Axel Thue and the Prouhet–Thue–Morse sequence

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Axel Thue (1863–1922)

Axel Thue reading without glasses:

"Om Pendelets Betydning for Geometrien".

Axel Thue reading with glasses:

"Om Poncelets Betydning for Geometrien".

Thue's Theorem (1909)

The Diophantine equation

$$f(x, y) = m$$

has only finitely many integral solutions (x, y). Here $f(x, y) \in \mathbb{Z}[x, y]$ is an irreducible (homogeneous) polynomial of degree at least 3, and m is a non-zero integer.

Example

$f(x, y) = x^4 - 12x^2y^2 - 8xy^3 + 4y^4 = 1$ has only the solutions $(x, y) = (\pm 1, 0), (\pm 1, \mp 1), (\pm 1, \pm 3)$ and $(\pm 3, \pm 1)$.

Liouville (1844) – Thue (1909) – Roth (1955) Theorem

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be an algebraic number, and let $\epsilon > 0$. Then the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}$$

has only finitely many solutions.

Die von Thue im 1912 bewiesenen Sätze wurden 1938 von Pisot auf anderen Wege wiedergefunden und verschärft. In der neueren Litteratur hat man solche Zahlen als "Pisot–Vijayaraghavan numbers" bezeichnet, sehr zum Unrecht gegen der eigentlichen Entdecker.

— Carl Ludwig Siegel

Basic Concepts

Finite alphabet A (e.g. $A = \{a, b, c\}$). Infinite sequence ω over A, i.e. $\omega \in A^{\mathbb{N}}$.

Example

$$\omega = ab\,\underbrace{bcacbc}_{W}\,cc\,\underbrace{abc}_{W_1}\,\underbrace{abc}_{W_2}\,b\,\underbrace{bc}_{u_1}\,\underbrace{bc}_{u_2}\,\underbrace{bc}_{u_3}\,a\ldots$$

$W = bcacbc$ is a subword of ω.

ω contains a square, namely W_1W_2, where $W_1 = W_2 = abc$.

ω contains a cubic, namely $u_1u_2u_3$, where $u_1 = u_2 = u_3 = bc$.

Thue's questions:

(i) Does there exist a cube-free binary sequence ω? [$\omega \in A^{\mathbb{N}}$, where $A = \{0, 1\}$]

(ii) Does there exist a square free sequence on three symbols?

The Thue substitution σ and the Prouhet–Thue–Morse (PTM) sequence ω

$$\sigma: \begin{cases} 0 \to 01 \\ 1 \to 10 \end{cases} \qquad \mathbf{t} = \lim_n \overrightarrow{\sigma^n(0)}$$

$$0 \to \underbrace{01}_{=\sigma(0)} \to \underbrace{0110}_{=\sigma^2(0)} \to \underbrace{01101001}_{=\sigma^3(0)} \to \underbrace{0110100110010110}_{=\sigma^4(0)} \to \cdots \quad \mathbf{t}$$

Other ways to generate $\mathbf{t} = t_0 t_1 t_2 t_3 \cdots$

(i) $t_0 = 0$, $t_{2n} = t_n$, $t_{2n+1} = 1 - t_n$

(ii) $n = a_k a_{k-1} \cdots a_1 a_0$ (binary representation). If the number of 1's is odd then $t_n = 1$, if even $t_n = 0$. [Example: $23 = 10111$, so $t_{23} = 0$.]

Theorem (Thue (1906, 1912))

The PTM-sequence $\mathbf{t} = t_0 t_1 t_2 t_3 \cdots$ is cube-free. Also, $\mathbf{t}$ has no overlapping-squares, i.e. $\mathbf{t}$ has no subwords of the form $0w0w0$ or $1w1w1$, where w is a subword of $\mathbf{t}$. The sequence $\tilde{\mathbf{t}} = v_0 v_1 v_2 v_3 \cdots \in \{0, 1, 2\}^{\mathbb{N}}$ is square-free, where v_n is the number of 1's between the n'th and the $(n+1)$'th occurrence of 0 in $\mathbf{t}$.

$$\mathbf{t} = 0110100110010110 \cdots$$
$$\tilde{\mathbf{t}} = 2102012 \cdots$$

Remark

The cube-freeness of ω is used to prove the Burnside problem for groups: Is every group G with a finite number of generators and satisfying the identity $x^n = 1$ finite?

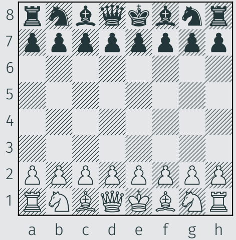
(Answer: No for large odd n; for example, for all odd n with $n \geq 665$.)

Dutch chess grandmaster Max Euwe (1901–1981)—world champion 1935–1937—applied the PTM-sequence to an interesting problem in chess:

The so-called German rule states that a draw occurs if the same sequence of moves occurs three times in succession. Euwe showed, using the cube-free property of **t**, that under such a rule infinite games of chess are possible.

Example

$\mathbf{t} = 01101001\cdots$. Map each 0 to the sequence of four moves (Ng1–f3, Ng8–f6, Nf3–g1, Nf6–g8) and each 1 to the four moves (Nb1–c3, Nb8–c6, Nc3–b1, Nc6–b8).



Some noteworthy properties of t $= t_0 t_1 t_2 t_3 \cdots$

(i) $\sum_{n=0}^{\infty} t_n 2^{-n}$ is a transcendental number. (Proved by K. Mahler in 1929.)

(ii) Let a and b be two distinct positive integers. Substitute a for 0 and b for 1 in $\mathbf{t} = 0110100110010110 \cdots$ and get the sequence: $abbabaabbaababba \cdots$. Then

$$a + \cfrac{1}{b + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \cdots}}}}$$

is a transcendental number (Queffélec (1998)).

(iii) $\prod_{n=0}^{\infty} \left(1 - x^{2^n}\right) = \sum_{n=0}^{\infty} (-1)^{t_n} x^n$

The Prouhet problem

Is it possible to find a partition of the set $\{0, 1, 2, \dots, 2^N - 1\}$ into two disjoint sets I and J, such that

$$\sum_{i \in I} i^k = \sum_{j \in J} j^k$$

for $k = 0, 1, 2, \dots, N - 1$?

Theorem (Prouhet (1851))

The PTM-sequence $\mathbf{t} = t_0 t_1 t_2 t_3 \cdots$ has the following property. Define

$$I = \{i \in \{0, 1, 2, \dots, 2^N - 1\} \mid t_i = 0\}$$
$$J = \{j \in \{0, 1, 2, \dots, 2^N - 1\} \mid t_j = 1\}.$$

Then for every $0 \leq k \leq N - 1$ we have

$$\sum_{i \in I} i^k = \sum_{j \in J} j^k.$$

Example

$N = 4$, $\{0, 1, 2, 3, \ldots, 2^4 - 1 = 15\}$

$\mathbf{t} =$	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
Positions:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

$$I = \{0, 3, 5, 6, 9, 10, 12, 15\}$$
$$J = \{1, 2, 4, 7, 8, 11, 13, 14\}$$

$$0^k + 3^k + 5^k + 6^k + 9^k + 10^k + 12^k + 15^k = 1^k + 2^k + 4^k + 7^k + 8^k + 11^k + 13^k + 14^k$$

for $k = 0, 1, 2, 3$.

Theorem (Prouhet)

Let $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2^2, \ldots, \mu_N = 2^{N-1}$. For $r = 0, 1$, let

$$S_r = \{a_1\mu_1 + a_2\mu_2 + \cdots a_N\mu_N \mid a_i = 0, 1; a_1 + a_2 + \cdots + a_N \equiv r \pmod 2\}.$$

Then

$$\sum_{i \in S_0} i^k = \sum_{j \in S_1} j^k$$

for $k = 0, 1, \ldots, N-1$.

Sketch of proof:

$$F(x) = \prod_{k=1}^{N}(1 - e^{\mu_k x}) = \sum (-1)^{a_1 + \cdots + a_N} e^{(a_1\mu_1 + \cdots + a_N\mu_N)x}$$

where the a's range independently over 0 and 1. Now $F(x)$ has a zero of n'th order at $x = 0$. So $F(x)$ and its first $N-1$ derivatives vanish at $x = 0$. Now

$$F^{(k)}(0) = \sum (-1)^{a_1 + \cdots a_N}(a_1\mu_1 + \cdots a_N\mu_N)^k = 0$$

for $k = 0, 1, \ldots, N-1 \ldots$

Magic squares

By a magic square of order T, we mean a T by T matrix whose entries are taken from the numbers $1, 2, 3, \ldots, T^2$, and such that the sum of the entries in any row, column, or diagonal is the same number.

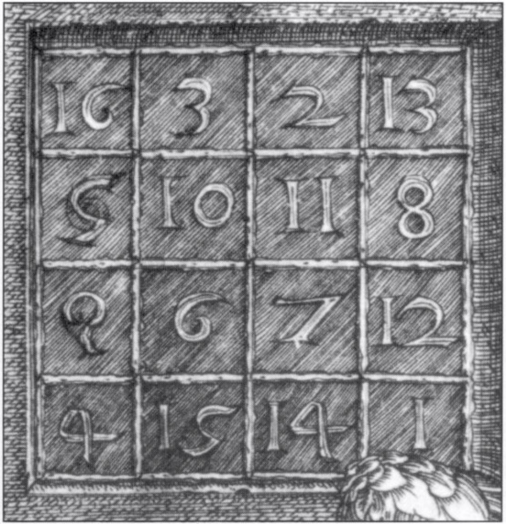
One also requires that each of the numbers $1, 2, 3, \ldots, T^2$ be used exactly once as an entry.

Example

Order 4 magic square that Albrecht Dürer immortalized in his 1514 engraving "Die Melancholie".

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Melencolia I (Albrecht Dürer, 1514)



Observation

The sum of the entries in any row, column or diagonal of a magic square of order T is

$$\frac{1}{2}T(T^2+1).$$

Facts

(i) It is known that one can construct magic squares of any order except 2.

(ii) Any magic square can be rotated and reflected to produce 8 trivially distinct squares, and the eight such squares are said to make up a single equivalence class.

(iii) There is exactly one 3×3 magic square, exactly 880 4×4 magic squares and exactly 275 305 224 5×5 magic squares.

Construction of magic squares of order $T = 2^M$ with special properties using the PTM-sequence t

We illustrate this by constructing a 4×4 magic square, i.e. putting $M = 2$:

$$\mathbf{t} = 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0$$

$$\color{red}{1} \quad 2 \quad 3 \quad \color{red}{4} \quad 5 \quad \color{red}{6} \quad \color{red}{7} \quad 8 \quad 9 \quad \color{red}{10} \quad \color{red}{11} \quad 12 \quad \color{red}{13} \quad 14 \quad 15 \quad \color{red}{16}$$

(positions of 1's, positions of 0's)

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

$\rightsquigarrow$

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Property: The sum of the squares of a row (respectively column) equals the sum of the squares of the complementary row (respectively column).

More generally, by Prouhet's theorem it follows that by this type of construction of magic squares of order $T = 2^M$, a row and its complementary row (respectively, a column and its complementary column) have equal sums of the k'th power of their entries for all $k \leq \max\{2, M-1\}$.

Bob and Joe duelling

Bob and Joe are in a duel.

They are both terrible shots, and equally so. On the other hand, they are deeply committed to fairness, and therefore they make the following deal:

Bob shoots first. Then Joe shoots as many times as he needs to obtain a probability of winning that exceeds the probability that Bob has won so far.

Then Bob shoots again until his probability of having won exceeds Joe's so far.

Joe shoots next following the same rule, and so they continue until someone finally succeeds in hitting the other.

To illustrate, suppose the duelers' hitting probability is 1/3.

- Bob shoots first, so his probability of winning by the end of round 0 is 1/3. Joe's probability of winning is so far zero, so he shoots next.
- For Joe to win in round 1, Bob has to have missed in round 0 and Joe has to hit. Therefore, Joe's probability of having won by the end of round 1 is $(2/3) \cdot (1/3) = 2/9$. This is still less than 1/3, so Joe shoots again in round 2.
- For Joe to win round 2, he must survive Bob's initial shot, miss in round 1, and hit in round 2. Hence the probability of winning by the end of round 2 is $(1/3) \cdot (2/3) + (1/3) \cdot (2/3)^2 = 10/27$. This is more than Bob's probability of 1/3, so Bob gets to shoot in round 3.
- Continuing this assignment one gets that the order of who gets to shoot will be $\underbrace{BJJBJB}\,JJB \cdots$.

  We see that the 6 first positions agree with the (PTM)-sequence, where $B \leftrightarrow 0$, $J \leftrightarrow 1$.

Theorem

Let p be the dueler's hitting probability. As $p \to 0$, the shooting order will "converge" to the (PTM)-sequence.

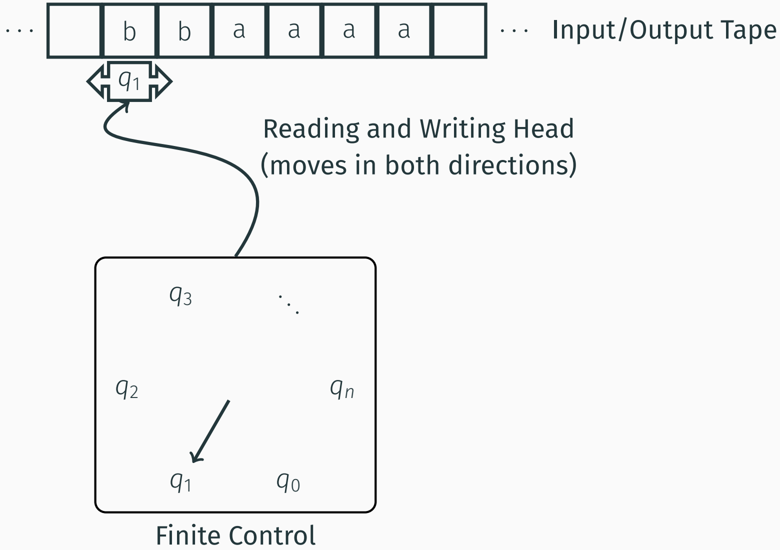
Thue—a precursor of Alan Turing (1912–1954) and Noam Chomsky (1928–)

In 1916 Thue introduced a so-called string reuniting system, later named a semi-Thue system.

His goal was two-fold.

- Firstly, he hoped to solve the word problem for finitely presented monoids (or semigroups).
- Secondly, he had a much more general goal: to add additional constructs to logic, so as to create systems that would allow general mathematical theorems to be expressed in a formal language, and then proven and verified in an axiomatic mechanical fashion, i.e. reduced to a set of defined manipulations on a set of strings over a finite alphabet.

Turing machine



- In 1947 E. Post and A.A. Markov independently showed that Thue's word problem is undecidable, i.e. there is no general algorithm for solving this problem. (According to Martin Davis this was the first unsolvability proof of a problem from classical mathematics.)

- In 1956 Chomsky introduced his hierarchy of formal languages and their associated grammars, and it was realized that his Type-0 grammars, the so-called unrestricted grammars, were isomorphic to the semi-Thue systems. This again was shown to be isomorphic to Turing machines.

- Thue (invented in 2001 by John Colagioia) is a programming language, based on the semi-Thue grammar, which recognizes Type-0 languages, and so is Turing-complete.

- A monoid (or semigroup) is a set S with an algebraic structure consisting of a single associative binary operation and having an identity element.

- A monoid S may be given a presentation by specifying a set of generators $\Sigma \subseteq S$ and a set of relations R on Σ^* (finite strings over Σ).

- If Σ and R are finite sets we say that S can be finitely presented, and we call $T = (\Sigma, R)$ a Thue system for the monoid S.

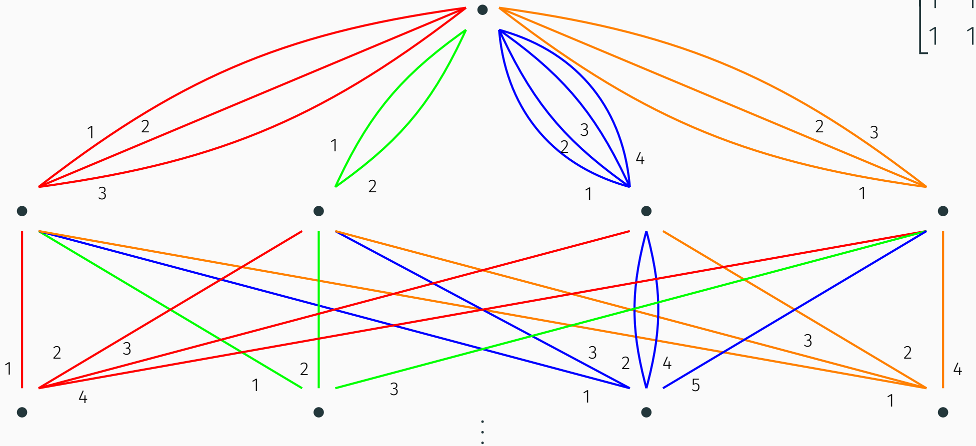
- The word problem for S can be stated in this way: given two strings u and v over Σ; can u be transformed into v by applying the "rules" imposed by the relation R?

Symbolic dynamics and the (PTM)-sequence t

- Form $\omega = \overleftarrow{\mathbf{t}}\,.\,\overrightarrow{\mathbf{t}} = \cdots 10010110.01101001 \cdots \in \prod_{-\infty}^{\infty}\{0,1\}$ where $\overrightarrow{\mathbf{t}} = \mathbf{t}$ and $\overleftarrow{\mathbf{t}}$ denotes $\mathbf{t}$ "reflected".
- Let $S\colon \prod_{-\infty}^{\infty}\{0,1\} \to \prod_{-\infty}^{\infty}\{0,1\}$ be the shift map, i.e. $S((x_n)) = (x_{n+1})$ for $(x_n) \in \prod_{-\infty}^{\infty}\{0,1\}$.
- Let $X = \overline{\{S^n\omega \mid n \in \mathbb{Z}\}} \subseteq \prod_{-\infty}^{\infty}\{0,1\}$. Then X is a Cantor set and (X,S) is a minimal symbolic system.
- The Bratteli–Vershik model for (X,S) is exhibited in the figure.

The Bratteli–Vershik model

Incidence matrix: $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$



etc.

- The (ordered) K_0-group of (X, S), denoted $K^0(X, S)$, is the inductive limit of

$$\mathbb{Z}^4 \xrightarrow{A} \mathbb{Z}^4 \xrightarrow{A} \mathbb{Z}^4 \xrightarrow{A} \cdots$$

  and the order unit is $[3, 2, 4, 3]$.

- $K^0(X, S)$ is a complete invariant for the orbit structure of (X, S).

- $K^0(X, S)$ is equal to $K_0(C(X) \rtimes_S \mathbb{Z})$, where $C(X) \rtimes_S \mathbb{Z}$ is the C*-crossed product associated to (X, S), and $K_0(C(X) \rtimes_S \mathbb{Z})$ is a complete isomorphism invariant for C*-algebras associated to minimal Cantor systems.

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