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The ubiquitous constant and its siblings

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A "Weltkonstante" in time-frequency analysis

Conjecture (before August 1, 2017)

$$\mathcal{A} \overset{(?)}{=} \frac{1}{\mathcal{L}_+}$$

Theorem (unpublished)

$$\mathcal{A} = \frac{1}{\mathcal{L}_+}$$

Markus Faulhuber.
Extremal Bounds of Gaussian Gabor Frames and Properties of Jacobi's Theta Functions.
Doctoral Thesis, University of Vienna, December 2016.

Gauss' constant and the ubiquitous constant

Gauss' constant is approximately

$$G = 0.83462684167407318628143\ldots$$

The inverse of Gauss' constant is given by

$$M = \frac{1}{G} = 1.19814023473559220743999\ldots$$

The constant

$$C_u = \frac{M}{\sqrt{2}} = \frac{1}{\sqrt{2}\,G} = 0.847213\ldots$$

is sometimes called the ubiquitous constant.

Jerome Spanier, Keith B. Oldham
An Atlas of Functions (1st edition). Hemisphere, 1987

Steven R. Finch.
Mathematical Constants. Cambridge University Press, 2003.

http://mathworld.wolfram.com/GausssConstant.html

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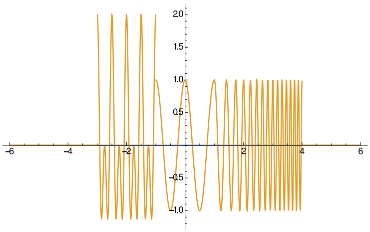
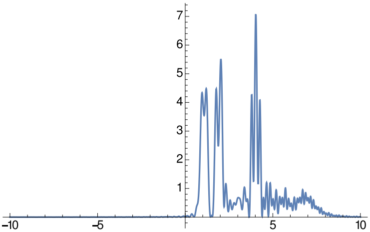
The Fourier transform

For $f \in L^2(\mathbb{R})$ we define the Fourier transform by

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega \cdot t} \, dt.$$

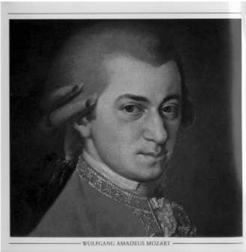


A multi-component signal (real part).



The Fourier transform of the signal (absolute value squared).

The musical score as a metaphor



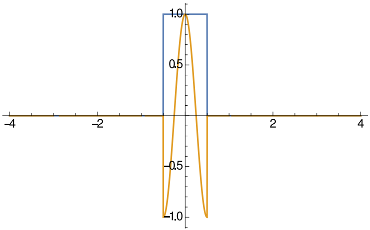
Eine Kleine Nachtmusik, K. 525

1

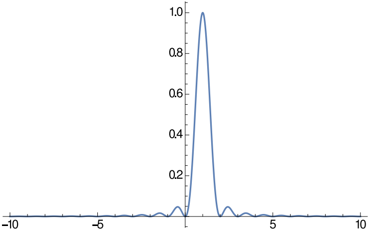
The short-time Fourier transform (STFT)

For $f \in L^2(\mathbb{R})$ the short-time Fourier transform (STFT) of f with respect to the window $g \in L^2(\mathbb{R})$ is given by

$$\mathcal{V}_g f(x, \omega) = \int_{\mathbb{R}} f(t) \overline{g(t-x)} e^{-2\pi i \omega \cdot t} \, dt.$$



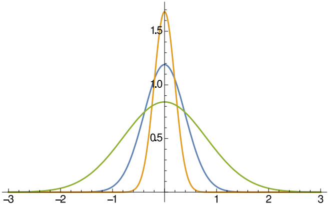
Windowed part of a multi-component signal using a rectangular window (real part).



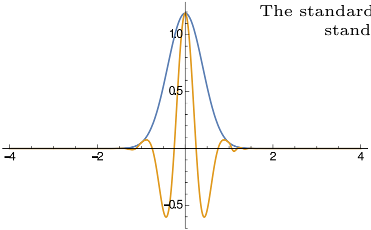
The Fourier transform of the windowed signal (absolute value squared).

The short-time Fourier transform (STFT)

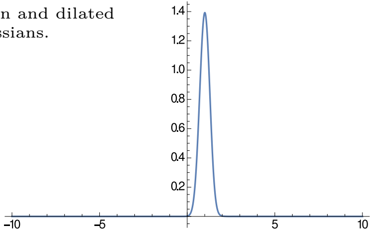
We denote the standard Gaussian by $g_0(t) = 2^{1/4} e^{-\pi t^2}$.



The standard Gaussian and dilated standard Gaussians.



Windowed part of a multi-component signal using a standard Gaussian window (real part).

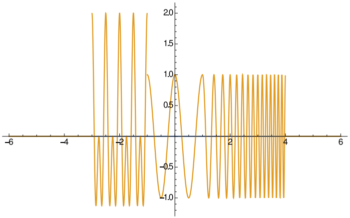


The Fourier transform of the windowed signal (absolute value squared).

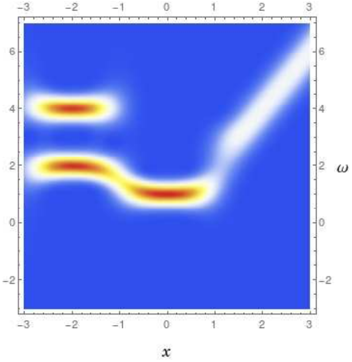
The spectrogram (SPEC)

For $f \in L^2(\mathbb{R})$ the spectrogram with respect to the window $g \in L^2(\mathbb{R})$ is given by

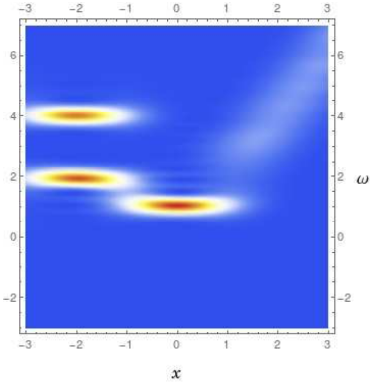
$$spec_g f(x,\omega) = |\mathcal{V}_g f(x,\omega)|^2.$$



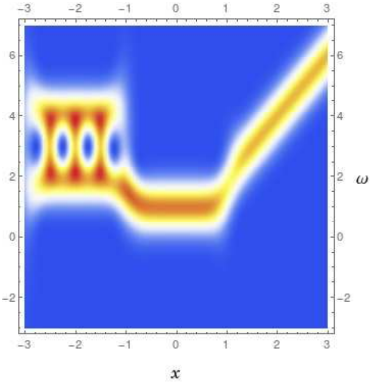
A multi-component signal (real part).



Spectrogram of the multi-component signal using the standard Gaussian.



Spectrogram with a dilated Gaussian (dilation factor 1/2).



Spectrogram with a dilated Gaussian (dilation factor 2).

An inversion formula

It is possible to recover f from its STFT. We have

$$f = \frac{1}{\|g\|^2} \int_{\mathbb{R}^2} \mathcal{V}_g f(x,\omega)\, g(t-x) e^{2\pi i \omega \cdot t}\, dx d\omega. \tag{1}$$

However, $L^2(\mathbb{R})$ is a separable Hilbert space and, hence, the representation of f given by (1) is highly redundant.

Is it possible to get some kind of generalized Fourier series of the form

$$f(t) = \sum_{x,\omega} c_{x,\omega}\, g(t-x) e^{2\pi i \omega \cdot t} \text{ ?}$$

Gabor systems

By $\pi(\lambda)$ we denote a time-frequency shift by $\lambda = (x, \omega)$;

$$\pi(\lambda)g(t) = M_\omega T_x g(t) = e^{2\pi i \omega t} g(t - x).$$

We have

$$\mathcal{V}_g f(\lambda) = \langle f, \pi(\lambda)g \rangle.$$

A Gabor system $\mathcal{G}(g, \Lambda)$ consists of time-frequency shifted copies of a window g with respect to a discrete index set $\Lambda \subset \mathbb{R}^2$;

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g \mid \lambda \in \Lambda\}.$$

Lattices

We say Λ is a lattice if it is generated by an invertible matrix $(v_1, v_2) = M \in GL(2, \mathbb{R})$;

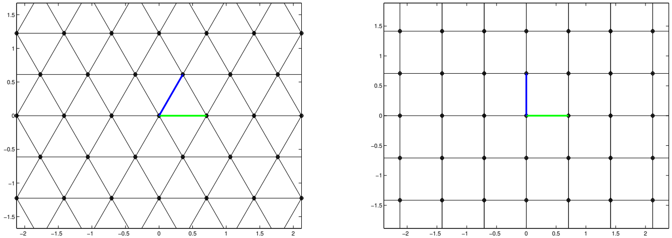
$$\Lambda = S\mathbb{Z}^2 = \{kv_1 + lv_2 \mid k, l \in \mathbb{Z}\}$$

The volume and the density of the lattice is given by

$$\text{vol}(\Lambda) = |\det(M)| \qquad \text{and} \qquad \delta = \frac{1}{\text{vol}(\Lambda)}$$

respectively.

Lattices



A hexagonal (or triangular) lattice and a square lattice of density 2.

Gabor frames

A Gabor system $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R})$ if and only if

$$A\|f\|_{L^2(\mathbb{R})}^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda) g \rangle|^2 \leq B\|f\|_{L^2(\mathbb{R})}^2, \qquad \forall f \in L^2(\mathbb{R})$$

with $0 < A \leq B < \infty$. In particular, any $f \in L^2(\mathbb{R})$ can be expanded into

$$f = \sum_{\lambda \in \Lambda} c_\lambda \, \pi(\lambda) g.$$

The frame operator

The frame operator is denoted by $S_{g,\Lambda}$ and acts on an element by the rule

$$S_{g,\Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g.$$

The upper frame bound ensures that $S_{g,\Lambda}$ is bounded (hence continuous) and the lower frame bound ensures that $S_{g,\Lambda}$ is (boundedly) invertible. Hence,

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g^\circ \rangle \, \pi(\lambda) g,$$

where $g^\circ = S_{g,\Lambda}^{-1} g$ is the canonical dual window to g.

The frame operator

The sharp frame bounds are connected to the frame operator in the following way;

$$\|S_{g,\Lambda}\|_{op} = B \qquad \text{and} \qquad \|S_{g,\Lambda}^{-1}\|_{op} = A^{-1}.$$

The number $\kappa = B/A$ is the condition number of the Gabor frame operator.

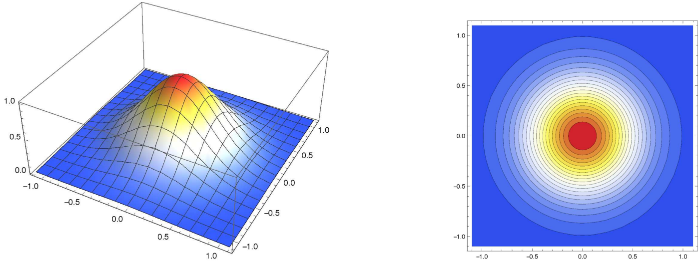
The Strohmer and Beaver conjecture

Recall that the standard Gaussian is given by

$$g_0(t) = 2^{1/4} e^{-\pi t^2}, \qquad \|g_0\|_{L^2(\mathbb{R})} = 1.$$

Its (auto-) spectrogram is given by

$$|\mathcal{V}_{g_0} g_0(x, \omega)|^2 = e^{-\pi(x^2+\omega^2)}$$



Time-frequency concentration of the standard Gaussian.

The Strohmer and Beaver conjecture

Conjecture (Strohmer and Beaver, 2003)

For the standard Gaussian g_0 *and* $\mathrm{vol}(\Lambda) < 1$ *fixed*

$$\kappa(\Lambda_h) \leq \kappa(\Lambda),$$

where Λ_h *is the hexagonal lattice and* Λ *is any lattice of same volume as* Λ_h.

The Strohmer and Beaver conjecture

Conjecture (Strohmer and Beaver, 2003)

For the standard Gaussian g_0 *and* $\Lambda_{(\alpha,\beta)} = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ *with* $\alpha\beta = r < 1$ *fixed*

$$\kappa(\sqrt{r}, \sqrt{r}) \leq \kappa(\alpha, \beta)$$

for all (α, β) *with* $\alpha\beta = r$.

"This conjecture is plausible for rectangular lattices, but is wrong if we allow more general lattices."

For standard Gaussian window and the square lattice of density 2, the optimal frame bounds are

$$A_s = 2 \sum_{k,l \in \mathbb{Z}} e^{-\pi\left((k+1/2)^2+(l+1/2)^2\right)}$$
$$= 2 \sum_{k,l \in \mathbb{Z}} e^{-\pi\left(k^2+l^2\right)} e^{2\pi i (k/2-l/2)} = 1.66925\ldots = 2G$$

and

$$B_s = 2 \sum_{k,l \in \mathbb{Z}} e^{-\pi\left(k^2+l^2\right)} = 2.36068\ldots = 2(2^{1/2}G) = 2C_u^{-1}$$

The condition number of the frame operator is $\kappa_s = 2^{1/2}$.

The appearance of the siblings

For standard Gaussian window and the hexagonal lattice of density 2, the optimal frame bounds are

$$A_h = 2 \sum_{k,l \in \mathbb{Z}} e^{-\frac{2}{\sqrt{3}}\pi\left((k+1/3)^2+(k+1/3)(l+1/3)+(l+1/3)^2\right)}$$
$$= 2 \sum_{k,l \in \mathbb{Z}} e^{-\frac{2}{\sqrt{3}}\pi\left(k^2+kl+l^2\right)} e^{2\pi i(k/3-l/3)} = 1.84074\ldots = \mathcal{A}$$

and

$$B_h = 2 \sum_{k,l \in \mathbb{Z}} e^{-\frac{2}{\sqrt{3}}\pi\left(k^2+kl+l^2\right)} = 2.31919\ldots = 2^{1/3}\mathcal{A}$$

The condition number of the frame operator is $\kappa_h = 2^{1/3}$.

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The heat kernel on a torus

We are interested in tori of the form

$$\mathbb{T}^2_\Lambda = \mathbb{R}^2/\Lambda$$

where $\Lambda \subset \mathbb{R}^2$ is a lattice. The area of the torus equals the volume of the lattice. We say a torus is rectangular if the underlying lattice is separable, i.e.,

$$\mathbb{T}^2_{(\alpha,\beta)} = \mathbb{R}^2/(\alpha\mathbb{Z} \times \beta\mathbb{Z}).$$

The heat kernel on a torus

We denote the Laplace–Beltrami operator on the torus by Δ_Λ or $\Delta_{(\alpha,\beta)}$. The heat semi group is

$$P_{\Lambda,t} = \{e^{t\Delta_\Lambda}\}_{t\geq 0},$$

which is a family of positive definite, bounded, self adjoint operators and acts on functions by the rule

$$P_{\Lambda,t} f(z) = \int_{\mathbb{T}^2_\Lambda} p_\Lambda(z - z'; t)\, f(z')\, dz', \qquad z, z' \in \mathbb{T}^2_\Lambda,\ t \geq 0.$$

The heat kernel on a torus

The function p_Λ is called the heat kernel of Δ_Λ. Its explicit formula is

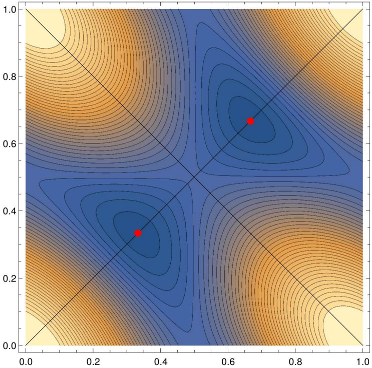
$$p_\Lambda(z;t) = \frac{1}{4\pi t} \sum_{\lambda \in \Lambda} e^{-\pi \frac{|z-\lambda|^2}{4\pi t}}.$$

The minimal and the maximal temperature of the heat kernel are

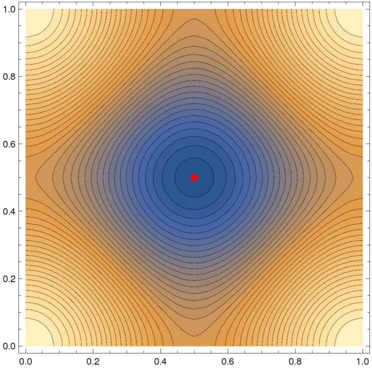
$$m_\Lambda = \min_{z \in \mathbb{T}^2_\Lambda} p_\Lambda(z;t)$$

$$M_\Lambda = \max_{z \in \mathbb{T}^2_\Lambda} p_\Lambda(z;t).$$

The heat kernel on a torus



The heat kernel on the torus with "hexagonal metric". The minima are marked, the maximum is achieved at the corners.



The heat kernel on the torus with standard metric. The minimum is achieved in the middle, the maximum is achieved at the corners.

Let $t = \frac{1}{4\pi}$ and $\mathrm{vol}(\Lambda) = 1$. We denote the hexagonal lattice by Λ_h.

$$m_{\Lambda_h} = 0.920371\ldots = \frac{\mathcal{A}}{2}, \quad M_{\Lambda_h} = 1.1596\ldots = 2^{1/3}\frac{\mathcal{A}}{2}.$$

Also, for the square lattice Λ_s we have

$$m_{\Lambda_s} = 0.834627\ldots = G, \quad M_{\Lambda_s} = 1.18034\ldots = C_u^{-1}.$$

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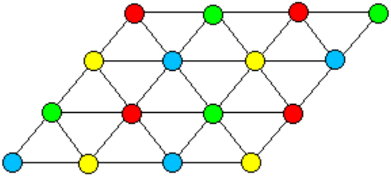
4-coloring the triangular lattice

We consider an $n \times n$ grid with triangular structure and wraparound.



One possible 4-coloring of a triangular lattice with 16 vertices.

Let v_n denote the number of ways of coloring a triangular lattice with 4 colors so that neighboring points have different colors. Then

$$\lim_{n\to\infty} v_n^{1/n^2} = C_{B4CC} = \frac{3}{4\pi^2}\Gamma(\tfrac{1}{3})^2 = 1.46069984862\ldots$$

$$2^{1/3} C_{B4CC} = \mathcal{A}$$

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Landau's "Weltkonstante"

Theorem (262)

There exists exactly one $\pi > 0$ *such that*

$$\cos \frac{\pi}{2} = 0,$$

$$\cos x > 0 \text{ for } 0 \leq x \leq \frac{\pi}{2}.$$

In other words,

$$\cos y = 0$$

has a positive solution, and in fact a smallest one.

Definition (61)

The universal constant of Theorem 262 will be denoted henceforth by π.
[Die Weltkonstante aus Satz 262 werde dauernd mit π bezeichnet.]

Edmund Landau.
Differential and Integral Calculus, Chelsea Publishing Company, 1951.
translated by M. Hausner and M. Davis.
[Einführung in die Differentialrechnung und Integralrechnung, 1934].

Landau's "Weltkonstante"

Theorem (Landau 1929)

Let $f \in \mathcal{H}(\mathbb{D})$ with $|f'(0)| = 1$. Then there exists a disc $D_f(r)$ of radius $r > 0$ such that $D_f(r) \subset f(\mathbb{D})$.

Definition (Landau's "Weltkonstante")

For $f \in \mathcal{H}(\mathbb{D})$ we define

$$\ell(f) = \sup\left\{r \in \mathbb{R}_+ \colon D_f(r) \subset f(\mathbb{D})\right\}$$

and

$$\mathcal{L} = \inf\left\{\ell(f) \colon f \in \mathcal{H}(\mathbb{D}),\, |f'(0)| = 1\right\}.$$

$\mathcal{L}$ is called Landau's constant.

Problem (Landau 1929)

What is the exact value of $\mathcal{L}$?

We have the following estimates for Landau's constant

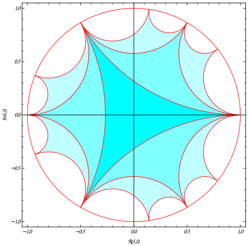
$$\frac{1}{2} < \mathcal{L} \leq \mathcal{L}_+ = \frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{6}\right)} = 0.54326\ldots$$

Conjecture (Rademacher 1943)

$$\mathcal{L} = \mathcal{L}_+ = \frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{6}\right)} = 0.54326\ldots$$

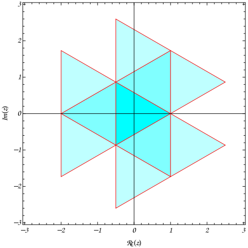
Equivalently we have

$$\mathcal{L}^{-1} = \mathcal{L}_+^{-1} = \frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)} = 1.84074\ldots = \mathcal{A}.$$



Tessellation of the unit disc.

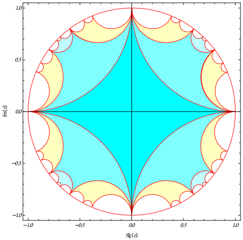
$$\xrightarrow{\Phi}$$



Tessellation of the plane.

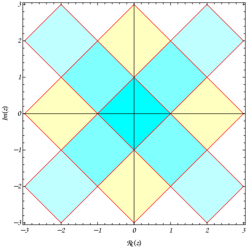
Φ maps $\mathbb{D}$ to $C \setminus \Lambda_h$, where Λ_h is a hexagonal lattice with covering radius 1.

$$|\Phi'(0)| = \mathcal{L}_+^{-1} \overset{(!)}{=} \mathcal{A} = 1.84074\ldots.$$



Tessellation of the unit disc.

$$\xrightarrow{\phi}$$



Tessellation of the plane.

ϕ maps $\mathbb{D}$ to $C \backslash \Lambda_s$, where Λ_s is a square lattice with covering radius 1.

$$|\phi'(0)| = \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)} \overset{(!)}{=} 2G = 1.66925\ldots$$

https://www.math.purdue.edu/~eremenko/uns1.html

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Special functions

Gauss' hypergeometric function ${}_2F_1$ is defined as

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1, $$

where

$$ (z)_k = \frac{\Gamma(z+k)}{\Gamma(z)}. $$

For $|q| < 1$ we define the "Nullwerte" of Jacobi's theta functions

$$\theta_2(q) = \sum_{k\in\mathbb{Z}} q^{\left(k+\frac{1}{2}\right)^2}, \quad \theta_3(q) = \sum_{k\in\mathbb{Z}} q^{k^2}, \quad \theta_4(q) = \sum_{k\in\mathbb{Z}} (-q)^{k^2}$$

Often q is replaced by $e^{\pi i \tau}$ with $\tau \in \mathbb{H}$.

The elliptic modulus and the complementary elliptic modulus are defined as

$$k = \frac{\theta_2^2}{\theta_3^2} \quad \text{and} \quad k' = \frac{\theta_4^2}{\theta_3^2}.$$

They satisfy the property

$$k^2 + k'^2 = 1.$$

Theorem (Ramanujan)

For $|q| < 1$, $k = \frac{\theta_2(q)^2}{\theta_3(q)^2}$, we have

$$ {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) = \theta_3(q)^2. $$

By setting $q = e^{-\pi}$, it follows that $k^2 = k'^2 = \frac{1}{2} = \kappa_s^{-2}$.
Therefore,

$$ {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2}) = 1.18034\ldots = C_u^{-1} $$

and

$$ \frac{1}{\sqrt{2}}\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2}) = 0.834627\ldots = G. $$

For $|q| < 1$, we define the cubic analogues to Jacobi's theta functions.

$$a(q) = \sum_{k,l \in \mathbb{Z}} q^{k^2+kl+l^2}$$

$$b(q) = \sum_{k,l \in \mathbb{Z}} q^{(k+1/3)^2+(k+1/3)(l+1/3)+(l+1/3)^2}$$

$$c(q) = \sum_{k,l \in \mathbb{Z}} q^{k^2+kl+l^2} \zeta_3^{k-l}$$

where $\zeta_3^3 = 1$, $\zeta_3 \neq 1$. Setting $s = \frac{b(q)}{a(q)}$ and $s' = \frac{c(q)}{a(q)}$, we have

$$s^3 + s'^3 = 1.$$

Theorem (Ramanujan; Borwein and Borwein)

For $|q| < 1$, $s = \frac{b(q)}{a(q)}$, we have

$$ {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; 1; s^3) = a(q) $$

By setting $q = e^{-\frac{2}{\sqrt{3}}\pi}$, it follows that

$$ s^3 = s'^3 = \frac{1}{2} = \kappa_h^{-3}. $$

Also,

$$ {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; 1; s^3) = 1.1596\ldots = 2^{1/3}\frac{\mathcal{A}}{2} $$

and

$$ \tfrac{1}{\sqrt[3]{2}}\, {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; 1; s^3) = 0.920371\ldots = \frac{\mathcal{A}}{2} = \frac{1}{2\mathcal{L}_+}. $$

Jonathan Borwein and Peter Borwein.
A Cubic Counterpart of Jacobi's Identity and the AGM
Transactions of the American Mathematical Society, 323(2):691-701, 1991.

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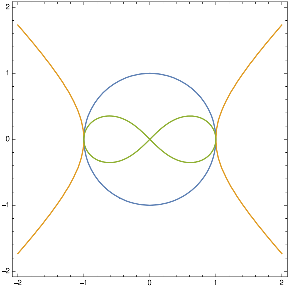
The lemniscate



For a parameter $a > 0$, the lemniscate is defined by the implicit equation

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

For $a = \frac{1}{\sqrt{2}}$ we have the “unit” lemniscate.

The lemniscate

The arc length of the lemniscate with parameter a is given by

$$4\sqrt{2}a\,F(1),$$

where the function F is given by

$$F(x) = \int_0^x \frac{dt}{\sqrt{1-t^4}}.$$

It follows that the arc length of the unit lemniscate is

$$4F(1) = 4\int_0^1 \frac{dt}{\sqrt{1-t^4}} = 2\varpi \approx 5.24412$$

The lemniscate

Gauss used the letter ϖ to denote the close connection to π;

$$2\varpi = 2\int_0^1 \frac{dt}{\sqrt{1-t^4}}.$$

The ratio $G = \frac{\varpi}{\pi}$ is called Gauss' constant[1]. Its exact value can be expressed in terms of the Beta or Gamma function;

$$G = \frac{B(\frac{1}{4},\frac{1}{2})}{2\pi} = \frac{\Gamma(\frac{1}{4})^2}{2\pi\sqrt{2\pi}} = 0.83463\ldots$$

[1] In German literature $\varpi = \pi G$ is called "lemniskatische Konstante" [lemniscate constant]

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The AGM

For two numbers $a, b \in \mathbb{R}_+$, the arithmetic mean is given by $\frac{a+b}{2}$ and the geometric mean is given by $\sqrt{ab}$.
We set

$$a_0 = a, \qquad b_0 = b$$

and start the iterative process

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}.$$

We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = M_2(a,b),$$

which is called the arithmetic-geometric mean of a and b.

The AGM: an example

We set

$$a_0 = 1 \qquad \text{and} \qquad b_0 = \frac{1}{\sqrt{2}} = 0.70710678118654752440 08$$

Then, we have

$$\begin{aligned} a_1 &= 0.853553390593273762200 & b_1 &= 0.840896415253714543031 \\ a_2 &= 0.847224902923494152615 & b_2 &= 0.847201266746891460403 \\ a_3 &= 0.847213084835192806509 & b_3 &= 0.847213084752765366704 \\ a_4 &= 0.847213084793979086607 & b_4 &= 0.847213084793979086605 \end{aligned}$$

and

$$M_2\left(1, \tfrac{1}{\sqrt{2}}\right) = C_u.$$

The AGM and complete elliptic integrals

Theorem (Gauss' diary, Entry 102, December 23, 1799)

$$M_2(a,b)\,\int_0^{\pi/2} \frac{d\varphi}{\sqrt{a^2\cos(\varphi)^2+b^2\sin(\varphi)^2}} = \frac{\pi}{2}.$$

Note:

$$a^2\cos(\varphi)^2+b^2\sin(\varphi)^2 = a^2(1-\underbrace{(1-\tfrac{b^2}{a^2})}_{k^2=1-k'^2}\sin(\varphi)^2)$$

Also,

$$\begin{aligned} K(k) := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin(\varphi)^2}} &= \int_0^1 \frac{dt}{\sqrt{(1-k^2t^2)(1-t^2)}} \\ &= \frac{\pi}{2}\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;k^2\right) = \frac{\pi}{2}\,\frac{1}{M_2(1,k')}. \end{aligned}$$

It follows that

$$G = \frac{\varpi}{\pi} = \frac{1}{M_2(1,\sqrt{2})} \qquad \text{and} \qquad C_u = M_2\left(1,\tfrac{1}{\sqrt{2}}\right).$$

The AGM: a cubic counterpart

For $a_0 = a$ and $b_0 = b$, we define the iterative process

$$a_{n+1} = \frac{a_n + 2b_n}{3}, \qquad b_{n+1} = \sqrt[3]{b_n \, \frac{a_n^2 + a_n b_n + b_n^2}{3}}$$

The limit of the sequences is the same

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = M_3(a,b).$$

We have, for $0 < s < 1$ and $s^3 + s'^3 = 1$,

$$\frac{1}{M_3(1,s)} = {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; s'^3\right)$$

Jonathan Borwein and Peter Borwein.
A Cubic Counterpart of Jacobi's Identity and the AGM
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The AGM and frame bounds

Consider the following sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of Gaussian Gabor systems with square lattices of density 2^n and

$$\mathcal{G}_n = \left\{ \tfrac{1}{\sqrt{2^n}} g_0, \tfrac{1}{\sqrt{2^n}}\mathbb{Z} \times \tfrac{1}{\sqrt{2^n}}\mathbb{Z} \right\}.$$

Then,

$$A_1 = G \qquad \text{and} \qquad B_1 = C_u^{-1} = \sqrt{2}G.$$

It follows that

$$M_2(G, \sqrt{2}G) = G \underbrace{M_2(1, \sqrt{2})}_{G^{-1}} = 1.$$

Furthermore,

$$A_{n+1} = \sqrt{A_n B_n} \qquad \text{and} \qquad B_{n+1} = \frac{A_n + B_n}{2}.$$

The AGM: a new field of analysis

Gauß' Tagebuch 1799. 23

98) Terminum medium arithmetico-geometricum inter 1 et $\sqrt{2}$ esse $= \frac{\pi}{\varpi}$ usque ad figuram undecimam comprobavimus, quare demonstrata prorsus novus campus in analysi certo aperietur.

Mai. 30. Brunsvigae.

Vergl. die Berechnung in III, p. 364.

[**Gauss' diary, Entry 98, May 30, 1799**]:
We have established that the arithmetic-geometric mean between 1 and $\sqrt{2}$ is $\frac{\pi}{\varpi}$ to the eleventh decimal place; the demonstration of this fact will surely open an entirely new field of analysis.

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