Waring's Problem

Petter Andreas Bergh

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Two squares

Which positive numbers can be written as the sum of two integer squares?

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$$
\begin{aligned}
1 &= 1^2 + 0^2 \\
2 &= 1^2 + 1^2 \\
3 &= ? \\
4 &= 2^2 + 0^2 \\
5 &= 2^2 + 1^2 \\
6 &= ? \\
7 &= ?
\end{aligned}
$$

Two squares

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$$\begin{aligned} 1 &= 1^2 + 0^2 \\ 2 &= 1^2 + 1^2 \\ 3 &= ? \\ 4 &= 2^2 + 0^2 \\ 5 &= 2^2 + 1^2 \\ 6 &= ? \\ 7 &= ? \end{aligned}$$

Boils down to looking at prime numbers:

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$$

Two squares

Three types of prime numbers:

$p = 2$	$p \equiv 1 \pmod 4$	$p \equiv 3 \pmod 4$

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$2 = 1^2 + 1^2$	$5 = 2^2 + 1^2$	
	$13 = 3^2 + 2^2$	
	$17 = 4^2 + 1^2$	
	$29 = 5^2 + 2^2$	
	$\vdots$	

Two squares

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2	$=$	$1^2 + 1^2$	5	$=$	$2^2 + 1^2$	3	$=$	$?$
			13	$=$	$3^2 + 2^2$	7	$=$	$?$
			17	$=$	$4^2 + 1^2$	11	$=$	$?$
			29	$=$	$5^2 + 2^2$	19	$=$	$?$
				$\vdots$			$\vdots$	

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Observation

No number $n \equiv 3 \pmod 4$ can be the sum of two squares.

Two squares

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Observation

No number $n \equiv 3 \pmod 4$ can be the sum of two squares.

Proof: if $a \in \mathbb{Z}$ then $a^2 \equiv 0, 1 \pmod 4$ so $a^2 + b^2 \equiv 0, 1, 2 \pmod 4$ for all $a, b \in \mathbb{Z}$.

Two squares

Theorem [Fermat-Girard-Euler]

An odd prime p is the sum of two squares if (and only if) $p \equiv 1 \pmod 4$.

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- Stated by Fermat in 1640 (no proof).
- Observed earlier by Girard.
- First proof: Euler in 1752–1755.
- Completely elementary proof: Heath-Brown in 1984.

Two squares

Theorem [Fermat-Girard-Euler]

An odd prime p is the sum of two squares if (and only if) $p \equiv 1 \pmod 4$.

Zagier's one-sentence proof (1990):

The involution on the finite set $S = \{(x, y, z) \in \mathbb{N}^3 \mid p = x^2 + 4yz\}$ defined by

$$(x, y, z) \mapsto \begin{cases} (x + 2z, z, y - x - z) & \text{if } x < y - z \\ (2y - x, y, x - y + z) & \text{if } y - z < x < 2y \\ (x - 2y, x - y + z, y) & \text{if } 2y < x \end{cases}$$

has exactly one fixed point, so $|S|$ is odd and the involution defined by $(x, y, z) \mapsto (x, z, y)$ also has a fixed point.

Two squares/three squares

Theorem - two squares

A positive number is the sum of two squares if and only if every prime divisor $p \equiv 3 \pmod 4$ appears an even number of times.

Two squares/three squares

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Theorem - three squares [Legendre, 1797/1798]

A positive number is the sum of three squares if and only if it is *not* of the form $4^n(8m+7)$ for integers $n, m \geq 0$.

Smallest non-example: 7

Four squares

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$$\begin{aligned}
1 &= 1^2+0^2+0^2+0^2 & \qquad 6 &= 2^2+1^2+1^2+0^2\\
2 &= 1^2+1^2+0^2+0^2 & 7 &= 2^2+1^2+1^2+1^2 \text{ (need four)}\\
3 &= 1^2+1^2+1^2+0^2 & 8 &= 2^2+2^2+0^2+0^2\\
4 &= 2^2+0^2+0^2+0^2 & 9 &= 3^2+0^2+0^2+0^2\\
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\end{aligned}$$

Boils down to looking at prime numbers:

$$(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+w^2) = \begin{array}{ll} (ax+by+cz+dw)^2 & + \\ (ay-bx+cw-dz)^2 & + \\ (az-bw-cx+dy)^2 & + \\ (aw+bz-cy-dx)^2 & \end{array}$$

Four squares

Theorem [Lagrange, 1770]

Every positive number is the sum of four integer squares.

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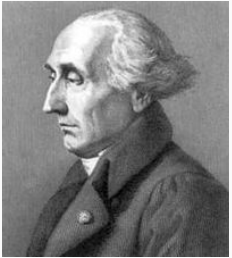

- Proved by Lagrange in 1770.
- Proof accessible to first year students.
- Problem goes back to Diophantus (3rd century).
- Jacobi (1834): simple formula for the number of four squares representations.

Waring's Problem

Waring, 1770:

Given $k \geq 1$, is there a number t_k such that every positive number is the sum of t_k kth powers of non-negative integers?

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- Edward Waring (1736–1798).
- English mathematician, Cambridge.
- Lucasian Professor of Mathematics 1760–1798.
- Other conjecture: every odd number ≥ 3 is either prime or the sum of three primes.

Waring's Problem

Waring, 1770:

Given $k \geq 1$, is there a number t_k such that every positive number is the sum of t_k kth powers of non-negative integers?

- Trivially we may take $t_1 = 1$.
- Lagrange's four squares theorem: $t_2 = 4$ works (and is minimal: 7 is not the sum of 3 squares: $7 = 2^2 + 1^2 + 1^2 + 1^2$).
- Wieferich, 1909: every positive number is the sum of 9 cubes, so $t_3 = 9$ works (and is minimal: 23 is not the sum of 8 cubes: $23 = 2^3 + 2^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$).

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Theorem [Hilbert, 1909]

Yes: for every $k \geq 1$ there is a number t_k such that every positive number is the sum of t_k kth powers of non-negative integers.

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- Complicated analytic proof.
- Gives no explicit bound for t_k.
- Modern proofs: Hardy-Littlewood, Vinogradov, elementary one by Linnik.

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Definition

$g(k)$ = smallest number such that every number is the sum of $g(k)$ kth powers.

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- $g(1) = 1$.
- $g(2) = 4$ (Lagrange, 1770).
- $g(3) = 9$ (Wieferich, 1909).
- $g(4) = 19$ (Balasubramanian, Deshouillers, Dress, 1986).
- $g(5) = 37$ (Chen, 1964).
- $g(6) = 73$ (Pillai, 1940).

Waring's Problem

Formula for $g(k)$?

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$$n = \underbrace{2^k + \cdots + 2^k}_{\left[(3/2)^k\right]-1} + \underbrace{1^k + \cdots + 1^k}_{2^k-1}$$

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Waring's Problem

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g(2) = 4 &= [(3/2)^2] + 2^2 - 2 & \qquad g(5) = 37 &= [(3/2)^5] + 2^5 - 2 \\
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Theorem [Dickson, Pillai, Niven, 1936–1944]

If $(3/2)^k - \left[(3/2)^k\right] \leq 1 - \left[(3/2)^k\right]/2^k$ **then** $g(k) = \left[(3/2)^k\right] + 2^k - 2$.

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- Condition confirmed for $k \leq 471600000$.
- Mahler, 1957: there exists a k_0 (undetermined) such that condition holds for $k \geq k_0$.

Waring's Problem - asymptotic version

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$G(k) =$ smallest number such that every **sufficiently large** number is the sum of $G(k)$ kth powers.

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- $G(k) \leq g(k)$.
- $G(1) = 1 = g(1)$.
- $G(2) = 4 = g(2)$ since four squares is needed for all $n \equiv 7 \pmod 8$.
-
  - $g(3) = 9$.
  - Dickson, 1939: every number except 23 and 239 requires only 8 cubes, so $G(3) \leq 8$.
  - Linnik, 1943: $G(3) \leq 7$.
  - $4 \leq G(3) \leq 7$, but precise value unknown.

Little is known about $G(k)$ in general.

Waring's Process - asymptotic version

Theorem

$G(k) \geq k+1$ for all $k > 1$.

Waring's Problem - asymptotic version

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Friday afternoon proof:

- $P(N) = \#\{n \leq N \mid \text{there exist } x_1, \ldots, x_k \in \mathbb{Z} \text{ with } n = x_1^k + \cdots + x_k^k\}$.

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- $P(N) \leq S(N)$.
- $S(N) = 1/k!\,\left(\left[N^{1/k}\right]+1\right)\left(\left[N^{1/k}\right]+2\right)\cdots\left(\left[N^{1/k}\right]+k\right) \sim N/k!$ for $N \gg 0$.

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- If $G(k) \leq k$, then there exists an m with $P(N) \geq N - m$ for all N.
  Contradiction: $N - m \leq P(N) \leq S(N) \sim N/k!$.

Generalized Waring's Problem - polynomials

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- $f(x)$ integer valued polynomial with
  - ▷ positive degree (so not a constant) and positive leading coefficient,
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Theorem [Kamke, 1921]

(1) The set $S = \{0, f(0), f(1), f(2), \ldots\}$ is an **asymptotic basis of finite order**: there exists a t such that every sufficiently large number is the sum of t elements in S

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(2) If $0, 1 \in U = \{f(0), f(1), f(2), \ldots\}$, then U is a **basis of finite order**: there exists a v such that every positive number is the sum of v elements in U.

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Hilbert-Waring theorem: $f(x) = x^k$.

Waring's Problem - algebraic number fields

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- Algebraic number field $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.
- Ring of integers $R_F = \{\alpha \in F \mid \alpha \text{ root in some monic integer polynomial}\}$.
- R_F is a generalization of $\mathbb{Z}$ (if $F = \mathbb{Q}$ then $R_F = \mathbb{Z}$).

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Example

If $F = \mathbb{Q}(\sqrt{3})$ then $R_F = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$.

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$$\left(a + b\sqrt{3}\right)^2 = (a^2 + 3b^2) + 2ab\sqrt{3}.$$

So no $u + v\sqrt{3} \in R_F$ with v odd can be the sum of squares.

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Theorem [Siegel, 1944-1946]

For $k \geq 1$, let A_k be the set of all $\alpha \in R_F$ which **can** be written as a sum of kth powers in R_F. Then there is a t_k such that every $\alpha \in A_k$ is the sum of t_k kth powers in R_F.