Aperiodic tilings and quasicrystals
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Outline

1. Introduction
2. History and motivations
3. Building aperiodic tilings
4. Symbolic coding of tilings
What is a tiling?

A tiling is a covering of the space by geometric shapes (*tiles*) such that:
- they *cover* the space;
- there is *no overlap*.

Often, we ask for finitely many tiles types up to some kind of motion (for us: translation).
An example: tiling with squares
An example: tiling with hexagons
An example: another periodic tiling
Can one produce a **regular** tiling with 5-fold symmetry?
An example: the quasiperiodic Penrose tiling
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An example: the quasiperiodic Shield tiling
Aperiodic order?

The last three examples are not periodic, yet highly repetitive.
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Crystals are ordered

The groups of *symmetries* of periodic tilings of the plane (or of the space) are entirely classified. So perfect crystals in which atoms are arranged periodically are classified also.
History and motivations

The first quasicrystal

In 1982, Dan Shechtman observed a diffraction pattern of a material with:
- sharp peaks (order);
- a forbidden 10-fold symmetry (non periodic).

First documented observation of aperiodic order. Need for new models!
The diffraction pattern: 10 fold???

On the left, the diffraction pattern of a zinc–manganese–holmium alloy.
On the right, Dan Shechtman’s logbook.
Shechtman’s observation was first thought to be an experimental error.

“There are no quasicrystals, just quasi-scientists!”

Linus Pauling, Nobel Prize winner
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In 1992, the International Union of Crystallography revised the definition of crystal.
Quasicrystals

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In 2011, Dan Shechtman won the Nobel Prize in chemistry.
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Penrose tilings were invented before quasicrystals. The reason? They were pretty:


In 1997, Kleenex sold toilet paper printed with Penrose tilings. Outrage ensued:

“...when it comes to the population of Great Britain being invited by a multinational [corporation] to wipe their bottoms on what appears to be the work of a Knight of the Realm without his permission, then a last stand must be made.”
Wang tiles and the tiling problem

Even before, aperiodic tilings appeared to answer this question:
Given a set of tiles, does it tile the plane? (answer: the general problem is undecidable.)

The tiles above (with matching rules) tile the plane, but only aperiodically.
Is the number of tiles optimal?
An optimal set of Wang tiles

In 2015, a computer search found a set of 11 aperiodic Wang tiles with 4 colours. This is optimal.

```
 3 1
 2 1
 1 3
 2 3
 4 2
 2 0
 0 0
 0 1
 3 2
 2 2
 1 4
 1 2
 3 3
 3 1
```

An optimal set of Wang tiles
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Different methods

Several methods: substitution, cut-and-project, local matching rules...
The chair substitution
The chair substitution
Why is it aperiodic?

The chair substitution has the unique decomposition property.
Why is it aperiodic?

The chair substitution has the unique decomposition property.
Why is it aperiodic?

This substitution rule does not have the unique decomposition property.
Other example: Penrose
Aperiodic tilings can be generated by local adjacency rules.
Islamic art


Gunbad-i Kabud tomb tower in Maragha, Iran (1197)
A set of basic “Girih tiles” used to produce Islamic patterns.
A section of the Topkapi scroll (15th–16th century).
Islamic art

The cut-and-project method

Idea: cut an “irrational” slice of a regular lattice; project it on a plane.

The resulting pattern will inherit some regularity from the lattice.
Example: Sturmian sequences
Example: Sturmian sequences
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Example: Sturmian sequences

![Sturmian sequence diagram](image)
Example: Sturmian sequences

In this case: one gets Sturmian sequences.

- Aperiodic if and only if the slope is irrational.
- Properties of this tiling related to **arithmetic properties** of the slope.
- For ex. the tiling is (the rewriting of) a substitution if and only if the slope is a quadratic irrational.
Meyer sets

These cut-and-project sets (or “model sets”) are examples of sets studied by Yves Meyer.

Original motivations: harmonic analysis. Which subsets $\Lambda \subset \mathbb{R}^d$ have characters which can be approximated by characters of $\mathbb{R}^d$?

Meyer sets have an approximate analogue of a reciprocal lattice.
Meyer sets

These cut-and-project sets (or “model sets”) are examples of sets studied by Yves Meyer.

**Theorem**

*If a point-set \( \Lambda \) is a Meyer set, and if \( \theta > 1 \) satisfies \( \theta \Lambda \subset \Lambda \), then \( \theta \) is either a Pisot number or a Salem number.*
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How many “chair” tilings are there?
Building chair tilings: growing a patch

\[ \begin{array}{c}
\text{Chair tile} \\
\text{Growing patch}
\end{array} \]
Building chair tilings: growing a patch
Building chair tilings: growing a patch
Building chair tilings: growing a patch
Building chair tilings: growing a patch
Building chair tilings: growing a patch
Building chair tilings: growing further
Building chair tilings: growing further
Building chair tilings: changing the head of the path
Diagram encoding: summary

For a good substitution:
- Infinite paths on the diagram encode tilings of the plane;
- Cofinal paths correspond to tilings which are translate of each other;
Diagram encoding: summary

For a good substitution:
- Infinite paths on the diagram encode tilings of the plane;
- Cofinal paths correspond to tilings which are translate of each other;
- The converse does not hold.