Irregular sampling of signals and images

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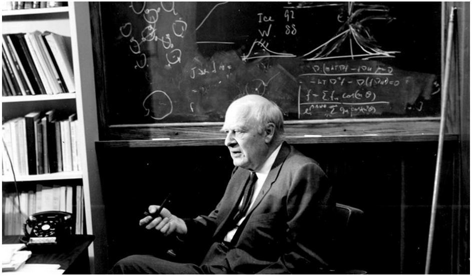

Lars Onsager in his office at Yale.

On the blackboard you can see a Fourier series expansion.

I did the best of my research at Yale, with Raphy Coifman.

Lars Onsager

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- -his courses on statistical mechanics were popularly known as “Advanced Norwegian I" and “Advanced Norwegian II."
- (Michael S. Longuet-Higgins)
- Now you will endure “Advanced French I".
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Organization of the talk

(a) The Nyquist-Shannon theorem

(b) Sparsity & Compressed sensing

(c) Irregular sampling

(d) Sampling on quasi-crystals

The Nyquist-Shannon theorem

The digital world

- Sampling a signal f on a grid $\Lambda = \{\dots, s_{-1}, s_0, s_1, \dots\}$ yields a sequence $X(f) = (f(s_j))_{j\in\mathbb{Z}}$ of numbers.
- The continuous world of signals or images is mapped into the discrete world of sequences.
- This is the digital revolution.
- Is it possible to retrieve f from $X(f)$?

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Band limited signals

- Let $f(t)$ be function of the real variable t. The Fourier transform $\widehat{f}(\omega)$ of f is defined by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} \exp(-2\pi i t\omega)\, f(t)\, dt$$

- A function f of the time variable t is a band limited signal if its Fourier transform $\widehat{f}(\omega)$ is supported by a finite interval $[-\omega_0, \omega_0]$.
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The Nyquist-Shannon theorem

- Let f be a band limited signal and let Λ be a regular grid: $\Lambda = \{0, \pm h, \pm 2h, \ldots\}$ where $h > 0$ is the step size.
- Let us assume that f is sampled on Λ.
- How f can be retrieved from its samples?
- The Nyquist-Shannon sampling rate is defined by $h_0 = \frac{1}{2\omega_0}$ where ω_0 is the largest frequency contained in the signal.
- First part of Shannon's theorem:
- If $h > h_0$ there are not enough measurements and an artifact occurs in the reconstruction: it is named aliasing.

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The Nyquist-Shannon theorem

- Second part of Shannon's theorem.

- Theorem (1)

Let us assume that the Fourier transform of $f \in L^2(\mathbb{R})$ vanishes outside $[-\omega_0, \omega_0]$ and set $h_0 = \frac{1}{2\omega_0}$.

Then for $0 < h \leq h_0$, f can be retrieved from its samples $f(kh)$, $k = 0, \pm 1, \pm 2, \ldots$.

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All children know it

- In many western movies a stagecoach is attacked by some fierce Indians. But in the movie the wheels of the stagecoach are rotating slowly the wrong way. This is ALIASING.
- Explanation: the sampling rate in a movie is 24 images per second. It is not sufficient to catch the rapid rotation of the wheels. This is SHANNON's theorem.
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Compact disc

- Digital audio signal processing is based on the Nyquist-Shannon theorem.
- The sampling rate used in CDs follows from from physiology and Shannon's theorem. Our hear cannot perceive sounds whose frequency is larger than 20 kHz.
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- The Nyquist-Shannon sampling theorem states that a sampling rate of more than twice the maximum frequency of the signal to be recorded is needed, resulting in a required rate of at least 40 kHz.
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Paley-Wiener space

- Let $f(x)$ be a square integrable function defined on $\mathbb{R}^n$. The Fourier transform of f is

$$\widehat{f}(\omega) = \int_{\mathbb{R}^n} \exp(-2\pi i x \cdot \omega) f(x)\, dx$$

- **Definition**
  Let $K \subset \mathbb{R}^n$ be a compact set and We write $f \in PW_K^2$ if f is square integrable and if $\widehat{f}$ vanishes outside K.
  - If $n = 1$ and $K = [-\omega, \omega]$, $f \in PW_K^2$ is a band limited signal and the cutoff frequency is ω.
  - PW stands for Paley-Wiener. If $p \in [1, \infty]$ we write $f \in PW_K^p$ if $f \in L^p(\mathbb{R}^n)$ and if $\widehat{f}$ vanishes outside K.

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Economical constraints

- In satellite imaging K depends on the optics of the instrument.
- The image is sampled on a lattice.
- The choice of this lattice is seminal in the economical success of the satellite.
- Coarse lattices are prohibited by the generalized Nyquist-Shannon theorem (see below).
- Fine lattices are too expensive.
- The optimal lattice depends on the geometry of K.

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Lattices

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- Equivalently

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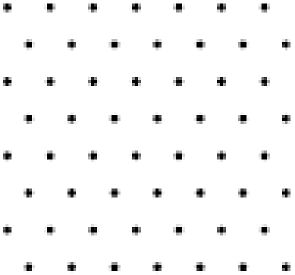

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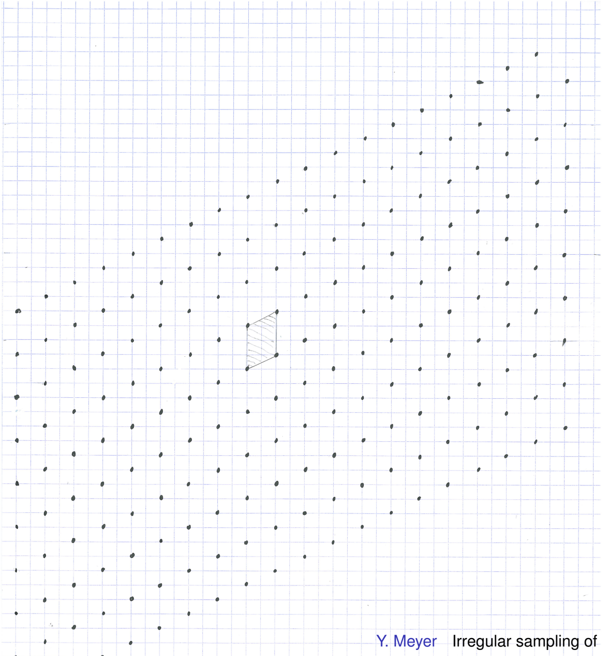

Generalized Shannon's theorem

- The dual lattice of a lattice $\Gamma \subset \mathbb{R}^n$ is defined as $\Gamma^* = \{y;\ \exp(2\pi i y \cdot x) = 1,\ \forall x \in \Gamma\}$.
- The following theorem is seminal in designing an optimal lattice to sample signals or images $f \in PW_K^2$.
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- Theorem (2)

  The two following properties are equivalent

  (a) *Every function $f \in PW_K^2$ can be recovered from its samples on Γ*

  (b) *For every $\gamma^* \in \Gamma^*$ we have*

$$(*) \qquad \gamma^* \neq 0 \Rightarrow |(K + \gamma^*) \cap K| = 0$$

- If K is Riemann integrable Theorem 2 is still valid for $f \in PW_K^p$.
- Y. Katznelson constructed a compact set K such that Shannon's theorem is no longer true when PW_K^2 is replaced by PW_K^p with $1 \leq p < 2$.

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In satellite imaging one is given K and the issue is to find the sparsest Γ for which $(*)$ holds. This corresponds to find Γ^* as dense as possible. This has been achieved by Bernard Escudié who elaborated the SPOT 5 satellite.

SPOT 5

- Satellite SPOT 5 was launched on April 2002 by an Ariane rocket. It was built by the CNES Agency. During fifteen years SPOT 5 provided Earth images with a resolution of 2.5 meters.
- SPOT 5 relied on a new sampling concept named “Supermode".
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- To our greatest surprise the opposite is true. Sampling on a simple quasi-crystal circumvents the limitations imposed by Shannon’s theorem.
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Compressed sensing

- The compressed sensing paradigm is the following statement:
- Let $\mathcal{C}$ be a collection of signals which have a sparse representation in a given orthonormal basis $\mathcal{B}$. It means that the expansion of $f \in \mathcal{C}$ in $\mathcal{B}$ only activates "a few vectors" depending on f. This shall be given a precise definition.
- Compressed sensing amounts to finding a universal sparse collection $\mathcal{G}$ of vectors such that every $f \in \mathcal{C}$ can be retrieved from the few samples $< f, g >$, $g \in \mathcal{G}$.

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- Let f be a signal or an image f and let K be the closure of the support of $\widehat{f}$.
- f is sparse in the Fourier domain if the Lebesgue measure $|K|$ of K is small.
- This definition is consistent with Landau's theorem (see below).
- The main message of this talk is the following:
- SPARSE SIGNAL SHOULD BE SAMPLED ON COARSE GRIDS.
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Irregular sampling

Delone sets

- A set of points $\Lambda \subset \mathbb{R}^n$ is a Delone set if:
- (a) Λ is uniformly discrete: There exists a positive β such that

$$\lambda, \lambda' \in \Lambda, \; \lambda \neq \lambda' \Rightarrow |\lambda - \lambda'| \geq \beta$$

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- On can ask a similar property where 2 is replaced by $p \in [1, \infty]$.
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- The converse implication:

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- It is even wrong when Λ is a lattice.
- $(**)$ is true if

1. Λ is a simple quasi-crystal,
2. $\geq$ is replaced by $>$ in $(**)$.

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Y. Katznelson

Y. Katznelson constructed a compact set K such that Landau's theorem fails if PW_K^2 is replaced by PW_K^p with $1 \leq p < 2$.

However Landau's theorem remains valid if K is Riemann integrable.

Universal sampling sets

- In “A universal sampling of band-limited signals", C.R. Math. Acad. Sci. Paris, 342 (2006) 927-931, A. Olevskii and A. Ulanovskii proved the following existence theorem:

- Theorem (4)

  For every positive β there exists a Delone set $\Lambda \subset \mathbb{R}$ with density β such that, for every compact set K with $|K| < \beta$, Λ is a set of stable sampling for PW_K^2.

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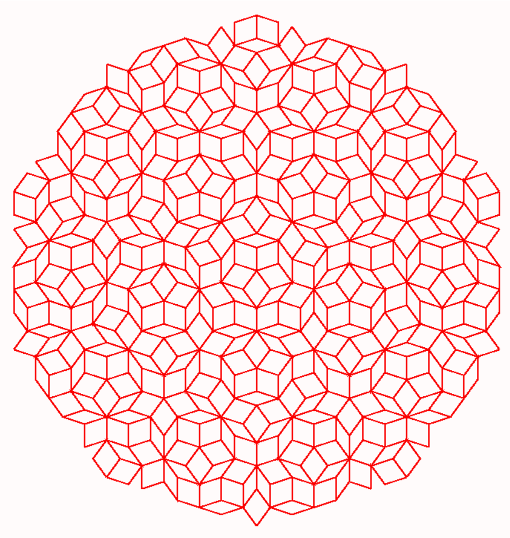

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- This is a Penrose paving (1974). It is a diamond tiling of the plane with isometric copies of two proto-tiles which here are two diamonds.
- The whole paving is rotational invariant by a $2\pi/5$ rotation around 0. The sampling set is the set of all vertices of these diamonds.
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- Here is the simplest example of a universal sampling set.
- Let $\{x\} \in [0,1)$ be the fractional part of the real number x. We have $x = k + \{x\}$, $k \in \mathbb{Z}$. The first example is

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Let $\theta(x)$ be the distance from x to the nearest integer $k \in \mathbb{Z}$. Consider

$$M = \{k + \theta(k\sqrt{2}),\ k \in \mathbb{Z}\}.$$

Is M a universal sampling set ? M is the union between two disjoint model sets.

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- Our second example of a universal sampling set depends on a parameter $\alpha > 0$ and is defined by

$$\Lambda_\alpha = \{\lambda = m + n\sqrt{2},\ |m - n\sqrt{2}| \leq \alpha,\ m, n \in \mathbb{Z}\}.$$

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- Before defining quasi-crystals we begin with model sets.
- Let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^m$ be a lattice. For $(x, t) \in \mathbb{R}^n \times \mathbb{R}^m$, we write $p_1(x, t) = x$, $p_2(x, t) = t$.
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  Let $Q \subset \mathbb{R}^m$ be a compact set (a window). Let us assume that Q is Riemann integrable with a positive Lebesgue measure. Then the model set $\Lambda_Q \subset \mathbb{R}^n$ is defined by

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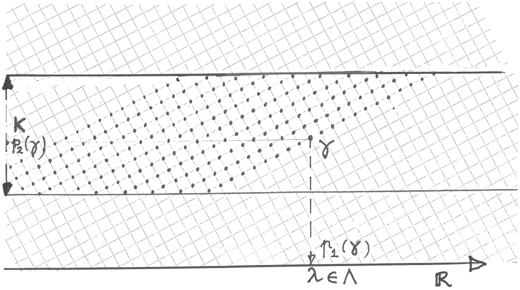

Cut and projection

Γ is a lattice, $\gamma \in \Gamma$

$\gamma = (p_1(\gamma), p_2(\gamma))$

$p_1 : \Gamma \to p_1(\Gamma)$ is one-to-one

$p_2(\Gamma)$ is dense in $\mathbb{R}$

$\Lambda = \{ p_1(\gamma) ; \gamma \in \Gamma, p_2(\gamma) \in K \}$.

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- A simple quasi-crystal is a model set Λ_Q for which $m = 1$ and Q is an interval.
- In “Quasicrystals are sets of stable sampling" Comptes Rendus Académie Sciences Paris (2008) vol. 346, pp 1235-1238, the following theorem is proved:

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- In other words Λ is a stable sampling set for PW_K^2 whenever $K \subset \mathbb{R}^n$ is a compact set such that $|K| < \mathrm{dens}\,\Lambda$.

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- I hoped that the preceding theorem could be extended to general model sets (more general windows). Here is a simple counter example.
- Let $\Lambda \subset \mathbb{R}^2$ be the set of all $x = (x_1, x_2)$ where $x_1 = k + \{k\sqrt{2}\}$, $x_2 = m + \{m\sqrt{2}\}$, $k, m \in \mathbb{Z}$.
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