Abel and abelian integrals

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Atiyah in his acceptance speech in Oslo 2004 on the occasion of receiving the Abel Prize:

Abel was really the first modern mathematician. His whole approach, with its generality, its insight and its elegance set the tone for the next two centuries. (...) Had Abel lived longer, he would have been the natural successor to the great Gauss.

Yuri Manin being interviewed in 2009:

Take for example the first volume of Crelle's Journal *(*Journal of Pure and Applied Mathematics*), which came out in 1826. Abel's article appeared there, on the unsolvability in radicals of the general equation of degree higher than four. A wonderful article! As a member of the editorial board of Crelle, I would accept it even today with great pleasure.*

Born Aug. 5 — 1802

1815

1816

1817

1818 ← New math teacher (Bernt Michael Holmboe)

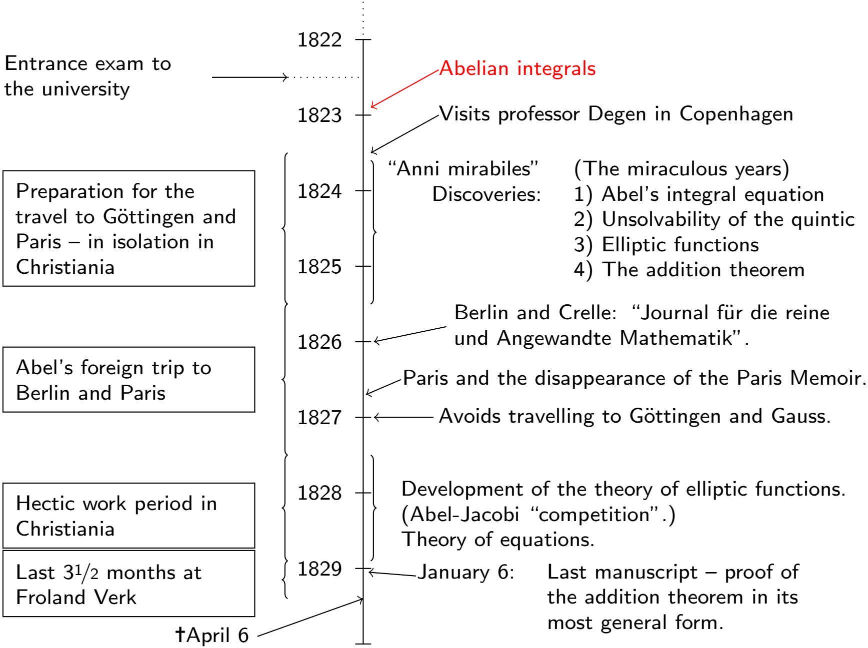
Pupil at the Cathedral School in Christiania (later Oslo) (1815–1821)

1819

1820 ← Death of Abel's father

1821 ← "Proof" of solvability (sic!) of the quintic

Examen Artium →


1822
Entrance exam to the university
Abelian integrals
1823
Visits professor Degen in Copenhagen
Preparation for the travel to Göttingen and Paris – in isolation in Christiania
1824
"Anni mirabiles" (The miraculous years)
Discoveries: 1) Abel's integral equation
2) Unsolvability of the quintic
3) Elliptic functions
4) The addition theorem
1825
Berlin and Crelle: "Journal für die reine und Angewandte Mathematik".
1826
Abel's foreign trip to Berlin and Paris
Paris and the disappearance of the Paris Memoir.
1827
Avoids travelling to Göttingen and Gauss.
1828
Hectic work period in Christiania
Development of the theory of elliptic functions.
(Abel-Jacobi "competition".)
Theory of equations.
1829
Last 3 1/2 months at Froland Verk
January 6: Last manuscript – proof of the addition theorem in its most general form.
†April 6

Question 1 If $f(x)$ is an elementary function, how can we determine whether or not its integral $\int f(x)\,dx$ is also an elementary function?

Question 2 If the integral is an elementary function, how can we find it?

Rephrased: If $f(x)$ is an elementary function, when is the solution $y = y(x)$ to the differential equation $\frac{dy}{dx} = f(x)$ an elementary function?

Comment It is easy to see that the derivative of an elementary function is elementary. We ask if the converse is true?

(1) *Théorie des transcendantes elliptiques* (1823-1825), 102 pages, (published posthumously).
(2) *Sur l'intégration de la formule différentielle* $\frac{\rho\, dx}{\sqrt{R}}$, *R et* ρ *étant des fonctions entières* (1826) Crelle, 41 pages.
(3) *Mémoire sur les fonctions transcendantes se la forme* $\int y\, dx$, *où* y *est une fonction algébrique de* x (1828), 11 pages (published posthumously).
(4) *Précis d'une théorie des fonctions elliptiques* (Chapter II) (1829) Crelle, 99 pages.
(5) Lettre à Legendre (November 1828), published in Crelle 1830, 9 pages.

Sur la comparison des transcendantes (1825), 12 pages (published posthumously). [Contains proof of Abel's Addition Theorem.]

Rational functions

$$R(x) = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0}{b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0} = \frac{P(x)}{Q(x)}$$

Algebraic functions

$$y^n + R_{n-1}(x) y^{n-1} + \cdots + R_1(x) y + R_0(x) = 0$$

Examples

(i) $y = R(x)$; $y - R(x) = 0$, $R(x)$ rational function

(ii) $y = \sqrt[n]{R(x)}$; $y^n - R(x) = 0$, $R(x)$ rational function

(iii) $y^5 - y - x = 0$, $y = y(x)$ has no explicit presentation

Rational functions $R(x, y)$ of x and y are of the form $\frac{P(x,y)}{Q(x,y)}$, where P and Q are polynomials in x and y.

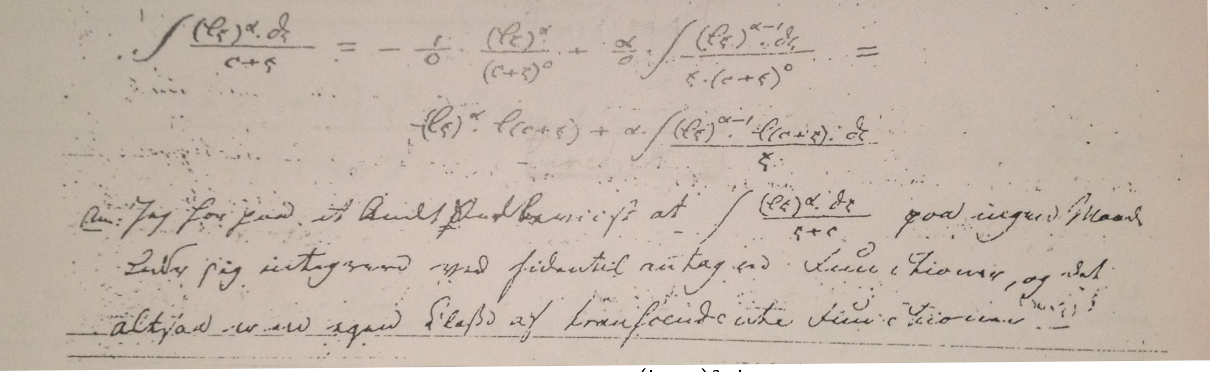
An elementary function is a member of the class of functions which comprises

(i) rational function
(ii) algebraic function, explicit or implicit
(iii) the logarithmic function $\log x$
(iv) the exponential function e^x
(v) all functions "built" up by a finite number of steps from the classes (i)-(iv).

Example

$$f(x) = \log\left(\frac{y}{\sqrt{1+x^4}}\right),$$

where $y = y(x)$ is defined (implicitly) by $y^5 - y - e^{x^2}\log x = 0$.



$$\int \frac{(lx)^{\alpha} \cdot dx}{c+x} = -\frac{1}{0} \cdot \frac{(lx)^{\alpha}}{(c+x)^{0}} + \frac{\alpha}{0} \int \frac{(lx)^{\alpha-1} \cdot dx}{x \cdot (c+x)^{0}} =$$

$$(lx)^{\alpha} \cdot l(c+x) + \alpha \int \frac{(lx)^{\alpha-1} \cdot l(c+x) \cdot dx}{x}$$

Anm: Jeg har paa et andet Sted beviist at $\int \frac{(lx)^{\alpha} \cdot dx}{x+c}$ paa ingen Maade lader sig integrere ved de hidentil antagne Functioner, og at det altsaa er en egen Classe af transcendente Functioner.

Jeg har paa et andet Sted beviist at $\int \frac{(\log x)^a \, dx}{c+x}$ *paa ingen Maade lader sig integrere ved de hidentil antagne Functioner, og at det altsaa er en egen Classe af transcendente Functioner.*

[*I have proved another place that* $\int \frac{(\log x)^a \, dx}{c+x}$ *in no way whatsoever can be integrated in terms of the up to now familiar functions, and hence this belongs to a separate class of transcendent functions.*]

M. J. Prelle and M. F. Singer, *Elementary first integrals of differential equations*, TAMS **279** (1983), 215-229:

Abstract We show that if a system of differential equations has an elementary first integral (i.e. a first integral expressible in terms of exponentials, logarithms and algebraic functions) then it must have a first integral of a very simple form.

M. F. Singer in a letter to Jesper Lützen:
I did not know that Abel had thought about this.

Integration methods 1. Partial fraction expansion

$$\frac{3x^4 - 12x^3 + 19x^2 - 16x - 6}{(x^2+1)(x-2)^2}$$
$$= \frac{1/2i}{x-i} - \frac{1/2i}{x+i} - \frac{2}{(x-2)^3} + \frac{4}{(x-2)^2} - \frac{1}{x-2}$$
$$\int \frac{3x^4 - 12x^3 + 19x^2 - 16x - 6}{(x^2+1)(x-2)^2}\, dx$$
$$= \frac{1}{2i}\log(x-i) - \frac{1}{2i}\log(x+i) + \frac{1}{(x-2)^2} - \frac{4}{x-2} - \log(x-2)$$

In general, the integral of a rational function has a rational part and a logarithmic part.

1. Partial fraction expansion (cont.)

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \arcsin x = \frac{1}{i} \log(ix + \sqrt{1 - x^2})$$

Trigonometric functions and the inverse arc-functions can be expressed by (complex) exponential and logarithmic functions.

2. Integration by parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Example

$$\int \log x\,dx = x\log x - \int \frac{1}{x}\cdot x\,dx = x\log x - x$$

$$f(x) = \log x, \quad g(x) = x$$

3. Integration by substitution

$$\int f(x)\,dx = \int f\left(\phi(t)\right)\phi'(t)\,dt$$

Example

$$\int \frac{x^2}{\sqrt[3]{x^3+1}}\,dx = \int \frac{1/3}{t^{1/3}}\,dt = \frac{1}{2}t^{2/3} = \frac{1}{2}\left(x^3+1\right)^{2/3}$$

$$x^3 + 1 = t$$

$$3x^2\,dx = \ dt$$

Abelian integral

$\int y\,dx$ is called an *abelian integral*, whenever $y = y(x)$ is an algebraic function.

Comment $\int R(x, y)\,dx$, where $y = y(x)$ is an algebraic function and $R(x, y)$ is a rational function in x and y, is an abelian integral.

Example

$$\int \frac{x + \sqrt[3]{1 + x^7}}{(x - 2)\sqrt[3]{1 + x^7}}\,dx = \int R(x, y)\,dx,$$

where $y = \sqrt[3]{1 + x^7}$, $R(x, y) = \frac{x+y}{(x-2)y}$.

$$\int \frac{x\,dx}{\sqrt{x^4+10x^2-96x-71}} =$$

$$-\frac{1}{8}\log\left[\begin{array}{l}\left(x^6+15x^4-80x^3+27x^2-528x+781\right)\cdot\sqrt{x^4+10x^2-96x-71}\\ -\left(x^8+20x^6-128x^5+54x^4-1408x^3+3124x^2+10001\right)\end{array}\right]$$

But

$$\int \frac{x\,dx}{\sqrt{x^4+10x^2-96x-72}}$$

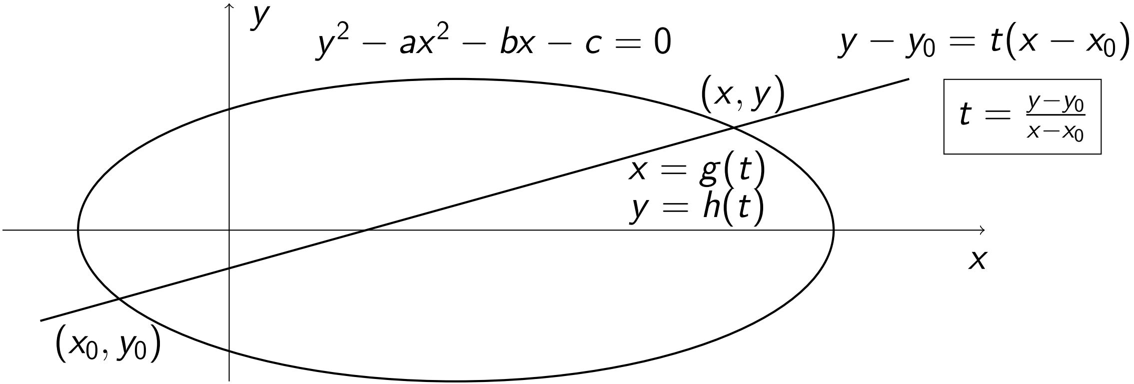
is <u>not</u> an elementary function.

R. H. Risch showed in 1969 that there exists an algorithm to decide whether the indefinite integral of an elementary function is elementary or not.

The crucial theorem that Risch's result is based upon is an analogue of Theorem 1 (due to Abel and Liouville), cf. below.

Let $R(x, y)$ be a rational function of x and y, where $y = \sqrt{ax^2 + bx + c}$. Substitute $x = g(t)$, $y = h(t)$ (rational functions) in $\int R(x, y)\, dx$:

$$\int R(g(t), h(t))g'(t)\, dt = r_0(t) + \sum_{i=1}^{k} c_i \log(r_i(t)) = R_0(x, y) + \sum_{i=1}^{k} c_i \log R_i(x, y)$$



Theorem 1 (*Abel and Liouville*) If the abelian integral $\int y\,dx$ is an elementary function then it must have the form

$$\int y\,dx = t + A\log u + B\log u + \cdots + F\log w$$

where $t, u, v, \ldots, w$ are algebraic functions of x and $A, B, \ldots, F$ are constants.

Theorem 2 (*Abel's Theorem*) The $t, u, v, \ldots, w$ functions in Theorem 1 are each *rational* functions of x and y.

Remark Abel proved much more: If $\int y\,dx$ in addition to the terms in Theorem 1 contains additive terms of elliptic integrals of any type, the analogue conclusion of Theorem 2 is still true.

The hierarchy of elementary functions

Order 0: algebraic functions
Order 1: algebraic functions of exp or log of functions of order 0
Order 2: algebraic functions of exp or log of functions of order 1.

Examples

$e^{x^2} + e^x\sqrt{\log x}$ (order 1)
y defined by $y^5 - y - e^x \log x = 0$ (order 1)
e^{e^x}, $\log\log x$ (order 2)

Analogy with classification of radicals (over $\mathbb{Q}$):
2, $\sqrt[3]{7}$, $\sqrt[5]{3+\sqrt{5}}$ (orders 0, 1, 2, respectively)

If

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

is elementary, then according to Abel's Theorem it has to look like (set $\Delta(x) = \sqrt{(1-x^2)(1-k^2x^2)}$):

$$\int \frac{dx}{\Delta(x)} = (p_0 + q_0\Delta(x)) + \sum_{k=1}^{n} A_k \log\left(p_k + q_k\Delta(x)\right)$$

where the p_k's and the q_k's are rational functions of x and the A_k's are constants.

XVII.

MÉMOIRE SUR LES FONCTIONS TRANSCENDANTES DE LA FORME $\int y\,dx$, OÙ y EST UNE FONCTION ALGÉBRIQUE DE x.

Soient $c_1\int y_1\,dx + c_2\int y_2\,dx + c_3\int y_3\,dx + \cdots + c_\mu\int y_\mu\,dx = P$,
où P fonction algébrique de x.

Soit

$$P^k + R_1 P^{k-1} + \cdots = 0,$$

irréductible, R_1 etc. étant des fonctions rationnelles de

$$x,\ y_1,\ y_2,\ y_3,\ \ldots y_\mu.$$

On aura

$$(k\,dP + dR_1)\,P^{k-1} + [(k-1)\,R_1\,dP + dR_2]\,P^{k-2} + \cdots = 0;$$

$$\frac{dP}{dx} = c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots,$$

$$k\,dP + dR_1 = 0,$$

$$P = -\frac{R_1}{k} + C;$$

par suite $k = 1$, $P = -R$. Donc

Théorème II.

$$c_1\int y_1\,dx + c_2\int y_2\,dx + \cdots + c_\mu\int y_\mu\,dx = P,$$

où P fonction rationnelle de $x,\ y_1,\ y_2,\ y_3,\ \ldots y_\mu$.

Maupertuis and team on expedition in Lapland in 1736/37 to determine that the earth is flattened near the pole. On their return to Paris, Voltaire came with this cutting remark:

Vous avez confirmé dans ces lieux pleins d'ennui ce qui Newton connut sans sortir de chez lui.
[*You have confirmed in these desolate places what Newton knew without leaving home.*]

About Norton's proof (in Hardy's book) of Abel's Theorem:

You have shown by a long an complicated proof what Abel gave a "one-line" proof of.

Let $\mathbb{C}(x)$ denote the field of rational functions over $\mathbb{C}$. Let $y = y(x)$ be an algebraic function given by

(i) $y^n + R_{n-1}(x)y^{n-1} + \cdots + R_1(x)y + R_0(x) = 0.$
Take the derivative of (i):

(ii) $ny^{n-1}y' + (R'_{n-1}y^{n-1} + (n-1)R_{n-1}y^{n-2}y') + \cdots + (R'_1 y + R_1 y') + R'_0 = 0.$

Conclusion y' is a rational function of x and y.

Let $a = a(x, y)$ be a rational function of x and y, where y is as above. Then

$$(a' =)\frac{da}{dx} = \frac{\partial a}{\partial x} + \frac{\partial a}{\partial y}\frac{dy}{dx} \qquad \text{is a rational function of } x \text{ and } y.$$

Theorem (*very special case of Abel's Theorem*) Let $y = y(x)$ be an algebraic function. Assume the abelian integral $u = \int y\,dx$ is an algebraic function. Then u is a *rational* function of x and y.

Proof: Let $\mathbb{C}(x)$ be the field of rational functions of x. Then $\mathbb{C}(x, y)$ ($\equiv$ rational functions of x and y) is a finite-dimensional field extension of $\mathbb{C}(x)$. Since u is algebraic over $\mathbb{C}(x)$, it is obviously algebraic over $\mathbb{C}(x, y)$.
(Abel: "...car cette supposition permise simplifiere beaucoup le raisonnement.")
Now u is a root of an *irreducible* polynomial

$$f(z) = z^k + a_{k-1}(x, y)z^{k-1} + \cdots + a_1(x, y) + a_0(x, y)$$

over $\mathbb{C}(x, y)$ (the minimal polynomial of u over K).

$$(*) \qquad u^k + a_{k-1}u^{k-1} + \cdots + a_1 u + a_0 = 0$$

Take the derivative and use that $u' = y$:

$(**)$

$$ku^{k-1}y + \left(a'_{k-1}u^{k-1} + (k-1)a_{k-1}u^{k-2}y\right) + \cdots + (a'_1 u + a_1 y) + a'_0 = 0$$

Now each a'_i is a rational function of x and y, and so $a'_i \in \mathbb{C}(x, y)$. From $(**)$ we get

$$(***) \qquad (ky + a'_{k-1})\, u^{k-1} + \cdots + (2a_2 y + a'_1)u + (a_1 y + a'_0) = 0.$$

Hence $ky + a'_{k-1} = 0$, and so

$$u = \int y\, dx = -\frac{1}{k}\int a'_{k-1}\, dx = -\frac{a_{k-1}}{k}, \qquad \text{q.e.d.}$$

$$\alpha = \sqrt[5]{75(5+4\sqrt{10})} + \sqrt[5]{225(35+11\sqrt{10})} + \sqrt[5]{225(35-11\sqrt{10})} + \sqrt[5]{75(5-4\sqrt{10})}$$

is a root of $f(x) = x^5 - 2625x - 61500 = 0$.
Let β be a root of

$$g(x) = \frac{x^7}{7!} + \frac{x^6}{6!} + \frac{x^5}{5!} + \frac{x^4}{4!} + \frac{x^3}{3!} + \frac{x^2}{2!} + \frac{x}{1!} + 1 = 0.$$

Why is $\alpha + \beta$ (or for that matter $\frac{\alpha^3\beta - 4\alpha^2\beta^2 + 7\beta^3}{2\alpha^2\beta^5 - 18\alpha\beta}$) again a root of a polynomial $h(x)$ over $\mathbb{Q}$; i.e. $h(x) \in \mathbb{Q}[x]$?

Abel's argument:

Let

$$h(x) = \prod_{i,j} (x - \alpha_i - \beta_j)\,,$$

where $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_5$ are the roots of $f(x)$ and $\beta = \beta_1, \beta_2, \ldots, \beta_7$ are the roots of $g(x)$.

Then $h(x) \in \mathbb{Q}[x]$ since $\prod_{i,j}(x - \alpha_i - \beta_j)$ is symmetric in the α_i's, as well as in the β_j's. Clearly $h(\alpha + \beta) = 0$

$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a field. It is also a vector space with scalar field $\mathbb{Q}$. In fact $\mathbb{Q}(\sqrt{2})$ is a vector space of dimension two. A basis is $\{1, \sqrt{2}\}$.

Recall the only non-trivial result about vector spaces: *All bases have the same cardinality*.
(First proved by Grassmann in his "Ausdehnungslehre" from 1862, but ignored. Rediscovered by Steinitz in 1913. Vector spaces were first formally defined in the 1920's.)

van der Waerden's (i.e. Artin's and E. Noether's) proof (1930):

$$\begin{array}{c} \mathbb{C} \\ | \\ \mathbb{Q}(\alpha, \beta) \\ | \\ \mathbb{Q} \end{array}$$

$\mathbb{Q}(\alpha, \beta)$ is a *vector space* over $\mathbb{Q}$, i.e. $\mathbb{Q}$ is the scalar field. By the Euclidean algorithm ("dividing out by irreducible polynomials") one shows that $\mathbb{Q}(\alpha, \beta)$ is of finite dimension n, say. Consider $\{1, \alpha+\beta, (\alpha+\beta)^2, \ldots, (\alpha+\beta)^n\}$. This is a linearly dependent set, and so there exists $c_0, c_1, \ldots, c_n$ in $\mathbb{Q}$ such that

$$c_0 + c_1(\alpha+\beta) + c_2(\alpha+\beta)^2 + \cdots + c_n(\alpha+\beta)^n = 0$$

<u>Conclusion</u>: $\alpha + \beta$ is a root of the polynomial

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \in \mathbb{Q}[x]$$

Instead of $\mathbb{Q}$, consider the field $\mathbb{C}(x)$. A similar argument as above proves that if $y_1 = y_1(x), \ldots, y_m = y_m(x)$ are algebraic functions of x, then $R(y_1, y_2, \ldots, y_m)$ is an algebraic function of x. Here $R(y_1, y_2, \ldots, y_m)$ is a rational function of $y_1, y_2, \ldots, y_m$.

Abel's introduction of the fundamentally important Galois resolvent: There exists a linear combination

$$\theta = c_1 y_1 + c_2 y_2 + \cdots + c_m y_m,$$

where $c_1, c_2, \ldots, c_m$ are constants, such that each $y_i = y_i(x)$ is a rational function of x and θ.

Abel's addition theorem in its most general form

(i) $P(x, y) = y^n + p_{n-1}(x)y^{n-1} + \cdots + p_1(x)y + p_0(x) = 0.$

(ii) $Q(x, y) = q_{n-1}(x)y^{n-1} + \cdots + q_1(x)y + q_0(x) = 0$, where each $q_j(x)$ is a polynomial in x and the coefficients of $q_j(x)$ are polynomials in the parameters $a, a', a'', \ldots$. Elimination between (i) and (ii) yields

(iii) $\rho(x) = \rho(x, a, a', a'', \ldots) = 0$, where $\rho(x)$ is a polynomial in x with coefficients that are polynomials in $a, a', a'', \ldots$. Let μ be the degree of $\rho(x)$, and let $x_1, x_2, \ldots, x_\mu$ be the roots of (iii). Let $\psi(x) = \int_0^x f(x, y)\,dx$, where $f(x, y)$ is a rational function in x and y. Then:

(iv) $\psi(x_1) + \psi(x_2) + \cdots + \psi(x_\mu) = u + k_1 \log v_1 + \cdots + k_m \log v_m$, where $u, v_1, v_2, \ldots, v_m$ are rational functions of $a, a', a'', \ldots$ and $k_1, k_2, \ldots, k_m$ are constants.

By choosing $Q(x, y)$ appropriately one can show that for any points $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ on the curve $P(x, y) = y^n + p_{n-1}(x)y^{n-1} + \cdots + p_1(x)y + p_0 = 0$:

$$\sum_{i=1}^{m} \int_0^{(x_i, y_i)} f(x, y)\, dx = \sum_{i=1}^{g} \int_0^{(x_i', y_i')} f(x, y)\, dx +$$

rational/logarithmic terms.

where each (x_i', y_i') is an algebraic function of $(x_1, y_1), \ldots, (x_m, y_m)$, and g (the genus) only depends upon the curve $P(x, y) = 0$. ($f(x, y)$ is a rational function of x and y.)

Atle Selberg:

Det har alltid stått for meg som den rene magi. Hverken Gauss eller Riemann, eller noen annen, har noe som riktig kan måle seg med dette.

[*For me this has always appeared as pure magic. Neither Gauss nor Riemann nor anyone else have anything that really measures up to this.*]