WHY DO SOME PEOPLE HAVE PROBLEMS COMPREHENDING MATHEMATICS?

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07 September 2018

The presentation

- Brain science
- Semiotics
- Didactical design

Mathematics

"The knowledge of mathematical things is almost innate in us… this is the easiest of sciences, a fact which is obvious in that no one's brain rejects it; for layman and people who are utterly illiterate know how to count and reckon."

R. Bacon (1267, cited in Dehaene, 2011, p. 260)

"Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

B. Russell (1918, p. 75)

MATHEMATICAL THINKING – ITS PREREQUISITE

- Mathematical thinking is not a "natural" human activity — it is a *cultural device* that has to be developed from mental systems meant to support primary mathematical abilities:
  - Numerosity
  - Ordinality
  - Counting
  - Simple arithmetic

Sets up to 4 items

(Geary, 1994, 1995; Schoenfeld, 1985)

THE APPROXIMATE NUMBER SYSTEM

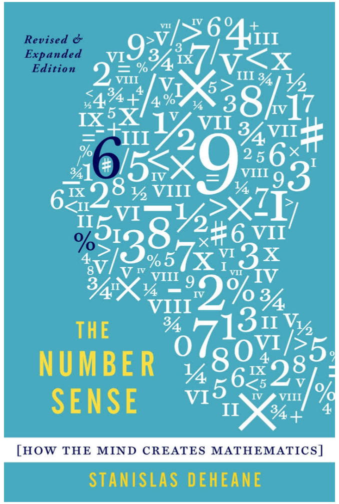
- a brain system that supports the **estimation of the magnitude of a group** without relying on language or symbols
- improves throughout childhood and reaches a final adult level of approximately 15% accuracy
- a child's precision level of this brain system predicts later mathematical achievement in school (!)

(Gilmore et al., 2010; Mazzocco et al., 2011)

Learning disability in mathematics

Three possible sources:

- The approximate number system does not develop at a desirable pace
- Not associating symbols to the quantities represented
- Handling the cardinal value of numbers, but not understanding the logical relationships among those numbers



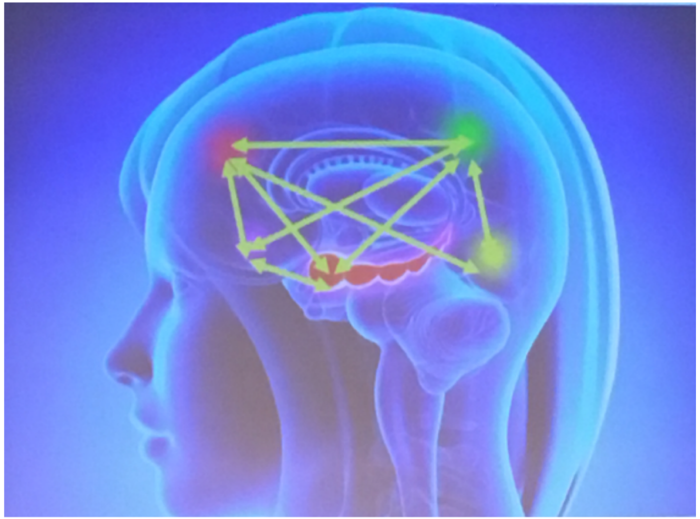
→ a *deficit in the* ***representation*** *of numerosity*

(Dehaene et al., 1999; Feigenson et al., 2004; Geary, 2013)

WHAT DOES THE BRAIN SCIENCE SAY?

When we calculate, 5 different areas of the brain light up and communicate, including 2 visual pathways:

- The dorsal pathway (green area) → visual or spatial representations of quantity, such as a *number line*.
- The ventral pathway (yellow area) → symbolic representations, such as *numerals*.



(Boaler et al., 2016; Milner & Goodale, 2006)

WHAT DOES THE BRAIN SCIENCE SAY?

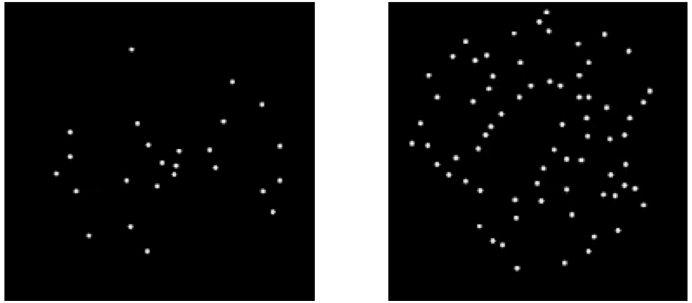
- People with amazing accomplishments have more communication between different areas of the brain
- What encourages brain connections is when we see mathematics in different ways: numbers, visuals, words, algebraic expressions, algorithms, gestures…

→ improving maths achievement means *improving communication*

(Hubbard et al., 2005; Park & Brannon, 2013; Boaler et al., 2016)

Is it possible to improve the approximate number system?

- 26 adults with no formal education (mean age 22.4) underwent 10 sessions with *non-symbolic approximate arithmetic* training tasks
- Control group of 26 adults with no formal education
- Add or subtract **visually presented dot arrays without counting** → mentally add (or subtract) the numerosity of two dot arrays (ranged from 9 to 36).



- Pre- and post-test sessions on symbolic arithmetic tasks

→ Experiment group had **significant improvement** on symbolic arithmetic

(Park & Brannon, 2013)

Non-symbolic and symbolic numerical abilities are related

- Kindergarten children (diverse backgrounds) were tested on their non-symbolic arithmetic abilities during the school year
- Performance of non-symbolic arithmetic predicted children's mathematics achievement at the end of the school year

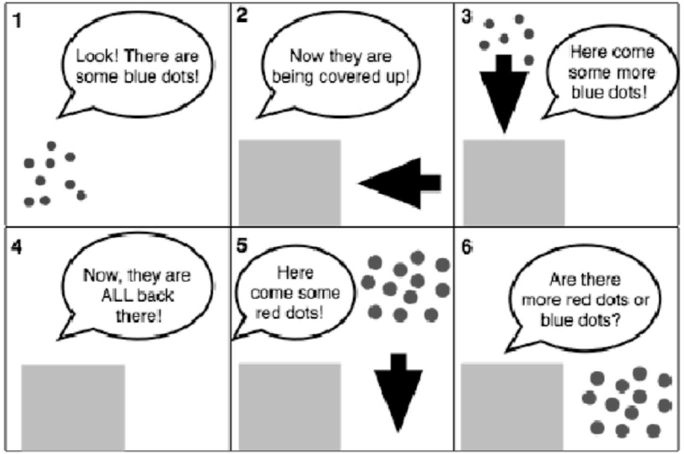


Figure 1.
The non-symbolic addition test for a sample problem in Exp. 1.

(Gilmore et al., 2010)

Engaging in mathematical thinking

Distinction between **“surface structures”** and **“deep structures”** in mathematics:

People try to communicate the deep structures (conceptual structures) with help of surface structures (by writing or speaking symbols).

(Skemp, 1982)

A **cognitive conflict** related to representation of mathematical objects:

“In order to do any mathematical activity, semiotic representations must necessarily be used even if there is the choice of the kind of semiotic representation. But the mathematical objects must never be confused with the semiotic representations that are used”.

(Duval, 2006, p. 107)

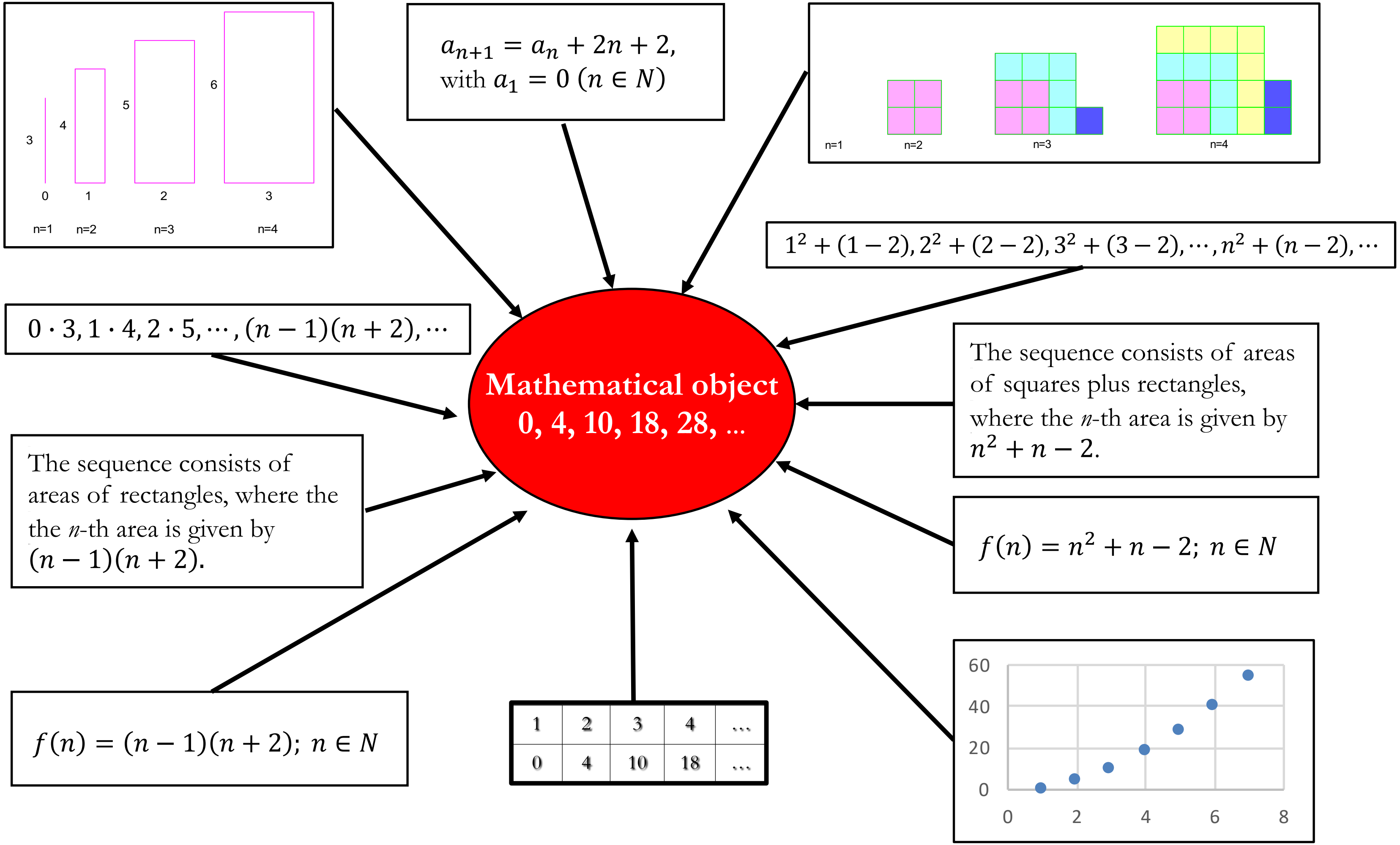
$$\lim_{x\to\infty}\frac{2}{x}=\frac{2}{\infty}=\frac{1}{4}$$

Characteristics of thinking processes in mathematics

1. The importance of semiotic representations
2. The large variety of semiotic representations used in mathematics

The crucial role played by language and other formal symbol systems in mathematics:

- Bourbaki (1948)
- Hilbert & Ackermann (1950)
- Whitehead & Russell (1910)



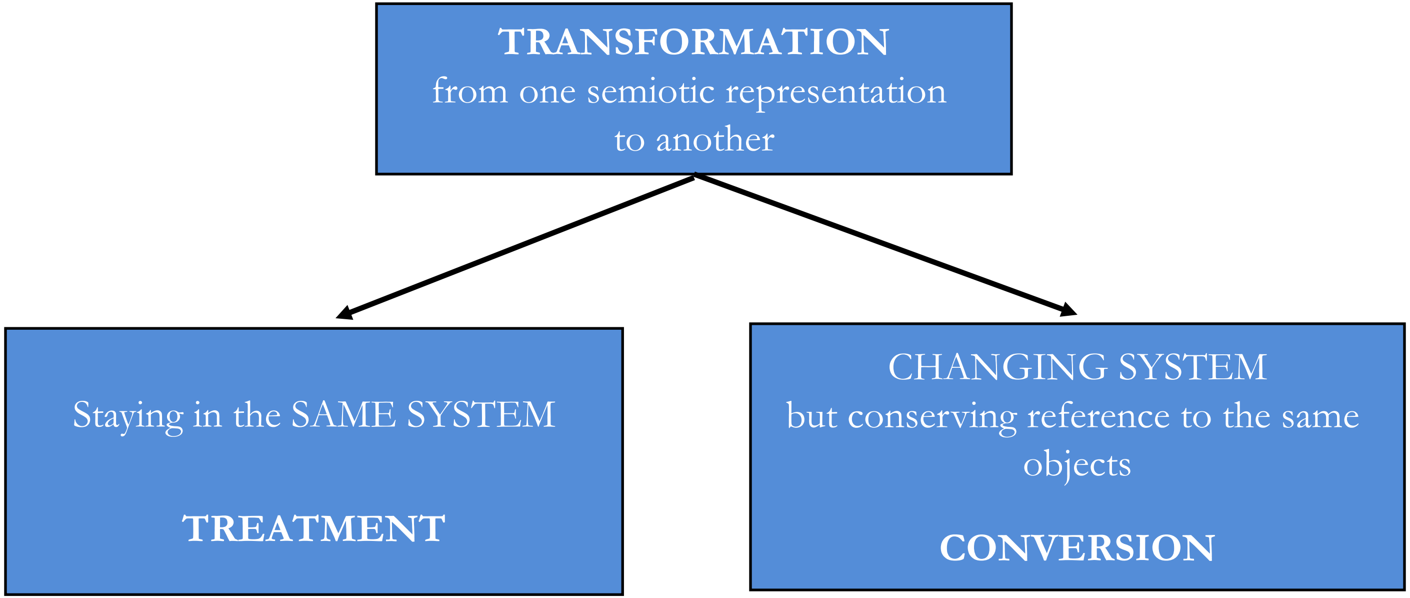
Mathematical object
0, 4, 10, 18, 28, ...
3
4
5
6
0
1
2
3
n=1
n=2
n=3
n=4
$a_{n+1} = a_n + 2n + 2,$
with $a_1 = 0 \ (n \in N)$
n=1
n=2
n=3
n=4
$1^2 + (1-2), 2^2 + (2-2), 3^2 + (3-2), \cdots, n^2 + (n-2), \cdots$
$0 \cdot 3, 1 \cdot 4, 2 \cdot 5, \cdots, (n-1)(n+2), \cdots$
The sequence consists of areas of squares plus rectangles, where the n-th area is given by $n^2 + n - 2$.
The sequence consists of areas of rectangles, where the the n-th area is given by $(n-1)(n+2)$.
$f(n) = n^2 + n - 2; \ n \in N$
$f(n) = (n-1)(n+2); \ n \in N$
1
2
3
4
...
0
4
10
18
...
60
40
20
0
0
2
4
6
8

A Semiotic perspective on mathematical activity

Four *registers* of semiotic representation (Duval, 2006):

- Natural language
- Notation systems
- Geometric figures
- Cartesian graphs

Two types of **transformations** of semiotic representations: treatments and conversions



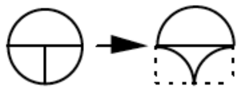
- Carrying out a calculation while remaining in the same notation system
- Solving a system of equations
- Completing a figure using criteria of symmetry

- Passing from the algebraic notation for an equation to its graphic representation
- Passing from the natural language statement of a relationship to its algebraic notation

Impact of the system of representation

$$0.20 + 0.125 = \ldots$$

$$\frac{1}{5} + \frac{1}{8} = \ldots$$



Independence of the area and the perimeter

CONVERSIONS

From a *mathematical* viewpoint:

- Interesting for efficiency reasons

From a *cognitive* viewpoint:

- Lead to the mechanisms that underlie understanding

"*Changing representation register* is the threshold of mathematical comprehension for learners at each stage of the curriculum."

(Duval, 2006, p. 128)

- What encourages brain connections is when we see mathematics in different ways: numbers, visuals, words, algebraic expressions, algorithms, gestures…

→ improving maths achievement means *improving communication*

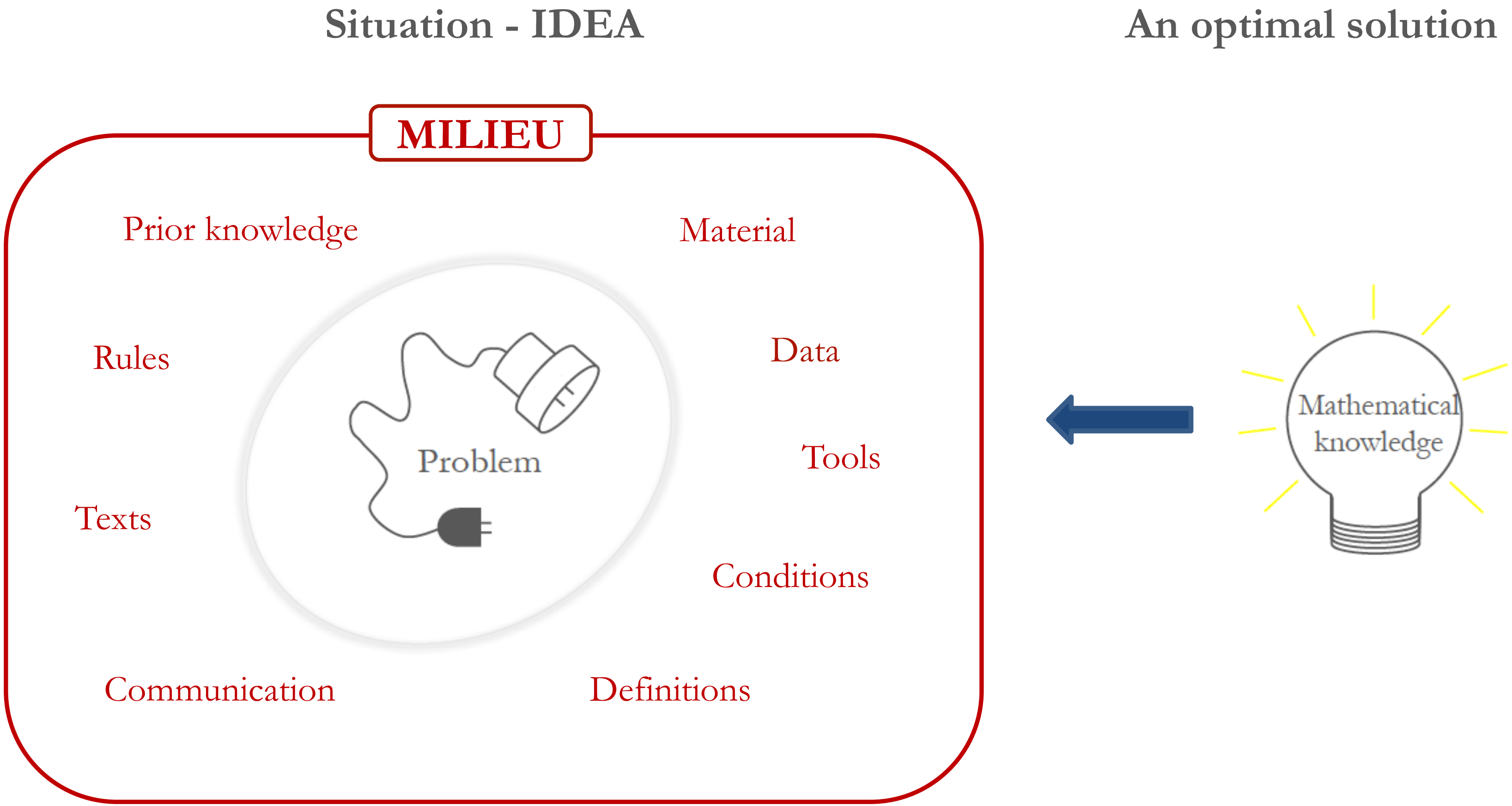
(Hubbard et al., 2005; Park & Brannon, 2013; Boaler et al., 2016)

Guidelines for mathematics instruction?

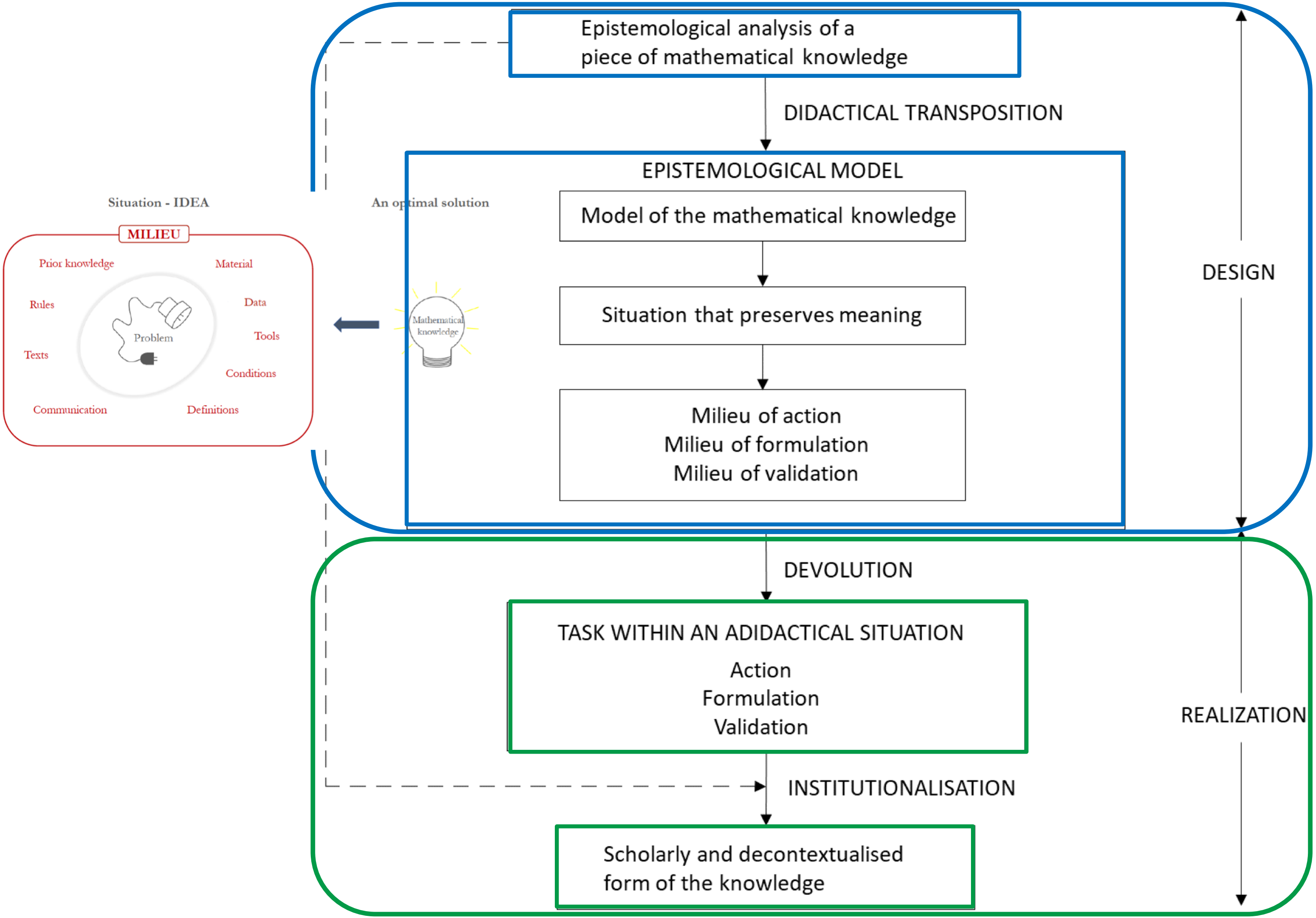
Theories rooted in *mathematics*, encompassing a methodology for *instructional design:*

- Theory of Didactical Situations in Mathematics (TDS)
  - Guy Brousseau
- The Anthropological Theory of the Didactic (ATD)
  - Yves Chevallard
- Realistic Mathematics Education (RME)
  - Hans Freudenthal

TDS - Methodological principle



MODEL FOR INSTRUCTIONAL DESIGN IN MATHEMATICS



(Strømskag, 2017)

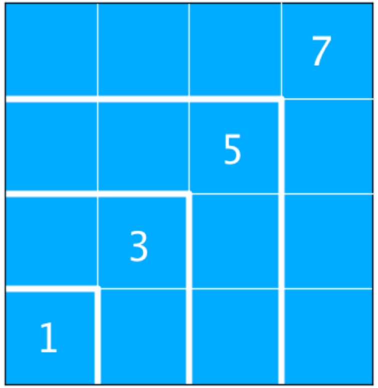
Example – Epistemological model

Target knowledge

$\sum_{i=1}^{n}(2i-1)=n^2$

Equivalence of the *n*-th square number and the sum of the first *n* odd numbers

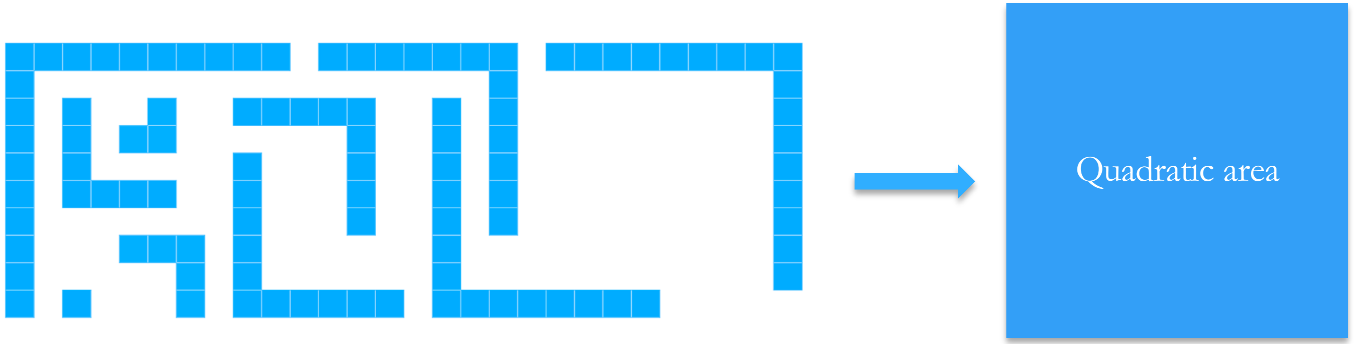
(1) Model of the target knowledge



(Not to be shown to the students)

(2) Model of the students' intended learning

→ situation that preserves meaning of $\sum_{i=1}^{n}(2i-1) = n^2$



IDEA:

Decide on the *size of a square*, and describe a method for choosing L-forms that *precisely covers* the chosen square.

CONDITIONS:

- L-forms with an ***odd number*** of unit squares (→ material milieu)
- Using L-forms of ***different size*** (→ contract)
- Only the ***size*** of the resulting square matters (→ contract)

(3) Milieus

EVOLUTION OF KNOWLEDGE

1. For a chosen size of a square, *choose* appropriate L-forms, and *arrange* them so as to cover the chosen square.

ACTION

Operating on concrete material → IMPLICIT STRATEGY

2. Describe a *method* for precise covering of a square of random size with L-forms of different sizes, and test it on another person.

FORMULATION

Need for semiotic references → EXPLICIT STRATEGY

3. Explain *why* your method will work for an arbitrary size of the square.

VALIDATION (Proof)

Need for mathematical symbols → MATHEMATICAL KNOWLEDGE

$$L1 + L2 + \cdots + L10 = 10^2$$

$$O_1 + O_2 + \cdots + O_x = x^2$$

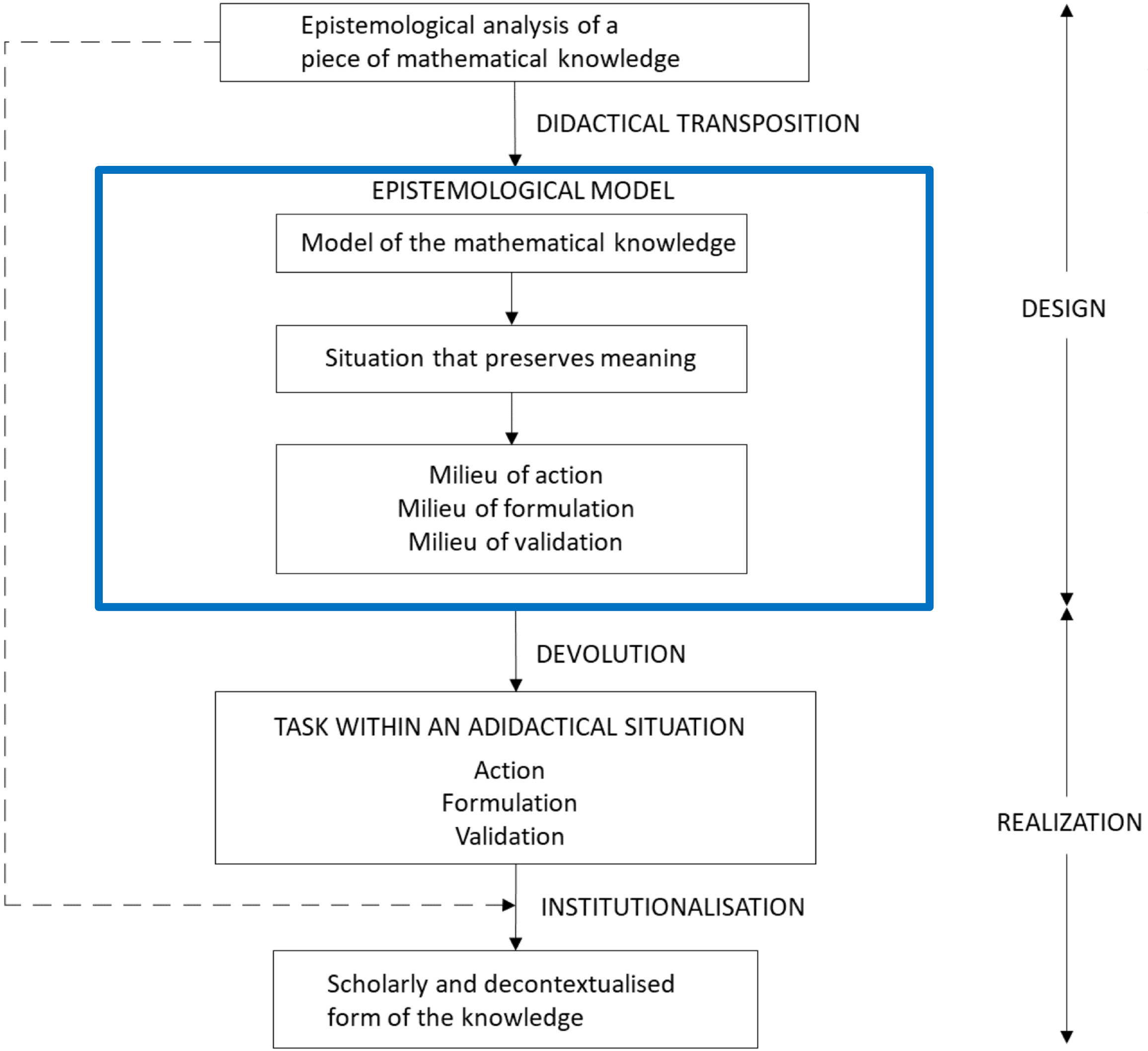
…

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

INSTITUTIONALISATION (decontextualisation)

Place, importance and future of the knowledge → SCHOLARLY KNOWLEDGE

MODEL FOR INSTRUCTIONAL DESIGN IN MATHEMATICS



A student teacher's experiment with instructional design

- Solveig V. Svinvik: Master's thesis (2018)
  - Experiments in upper secondary school, third year (Mathematics R2)
  - Target knowledge:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

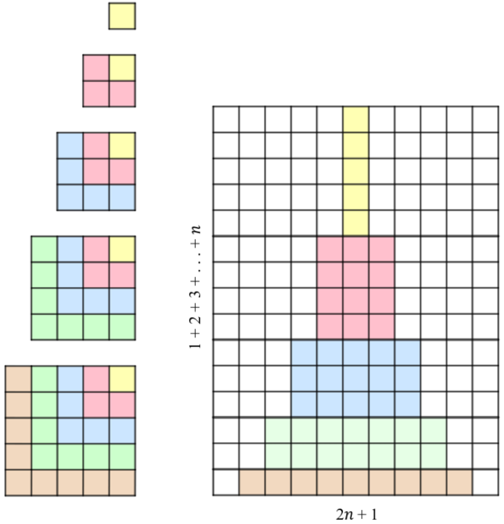
IDEA (Problem)

Illustrasjon av Oslo Plaza. Høyden på bygget er 117m, og bredden er 50m.

Gruppen deres har fått i oppgave av Kvadratoni å kartlegge hvor mye av ytterveggens *areal* (tilsvarer rammen) som vil kreves for å konstruere det unike tårnet som baserer seg på summen av et *tilfeldig* antall påfølgende kvadrater. Dette vil gjøre det lettere for Kvadratoni å finne den maksimale størrelsen på kunstverket som vil passe på ytterveggen til Oslo Plaza.

Kunstverket skal lages av glassfliser i ulike farger. Størrelsen på en glassflis (et enhetskvadrat) er 1x1 dm^2.

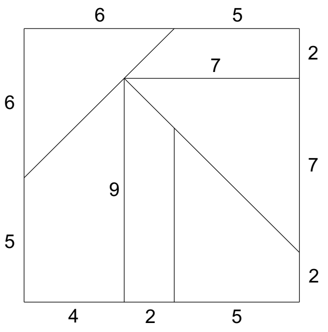
SOLUTION (material milieu → knowledge)

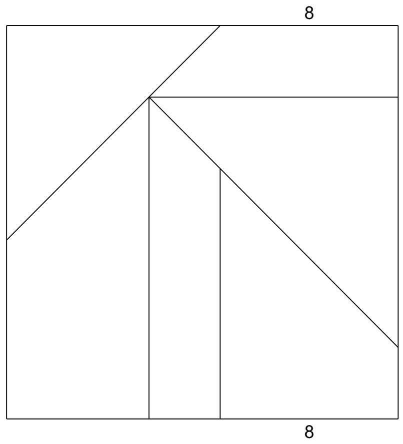


EXAMPLE – LINEARITY

Target knowledge: *Similarity is a multiplicative structure*

Model of the target knowledge





GOAL: Realising the necessity of fulfilling the requirement of linearity:

$f(a+b) = f(a) + f(b)$ and $f(ra) = rf(a)$ for any lengths $a, b \in R$ and any scalar $r \epsilon R$.

EXAMPLE: FRACTIONS

The thickness of a sheet of paper → rational numbers as measurements

Reinvention of *fractions* and *decimal numbers* among 5th graders in France.

(Brousseau, Brousseau & Warfield, 2014)

FINAL COMMENT

- What encourages brain connections is when we see mathematics in different ways: numbers, visuals, words, algebraic expressions, algorithms, gestures…
- Mathematics instruction should reflect this insight

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