Aperiodic tilings and quasicrystals

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Outline

1 Introduction

2 History and motivations

3 Building aperiodic tilings

4 Symbolic coding of tilings

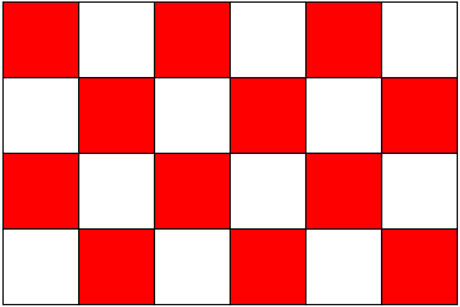
What is a tiling?

A tiling is a covering of the space by geometric shapes (*tiles*) such that:

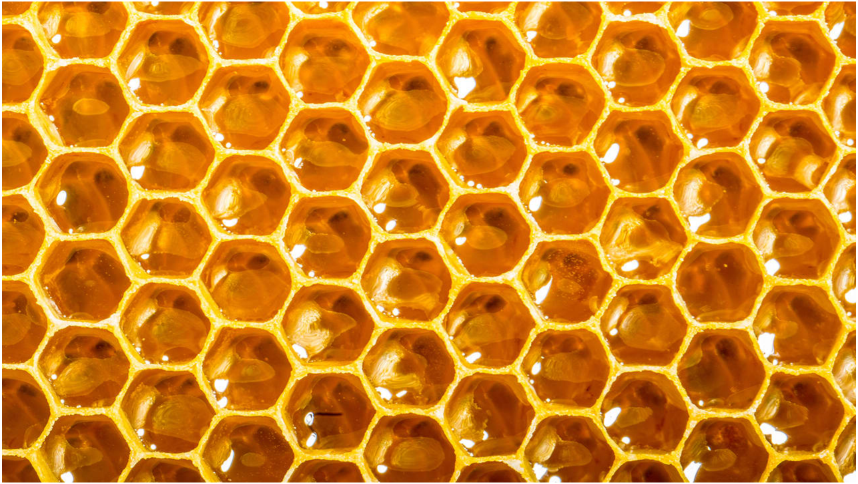
- they cover the space;
- there is no overlap.

Often, we ask for finitely many tiles types up to some kind of motion (for us: translation).

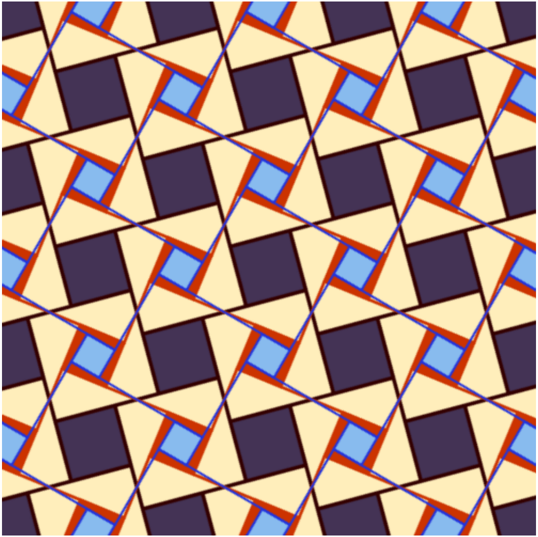
An example: tiling with squares



An example: tiling with hexagons



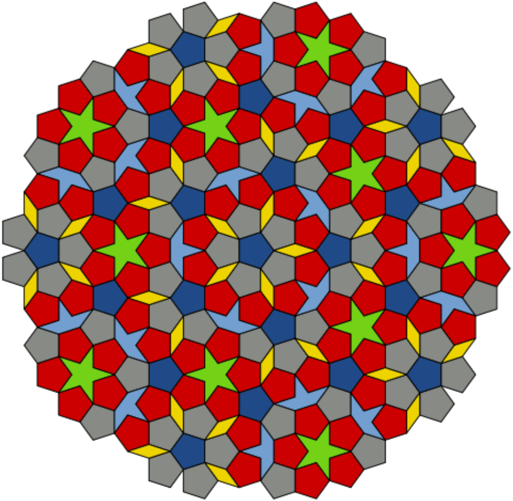
An example: another periodic tiling



Tilings with 5-fold symmetry?

Can one produce a regular tiling with 5-fold symmetry?

An example: the quasiperiodic Penrose tiling



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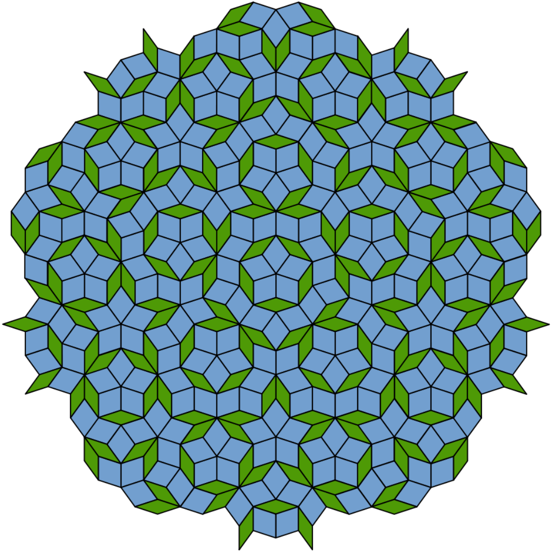

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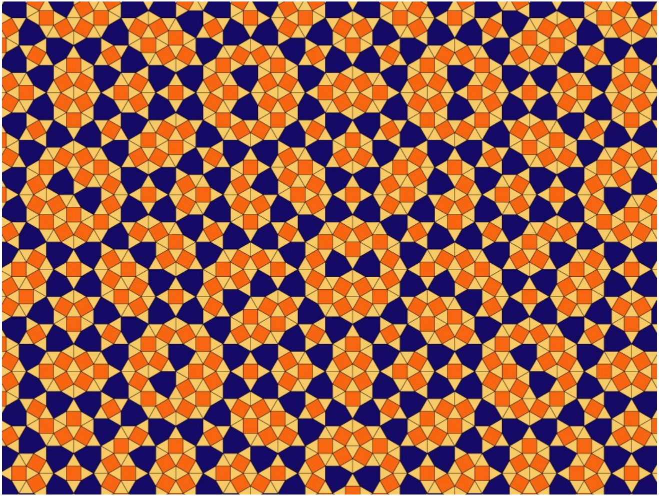

Aperiodic order?

The last three examples are not periodic, yet highly repetitive.

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Crystals are ordered

The groups of symmetries of periodic tilings of the plane (or of the space) are entirely classified.
So perfect crystals in which atoms are arranged periodically are classified also.

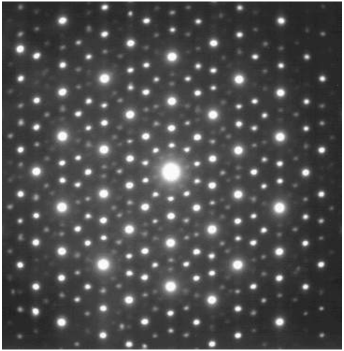
The first quasicrystal

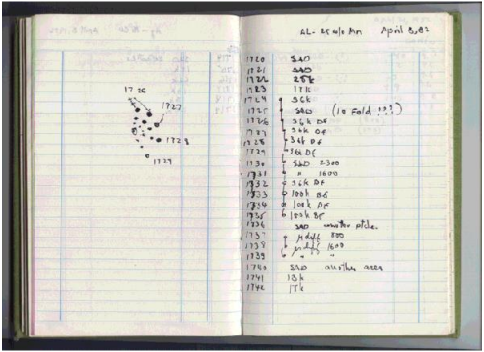
In 1982, Dan Shechtman observed a diffraction pattern of a material with:

- sharp peaks (order);
- a forbidden 10-fold symmetry (non periodic).

First documented observation of aperiodic order. Need for new models!

The diffraction pattern: 10 fold???





On the left, the diffraction pattern of a zinc–manganese–holmium alloy.
On the right, Dan Shechtman's logbook.

Quasicrystals

Shechtman's observation was first thought to be an experimental error.

"There are no quasicrystals, just quasi-scientists!"

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In 1992, the International Union of Crystallography revised the definition of crystal.

In 2011, Dan Shechtman won the Nobel Prize in chemistry.

Sir Roger Penrose and his tilings

Penrose tilings were invented before quasicrystals. The reason? They were pretty:

R. Penrose, *The Rôle of Aesthetics in Pure and Applied Mathematical Research*, Institute of Mathematics and its Applications Bulletin (1974).

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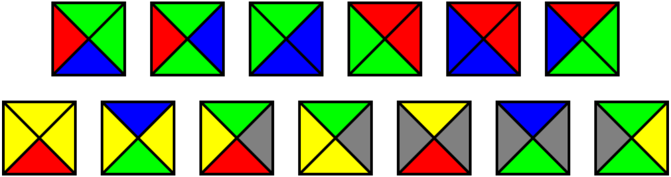
In 1997, Kleenex sold toilet paper printed with Penrose tilings. Outrage ensued:

"...when it comes to the population of Great Britain being invited by a multinational [corporation] to wipe their bottoms on what appears to be the work of a Knight of the Realm without his permission, then a last stand must be made."

Wang tiles and the tiling problem

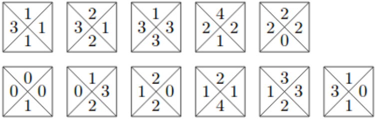
Even before, aperiodic tilings appeared to answer this question:
Given a set of tiles, does it tile the plane? (answer: the general problem is undecidable.)



The tiles above (with matching rules) tile the plane, but only aperiodically.
K. Culik, J. Kari, *On aperiodic sets of Wang tiles*, Lecture Notes in Computer Science (1997).
Is the number of tiles optimal?

An optimal set of Wang tiles

In 2015, a computer search found a set of 11 aperiodic Wang tiles with 4 colours. This is optimal.



E. Jandel, M. Rao, *An aperiodic set of 11 Wang tiles*, prepublication.

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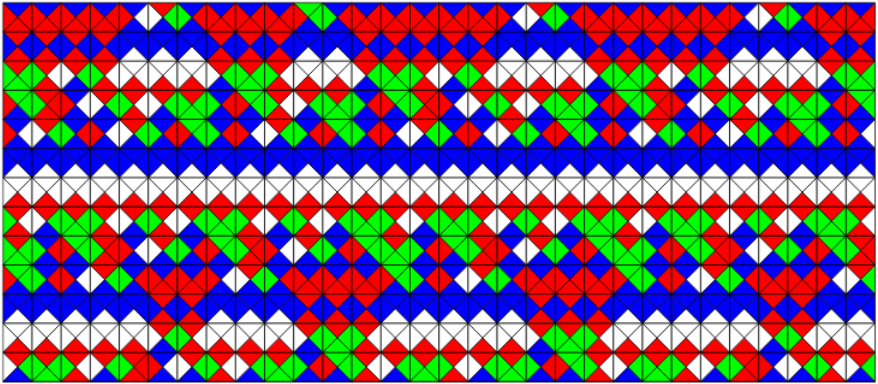

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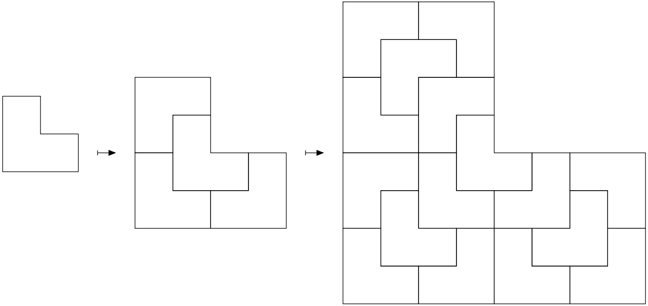
Different methods

Several methods: substitution, cut-and-project, local matching rules...

The chair substitution



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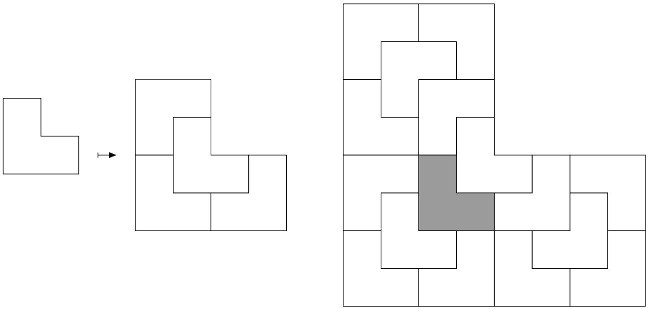
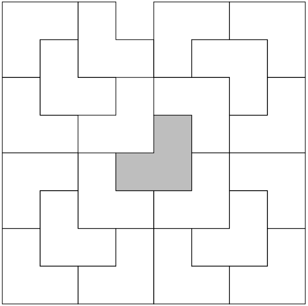


Why is it aperiodic?

The chair substitution has the unique decomposition property.



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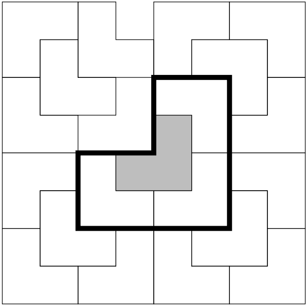
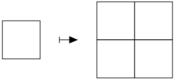
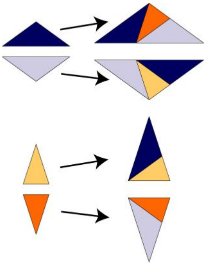


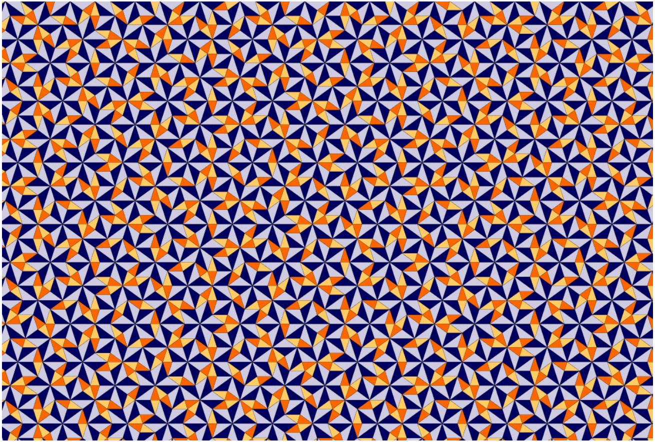
Why is it aperiodic?

This substitution rule does not have the unique decomposition property.



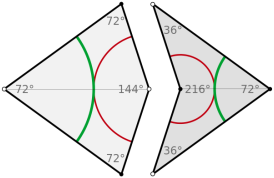
Other example: Penrose

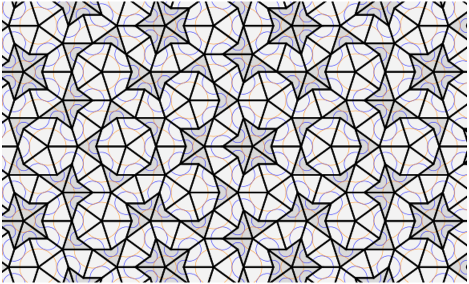




Tilings with local matching rules

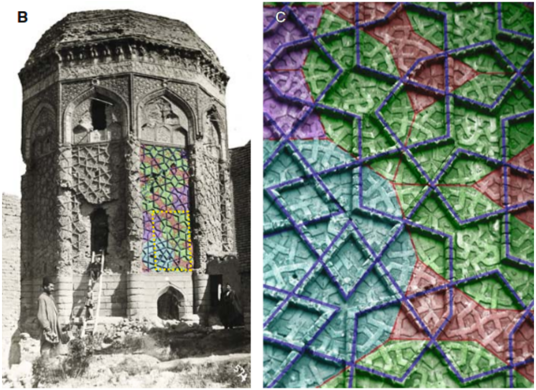
Aperiodic tilings can be generated by local adjacency rules.





Islamic art

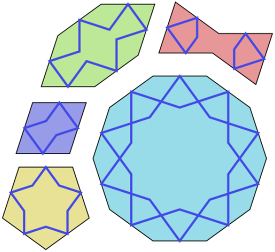
P. Lu and P. Steinhardt, *Decagonal and quasi-crystalline tilings in medieval Islamic architecture* Science **315** (2007), 1106–1110.



Gunbad-i Kabud tomb tower in Maragha, Iran (1197)

Islamic art

A set of basic “Girih tiles” used to produce Islamic patterns.



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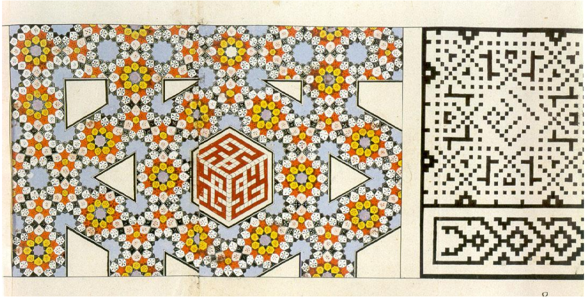
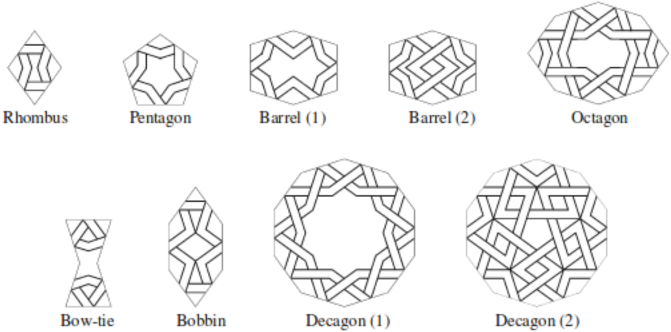


A section of the Topkapi scroll (15th–16th century).

Islamic art

P. Cromwell, *The search for quasi-periodicity in Islamic 5-fold ornament*, The Mathematical Intelligencer **31** (2009), 36–56.

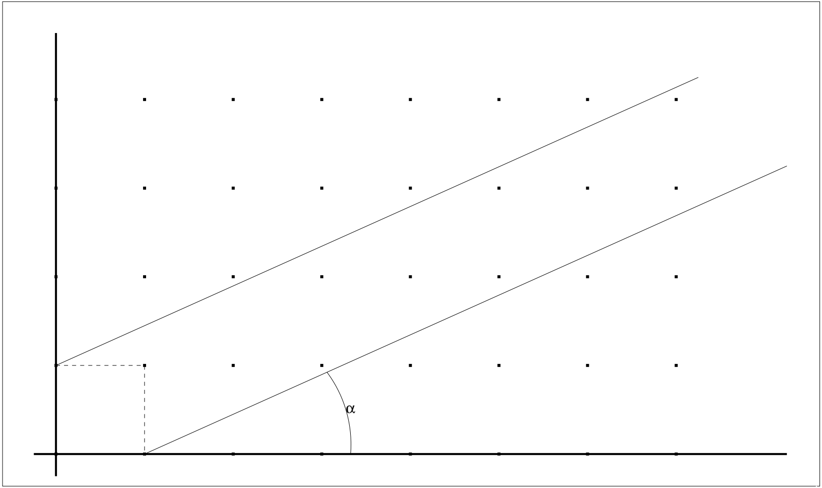


The cut-and-project method

Idea: cut an "irrational" slice of a regular lattice; project it on a plane.

The resulting pattern will inherit some regularity from the lattice.

Example: Sturmian sequences



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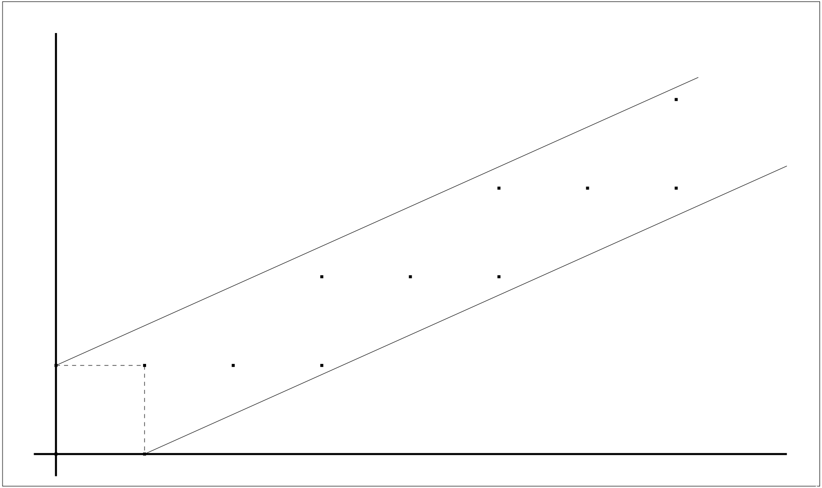

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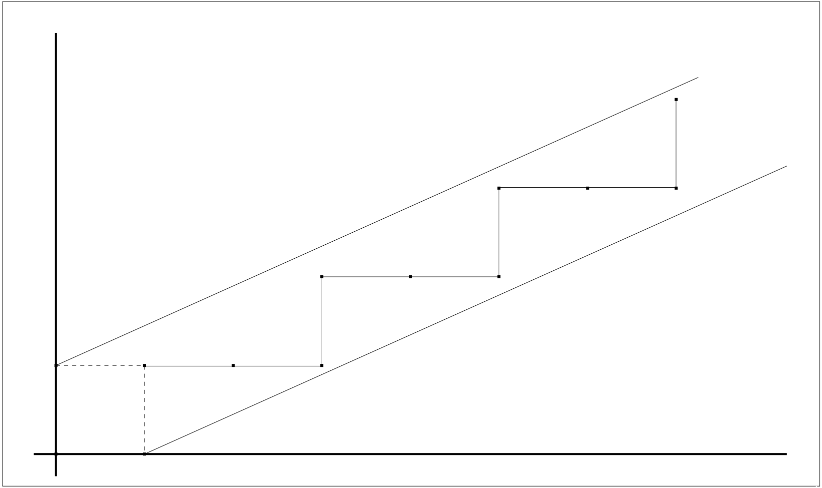

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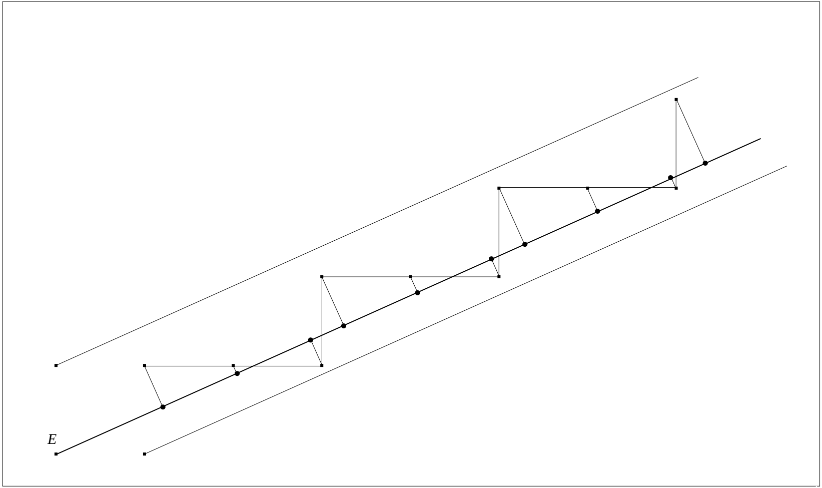

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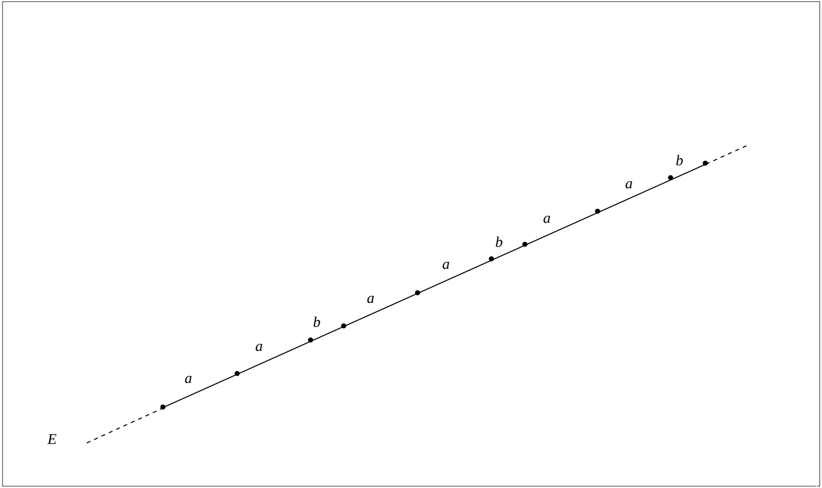

Example: Sturmian sequences

In this case: one gets Sturmian sequences.

- Aperiodic if and only if the slope is irrational.
- Properties of this tiling related to arithmetic properties of the slope.
- For ex. the tiling is (the rewriting of) a substitution if and only if the slope is a quadratic irrational.

Meyer sets

These cut-and-project sets (or “model sets”) are examples of sets studied by Yves Meyer.

Original motivations: harmonic analysis. Which subsets $\Lambda \subset \mathbb{R}^d$ have characters which can be approximated by characters of $\mathbb{R}^d$?

Meyer sets have an approximate analogue of a reciprocal lattice.

Meyer sets

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Theorem

If a point-set Λ is a Meyer set, and if $\theta > 1$ satisfies $\theta\Lambda \subset \Lambda$, then θ is either a Pisot number or a Salem number.

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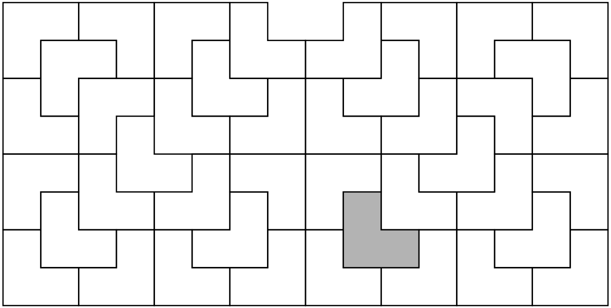
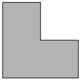
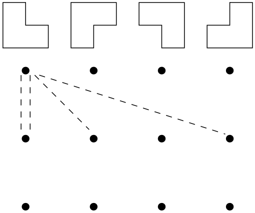
How many “chair” tilings are there?



Building chair tilings: growing a patch





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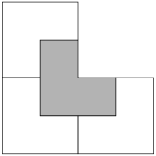

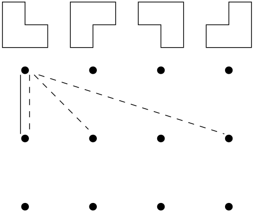

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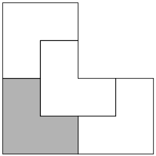

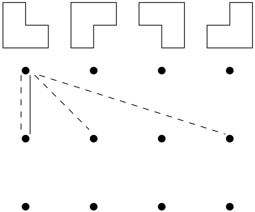

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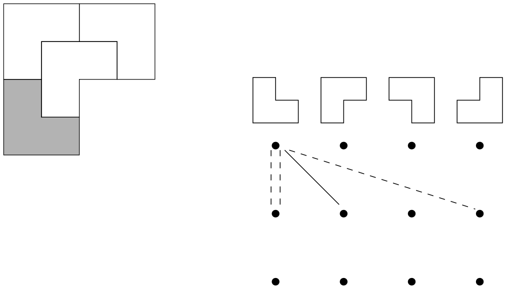

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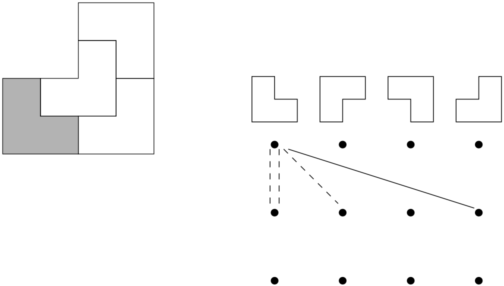

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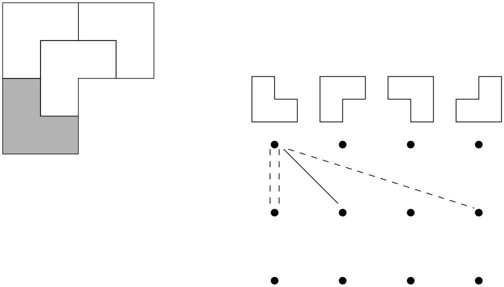
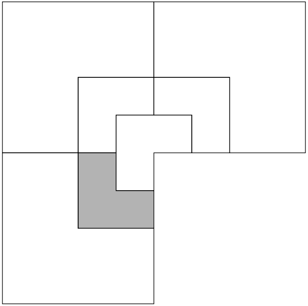
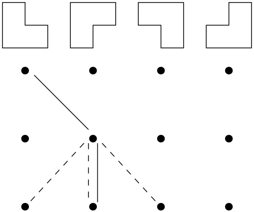


Building chair tilings: growing further





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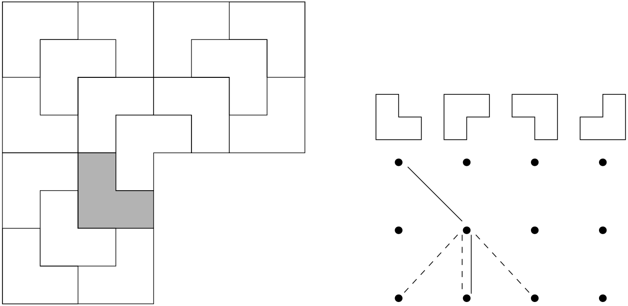

Building chair tilings: changing the head of the path

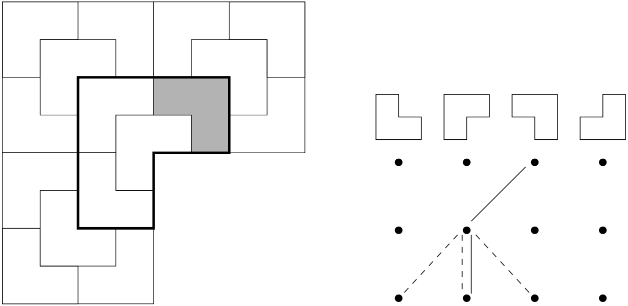

Diagram encoding: summary

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- Infinite paths on the diagram encode tilings of the plane;
- Cofinal paths correspond to tilings which are translate of each other;

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For a good substitution:

- Infinite paths on the diagram encode tilings of the plane;
- Cofinal paths correspond to tilings which are translate of each other;
- The converse does not hold.