Tradition and transmission: The Kahane legacy

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Introduction

Jean-Pierre Kahane

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- I used Google. Google pointed at a paper by Kahane entitled *Sur les fonctions moyenne-périodiques bornées* published in 1957 in Annales de l'Institut Fourier [2].
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- Can we infer the spectral properties of a point set $\Lambda \subset \mathbb{R}^n$ from the knowledge of its arithmetical structure ?
- More precisely Kahane investigated the structure of discrete sets $\Lambda \subset \mathbb{R}^n$ such that Every mean-periodic function whose spectrum is contained in Λ is necessarily an almost periodic function.
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- In 1968, I was attacking exactly the same problem. It paved the way to my construction of “model sets". I discovered the spectral properties of model sets. I wrote a book on these issues (North-Holland (1972) [4]). Today model sets are used to model quasi-crystals.
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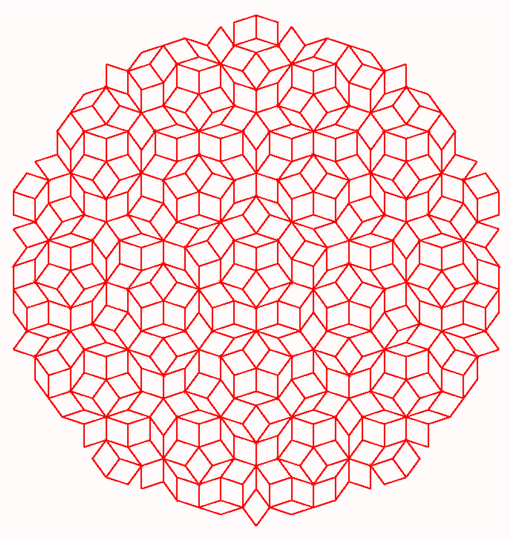

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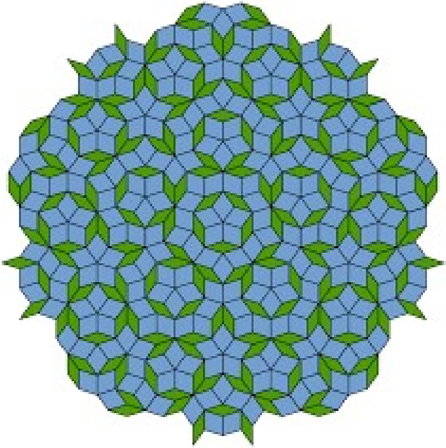

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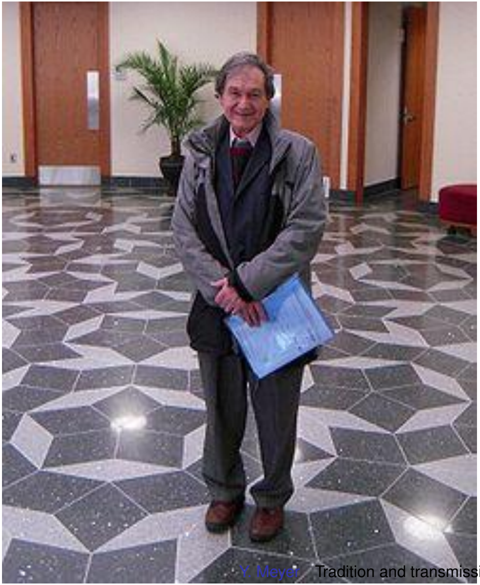

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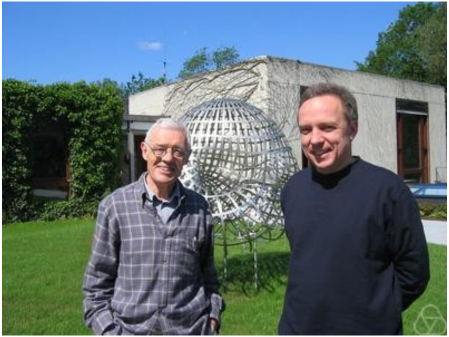
Robert Moody and Michael Baake, by Renate Schmid



Nicolaas de Bruijn, by Konrad Jacobs

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The Unity of Science

- The achievements of Kahane, Penrose, De Bruijn, Moody, Shechtman, Olevskii, Ulanovskii, and my contributions fit together like pieces of a jigsaw puzzle. This global structure was not immediately understood.
- Using distinct tools we were studying the same problem at the same time.
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- Mathematicians are players of an orchestra directed by a hidden conductor.
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- *This Comtian hierarchy is replaced by the orchestration of different instruments, each distinct but brought together to accomplish something bigger than any could do individually.*
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- (a) Almost-periodic functions (Harald Bohr)
- (b) Mean-periodic functions (Jean Delsarte and Laurent Schwartz)
- (c) Kahane's problem: almost-periodic versus mean-periodic
- (d) The new proof.

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Almost-periodic functions

Harald Bohr (1887-1951)

Eternal return

- The function $f(x) = \cos(x) + \cos(2x/3)$ of the real variable x is periodic with period 6π.
- The function $g(x) = \cos(x) + \cos(\sqrt{2}x)$ is not a periodic function. It is an almost periodic function. It repeats itself only approximatively since $\sqrt{2}$ is irrational.

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Eternal return

- Let f be continuous and bounded on $\mathbb{R}^n$ and let $\epsilon > 0$. Let $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. An ϵ almost period τ of f is defined by

$$\sup_{x\in\mathbb{R}^n} |f(x-\tau) - f(x)| \leq \epsilon \|f\|_\infty \tag{1}$$

- Then f is almost periodic in the sense of Harald Bohr if for every $\epsilon > 0$ there exists a $T(\epsilon) > 0$ such that for every x the ball $B(x, T(\epsilon))$ centered at x with radius $T(\epsilon)$ contains an ϵ almost period τ of f.

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Eternal return

- An almost periodic function is bounded and uniformly continuous. The Banach space $\mathcal{B}$ of almost periodic functions is equipped with the norm $\sup_{x\in\mathbb{R}^n}|f(x)| = \|f\|_\infty$
- Let $\Lambda \subset \mathbb{R}^n$ be a discrete collection of points.
- Let $\mathcal{B}_\Lambda$ denote the closure for the norm $\|f\|_\infty$ of the linear space of all trigonometric sums

$$(2) \qquad f(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i x\cdot\lambda)$$

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Mean-periodic functions

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- In the latter case a uniform convergence is demanded: what will happen in a billion of years is as relevant as what will happen tomorrow.
- In the case of mean periodic functions the present is more important than the future.

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Henri Cartan, Jean Delsarte, Simone and André Weil

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- $\mathcal{C}(\mathbb{R}^n)$ is equipped with the topology $\mathcal{T}$ of uniform convergence on compact sets.
- The topology $\mathcal{T}$ is much weaker (less regarding) than the topology used to define almost periodic functions.
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Kahane's program

This picture belongs to Hans Feichtinger

Jean-Pierre Kahane

Lost opportunities

Instead of working together we had sterile political disputes.

Jean-Pierre Kahane

- In his 1957 paper Jean-Pierre Kahane proposed the following definition:

- Definition

  A discrete set $\Lambda \subset \mathbb{R}^n$ satisfies the property $Q(\Lambda)$ if we have $\mathcal{C}_\Lambda = \mathcal{B}_\Lambda$.

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An equivalent definition was given by Kahane in [2]:

Lemma (1)

$Q(\Lambda)$ holds iff there exists a compact set K and a constant C such that for every trigonometric sum $f(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda\cdot x)$ whose frequencies belong to Λ we have

$$\|f\|_\infty \le C \sup_{x\in K} |f(x)| \tag{3}$$

We then say that $Q(K,\Lambda)$ holds.

Examples

Theorem (1)

Let $\alpha > 0$ be a real number. Then $\Lambda_\alpha = -\mathbb{N} \cup \alpha\mathbb{N} = \{\ldots, -4, -3, -2, -1, 0, \alpha, 2\alpha, 3\alpha, 4\alpha, \cdots\}$ satisfies Kahane's Q(Λ) property if and only if $\alpha \in \mathbb{Q}$.

Local structure of mean periodic functions

- Let $\Lambda \subset \mathbb{R}^n$ be uniformly discrete.
- Let $\mathcal{C}_\Lambda$ be the corresponding space of mean periodic functions (closure of trigonometric sums with frequencies en Λ).
- We investigate the local structure of a function $f \in \mathcal{C}_\Lambda$ by restricting it to a compact set K.
- If K is "small" the restriction to K of a function $f \in \mathcal{C}_\Lambda$ does not provide enough information to retrieve f.
- $\rho(K, \Lambda)$ denotes the following degeneracy:
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- Example: $\Lambda = \mathbb{Z}$ and $K = [0, \beta]$ where $0 \leq \beta < 1$.
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Local structure of almost periodic functions

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  - Obviously $R(K, \Lambda) \Rightarrow \rho(K, \Lambda)$.

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Sampling on quasi-crystals

- Simple quasi-crystals are defined now. Let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^m$ be a lattice. For $(x, t) \in \mathbb{R}^n \times \mathbb{R}^m$, we write $p_1(x, t) = x$, $p_2(x, t) = t$.
- Let us assume that p_1 once restricted to Γ is a 1-1 map with a dense range. The same is required on p_2.

Definition

Let $Q \subset \mathbb{R}^m$ be a compact set (a window). Let us assume that Q is Riemann integrable with a positive Lebesgue measure. Then the model set $\Lambda_Q \subset \mathbb{R}^n$ is defined by

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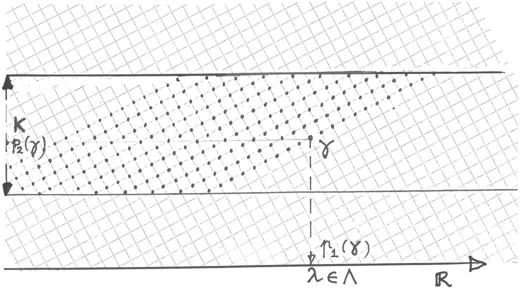

Cut and projection

Γ is a lattice, $\gamma \in \Gamma$

$\gamma = (p_1(\gamma), p_2(\gamma))$

$p_1 : \Gamma \to p_1(\Gamma)$ is one-to-one

$p_2(\Gamma)$ is dense in $\mathbb{R}$

$\Lambda = \{p_1(\gamma);\ \gamma \in \Gamma,\ p_2(\gamma) \in K\}.$

Sampling on quasi-crystals

A model set Λ_Q is a simple quasi-crystal if $m = 1$ and if Q is an interval.

Universal Sampling

Universal sampling sets

- We shall prove the following:
- Theorem (2)

Simple quasi-crystals are universal sampling sets.

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- ## Lemma (2)

  If Λ is a simple model set for every compact set K we have

$$(4) \qquad |K| < \operatorname{dens}\Lambda \Rightarrow R(K,\Lambda)$$

- ## Lemma (3)

  If Λ is a model set then for every compact sets K', L, K such that

  (a) *K' is contained in the interior of L and*
  (b) *L is contained in the interior of K*

  we have

$$(5) \qquad R(K,\Lambda) \Rightarrow S(\infty, L, \Lambda) \Rightarrow S(2, K', \Lambda)$$

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Other functional norms

- Property $S(p, K, \Lambda)$ is defined now.
- $S(p, K, \Lambda)$, $1 \leq p \leq \infty$, denotes the following property: Λ is a stable sampling set for PW_K^p.
- $f \in PW_K^p$ means $f \in L^p(\mathbb{R}^n)$ and $\widehat{f}$ is supported by K.
- Λ is a stable sampling set for PW_K^p iff

$$\|f\|_p \leq C\Big(\sum_{\lambda \in \Lambda} |f(\lambda)|^p\Big)^{1/p} \tag{6}$$

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- Lemma 2 follows from Beurling's theorem together with a result I proved 45 years ago (details are given in the Appendix).
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Appendix

- For proving Lemma 2 we use the fact that Λ is a simple quasi-crystal. For proving Lemma 3 we only need that Λ be a model set.
- Let us restate Lemma 3 as a theorem.

Theorem (3)

If Λ is a model set, if $1 \leq p \leq \infty$, and if K' is contained in the interior of K then $R(K, \Lambda)$ implies $S(p, K', \Lambda)$.

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Appendix

- Proof: If L is contained in the interior of K we have $R(K,\Lambda) \Rightarrow S(\infty, L, \Lambda)$. This is Proposition 3, page 178, of my 1973 Annals of Mathematics paper [3].
- On the other hand if K' is contained in the interior of L we have $S(\infty, L, \Lambda) \Rightarrow S(2, K', \Lambda)$. This is Theorem 3.32 by Olevskii and Ulanovskii in [5].

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The duality principle

- Let us return to Lemma 2. Lemma 2 follows from the duality principle of [3] (Proposition 4, page 181) combined with Beurling's theorem (Theorem 5 below).
- If $Q \subset \mathbb{R}^n$ is a compact set and $M \subset \mathbb{R}^n$ a discrete set, property $T(Q, M)$ holds if every bounded sequence defined on M is the restriction to M of a bounded function whose Fourier transform is supported by Q.

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The duality principle

Let us state the duality principle as a theorem.

Theorem (4)

Let Λ_Q be a model set defined by a window Q as above, let Γ^* denote the dual lattice, and let M_K be the "dual" model set defined by

$$M_K = \{p_2(\gamma^*); \gamma^* \in \Gamma^*, \, p_1(\gamma^*) \in K\}.$$

Then $T(Q, M_K)$ implies $R(K_0, \Lambda_Q)$ whenever K_0 is contained in the interior of K.

Arne Beurling's theorem

- When Q is an interval and the upper density of M is inferior to the length of Q the property $T(Q, M)$ results from Beurling's theorem.

- **Theorem (5)**

Let $M = \{m_k,\ k \in \mathbb{Z}\}$, be an increasing sequence of real numbers such that

$$\lim_{k \to +\infty} m_k = +\infty \text{ and } \lim_{k \to -\infty} m_k = -\infty.$$

Let I be an interval. Then the following two properties are equivalent:

(a) *Every bounded sequence $c_k,\ k \in \mathbb{Z}$, is the restriction to M of a bounded function F belonging to the Bernstein space B_I*

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