr}^f(a,b)|<M$,[7] AND

$$f((n+1)x)-(n+1)f(x)=\mathrm{cr}^f(nx,x)+f(nx)-nf(x)$$

gives by induction on n that $|f(nx)-nf(x)|<nM$.

[7] $\mathrm{cr}^f(a,b)=f(a+b)-f(a)-f(b)$

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You can add alis[8]

We can add linear functions. Can we add alis?
- If f,g alis, then $f+g$ is ali:

$$\begin{aligned}\mathrm{cr}^{f+g}(a,b)&=(f(a+b)+g(a+b))-(f(a)+g(a))-(f(b)+g(b))\\&=\mathrm{cr}^f(a,b)+\mathrm{cr}^g(a,b)\end{aligned}$$

- $f\approx f'$ and $g\approx g'$ implies $f+g\approx f'+g'$
- $\mathrm{Ali}(\mathbf{Z},\mathbf{Z})$ is an abelian group.

[8] $\mathbf{Z}\cong\mathrm{Hom}(\mathbf{Z},\mathbf{Z})\subseteq\mathrm{Ali}(\mathbf{Z},\mathbf{Z})=\{\text{ali } f\colon \mathbf{Z}\to\mathbf{Z}\}/\approx$.

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You can order alis[9]

$\mathrm{Ali}(\mathbf{Z},\mathbf{Z})$ is an ordered abelian group

If f is ali, exactly one of the following is true:
1. f is bounded ($f\approx 0$)
2. f is positive: $\{f(x)\,|\,x>0\}$ has no upper bound
3. f is negative: $\{f(x)\,|\,x>0\}$ has no lower bound

We write $f\succ g$ if $f-g$ is positive.

[9] $\mathrm{Ali}(\mathbf{Z},\mathbf{Z})=\{\text{ali } f\colon \mathbf{Z}\to\mathbf{Z}\}/\approx$.

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Can you multiply alis?[10]

- The product of two (almost) linear functions is usually not (almost) linear: $f(x)=x$, $f(x)\cdot f(x)=x^2$,

$$\mathrm{cr}^{f\cdot f}(a,b)=(a+b)^2-a^2-b^2=2ab$$

  is without bounds.
- The multiplication in $\mathbf{Z}$ $\leftrightarrow$ *composition* in $\mathrm{Hom}(\mathbf{Z},\mathbf{Z})$.
- Is the *composite* of two alis an ali?

[10] $\mathbf{Z}\cong\mathrm{Hom}(\mathbf{Z},\mathbf{Z})\subseteq\mathrm{Ali}(\mathbf{Z},\mathbf{Z})=\{\text{ali } f\colon \mathbf{Z}\to\mathbf{Z}\}/\approx$.

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Is the *composite* of two alis an ali?[11]

- Expanding, we see that

$$\mathrm{cr}^{f\circ g}(a,b)=\mathrm{cr}^f(g(a),g(b))-\mathrm{cr}^f(g(a)+g(b),\mathrm{cr}^g(a,b))-f(\mathrm{cr}^g(a,b)),$$

  consequently, if f and g are ali, so is the composite $fg=f\circ g$.

[11] $\mathbf{Z}\cong\mathrm{Hom}(\mathbf{Z},\mathbf{Z})\subseteq\mathrm{Ali}(\mathbf{Z},\mathbf{Z})=\{\text{ali } f\colon \mathbf{Z}\to\mathbf{Z}\}/\approx$.

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Two more elementary estimates

- If f ali[12] and n a positive integer. We proved by induction that

$$|f(nx) - nf(x)| < nM \quad {}^{13}$$

- Similarly, there are constants A, B, C, D such that

$$|f(x)| < A|x| + B$$

$$|xf(y) - yf(x)| < (|x| + |y|)C + D.$$

[12] $|\mathrm{cr}^f(a,b)| < M$ where $\mathrm{cr}^f(a,b) = f(a+b) - f(a) - f(b)$

[13] we used this to deduce that $f(nx) \approx nf(x)$

$fg \approx gf$

If f and g ali, then $fg \approx gf$.

Proof: Let $x \neq 0$. Setting $y = g(x)$ in $|xf(y) - yf(x)| < (|x| + |y|)C + D$ gives

$$|xfg(x) - g(x)f(x)| < (|x| + |g(x)|)C + D.$$

Swapping f and g and adding the resulting equations give

$$|x||fg(x) - gf(x)| < (2|x| + |f(x)| + |g(x)|)C + 2D.$$

Using that alis are bounded by linear functions and dividing by $|x|$ we are done.

$fg \approx gf$[14]

Corollary

$\mathrm{Ali}(\mathbf{Z},\mathbf{Z})$ is a commutative ring: pointwise addition and with composition as multiplication.

Proof: Zero and *one* from $\mathbf{Z}$.
Associativity is just associativity of composition.
Since $(f_1 + f_2)g = f_1g + f_2g$, *distributivity* follows by *commutativity* $fg \approx gf$.

[14] $\mathbf{Z} \cong \mathrm{Hom}(\mathbf{Z},\mathbf{Z}) \subseteq \mathrm{Ali}(\mathbf{Z},\mathbf{Z}) = \{\text{ali } f\colon \mathbf{Z} \to \mathbf{Z}\}/\approx$.

$1/2 \in \mathrm{Ali}(\mathbf{Z},\mathbf{Z}) = \{\text{ali } f\colon \mathbf{Z} \to \mathbf{Z}\}/\approx$

Let $f(x) = 2x$ (representing $2 \in \mathbf{Z} \cong \mathrm{Hom}(\mathbf{Z},\mathbf{Z}) \hookrightarrow \mathrm{Ali}(\mathbf{Z},\mathbf{Z})$), and define

$$g(2n) = n = g(2n+1).$$

Then g is ali[15] and $gf(x) = x$, so that g represents $1/2$!
Similarly, we get that $\mathrm{Ali}(\mathbf{Z},\mathbf{Z})$ contains the rationals $\mathbf{Q}$...
if q is a positive integer, $1/q$ is represented by $x \mapsto \lfloor x/q \rfloor$.

[15] $|\mathrm{cr}^g(a,b)| \leq 1$

$\mathrm{Ali}(\mathbf{Z},\mathbf{Z})$ is a field, I

Lemma

Assume the ali f is positive. Then the odd function given by

$$g(n) = \min\{q > 0 \,|\, f(q) \geq n\} \text{ for } n > 0$$

is ali and is the almost inverse of f.

- "$g \approx f^{-1}$": By definition $f(g(m)) \geq m \geq f(g(m) - 1)$, giving

$$0 \leq fg(m) - m \leq fg(m) - f(g(m) - 1),$$

  and since f is ali, the upper limit is bounded. Hence $fg(x) \approx 1 \cdot x$.
- g ali: next slide

$\mathrm{Ali}(\mathbf{Z},\mathbf{Z})$ is a field, II

Lemma

Assume the ali f is positive. Then the odd function given by

$$g(n) = \min\{q > 0 \,|\, f(q) \geq n\} \text{ for } n > 0$$

is ali and is the almost inverse of f.

- "$g \approx f^{-1}$": ok
- g ali: By definition $f(g(n)) \geq n \geq f(g(n) - 1)$, giving

$$fg(m+n) - f(g(m)-1) - f(g(n)-1) > 0 > f(g(m+n)-1) - fg(m) - fg(n).$$

  Since f ali, both sides have bounded difference to

$$f(g(m+n) - g(m) - g(n)),$$

  which itself consequently is bounded. Since f is positive, this means that the input $\mathrm{cr}^g(m,n) = g(m+n) - g(m) - g(n)$ is bounded.

A retraction

- Identifying $n \in \mathbf{Z}$ with the ali nx, we see that if f is ali,

$$\lfloor f \rfloor = \max\{n \in \mathbf{Z} \,|\, f \succeq n\}$$

  is well defined (the set is is nonempty and bounded above).
- This gives a retraction

$$\mathrm{Ali}(\mathbf{Z},\mathbf{Z}) \xrightarrow{f \mapsto \lfloor f \rfloor} \mathbf{Z}.$$

- $f \mapsto \lfloor f \rfloor$ is "ali": $\quad \lfloor f + g \rfloor - \lfloor f \rfloor - \lfloor g \rfloor \in \{0, 1\}$.

$\mathrm{Ali}(\mathbf{Z},\mathbf{Z})$ has LUB

Theorem

If $S \subseteq \mathrm{Ali}(\mathbf{Z},\mathbf{Z})$ is nonempty and bounded above, then S has a least upper bound given by the odd function

$$f_S(n) = \max\{\lfloor ng \rfloor \,|\, g \in S\}, \qquad n \geq 0.^a$$

[a] $\lfloor h \rfloor = \max\{n \in \mathbf{Z} \,|\, h \succeq n\}$

Follows by inequality manipulations similar to what we've done

$\mathrm{Ali}(\mathbf{Z},\mathbf{Z})$ is real

Theorem

If $S \subseteq \mathrm{Ali}(\mathbf{Z},\mathbf{Z})$ is nonempty and bounded above, then S has a least upper bound given by the odd function

$$f_S(n) = \max\{\lfloor ng \rfloor \,|\, g \in S\}, \qquad n \geq 0.^a$$

[a] $\lfloor h \rfloor = \max\{n \in \mathbf{Z} \,|\, h \succeq n\}$

Corollary

$\mathrm{Ali}(\mathbf{Z},\mathbf{Z})$ is a complete ordered field, and hence by Hölder:
$\mathrm{Ali}(\mathbf{Z},\mathbf{Z})$ is isomorphic to $\mathbf{R}$.

History and references

- Steven Schanuel (unpublished, early 80's). Independently Richard Lewis?
- Ross Street, An efficient construction of the real numbers. Gazette Australian Math. Soc., 12:5758, 1985 (later improved).
- John Harrison. Theorem Proving with the Real Numbers. Technical report, University of Cambridge Computer Laboratory, 1996.
- Norbert ACampo. A natural construction for the real numbers. arXiv:math.GN/0301015 v1, 3 January 2003.
- R.D. Arthan, The Eudoxus Real Numbers, arXiv:math/0405454 v1, 24 May 2004.

Fun facts

- Any ali is represented by an $f : \mathbf{Z} \to \mathbf{Z}$ with

$$|\mathrm{cr}^f(a,b)| = |f(a+b) - f(a) - f(b)| \leq 1.$$

- The asymptotic *slope function*

$$\mathrm{Ali}(\mathbf{Z},\mathbf{Z}) \to \mathbf{R}, \qquad f \mapsto \lim_{n\to\infty} f(n)/n$$

  is an isomorphism of complete ordered fields.
- $\mathrm{Ali}(\mathbf{Z}^m, \mathbf{Z}^n) \cong M_{n\times m}(\mathbf{R})$.
- $\underline{\mathrm{Ali}}_{\mathbf{Z}} \simeq \underline{\mathrm{Vect}}_{\mathbf{R}}$.

Theorem

If $S \subseteq \mathrm{Ali}(\mathbf{Z},\mathbf{Z})$ is nonempty and bounded above, then S has a least upper bound given by the odd function

$$f_S(n) = \max\{\lfloor ng \rfloor \mid g \in S\}, \qquad n \geq 0.^a$$

$^a \lfloor h \rfloor = \max\{n \in \mathbf{Z} \mid h \succeq n\}$

Proof. Let F be an upper bound for S. $\{\lfloor ng \rfloor \mid g \in S\}$ is nonempty and bounded by $\lfloor nF \rfloor$, so $f_S(n)$ is well defined.
Must show that f_S is ali!
Given a there is an $g_a \in S$ s.t. $f_S(a) = \lfloor a g_a \rfloor$, and for any $g \in S$, $\lfloor ag \rfloor \leq \lfloor a g_a \rfloor$.
Let $G = \max\{g_a, g_b, g_{a+b}\}$.
Notice: $f_S(a) = \lfloor aG \rfloor$, $f_S(b) = \lfloor bG \rfloor$ and $f_S(a+b) = \lfloor (a+b)G \rfloor = \lfloor aG + bG \rfloor$.
But then $\mathrm{cr}^{f_S}(a,b) = \lfloor aG + bG \rfloor - \lfloor aG \rfloor - \lfloor bG \rfloor \in \{0,1\}$!