SYMMETRYYRTEMMYSSYMMETRYYRTEMMYS
SYMMETRYYRTEMMYSSYMMETRYYRTEMMYS
SYMMETRYYRTEMMYSSYMMETRYYRTEMMYS
SYMMETRYYRTEMMYSSYMMETRYYRTEMMYS

# From Conway's Magic Theorem to Archimedes' Labyrinth

# and Beyond


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NTNU, March 2019

# My 15 minutes of fame ...

Bergensavisen 28/9:

# My 15 minutes of fame ...

Bergensavisen 29/9:

# My 15 minutes of fame ...

Science, Oct. 15, 2018:



ARNE RISTESUND

## This dizzying labyrinth will host next year's party for math's 'Nobel' prize

By Allyn Jackson | Oct. 9, 2018 , 1:10 PM

When mathematician Hans Munthe-Kaas of the University of Bergen in Norway was asked to help design a new botanical garden for his school, he had absolutely no idea what he could contribute.

# My 15 minutes of fame ...

A Korean Newspaper, Oct. 2018:

PHOTO 포토 뉴스 NEWS

## 아벨상 수상자가 갇힐 미로?

한스 문테코스 교수가 설계한 미로의 전체 도면.

2018년 9월 30일 노르웨이 베르겐대학교에 독특한 미로가 나타났습니다. 한스 문테코스 노르웨이 베르겐대학교 수학과 교수가 설계한 수학 미로입니다.

800㎡ 넓이에 펼쳐져 있는 이 미로는 아르키메데스 나선에서 영감을 받아 설계해 '아르키메데스 미로'라는 이름을 붙였습니다. 아르키메데스 나선은 중심으로부터의 거리가 회전각에 비례해 커지는 소용돌이 모양입니다. 어린 고사리를 비롯해 자연에서 찾아볼 수 있는 나선이죠.

문테코스 교수는 미로를 대칭적으로 만들기 위해 '평면의 결정군'이라 불리는 모자이크 패턴을 이용했습니다. 평면을 대칭적인 도형으로 반복해서 채우는 패턴으로, 총 17종류 중에서 육각형 격자를 기본으로 가진 패턴을 선택해 미로를 만들었습니다. 이 패턴은 스페인의 알람브라 궁전을 비롯해 중세 건축물에서 볼 수 있습니다.

내년 아벨상 시상식에서 방문객들은 이 미로 속에 흩어져 있는 단서를 찾아 퍼즐도 풀 예정이라고 합니다. 아벨상위원회의 의장직을 맡고 있는 문테코스 교수는 "아벨상은 수학자의 공로를 인정하는 목적 외에도 아이들의 수학적 흥미를 고취시키고 수학의 위상을 높이는 역할도 해야 한다"며, "이곳이 수학과 자연과학 커뮤니티가 되길 바란다"고 밝혔습니다. 그러면서 "내년 아벨상 수상자도 이 미로를 체험하길 소망한다"고 덧붙였습니다.

조혜인 기자

24 수학동아

수학뉴스201811.indd 24 2018-10-22 오후 4:48:01

한스 문테코스 교수가

설계한 미로의 전체 도면.

Google translate:

A complete drawing of a labyrinth

designed by Professor Hans Muntec.

# Botanical Garden - The Adiabata Project

# Botanical Garden - The Adiabata Project

# Designing a Labyrinth - Inspiration from Nature

- D'Arcy Thompson : *'On Growth and Form'* (1917) - (1942)

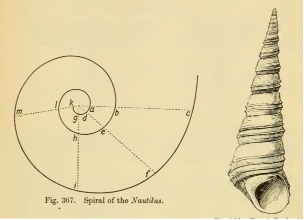
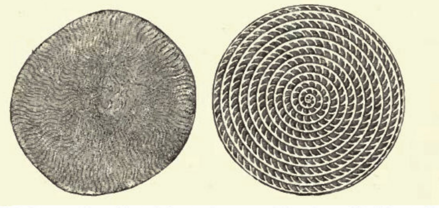
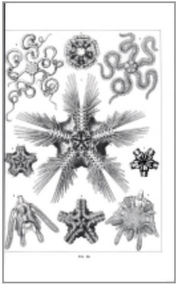


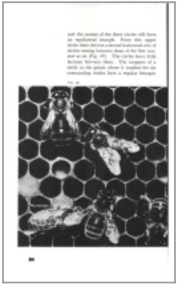
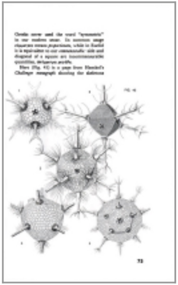
Fig. 367. Spiral of the *Nautilus*.



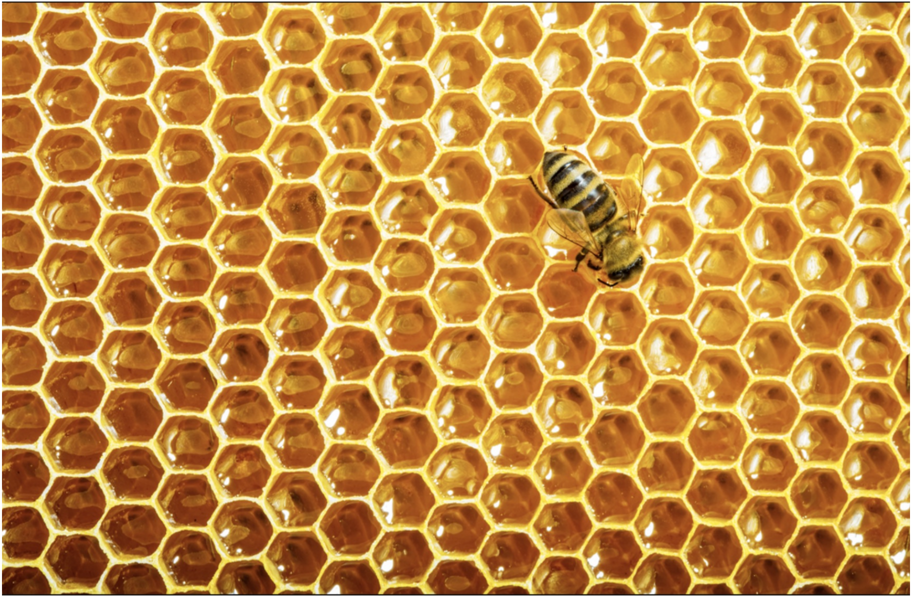
A Nummulite, viewed from above, and horizontally bisected.

- Hermann Weyl : *'Symmetry'* (1952)

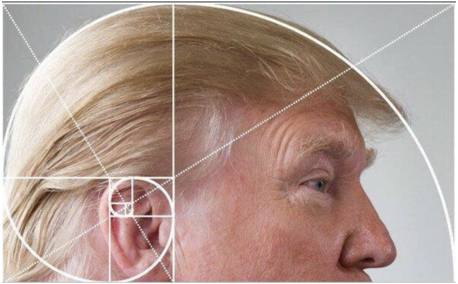


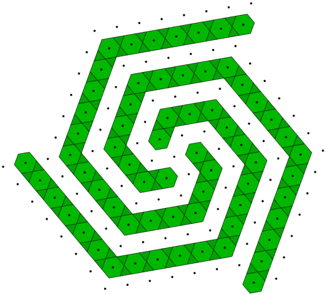




# 3 Elements: Honeycombs - Spirals - Symmetries



# 3 Elements: Honeycombs - Spirals - Symmetries

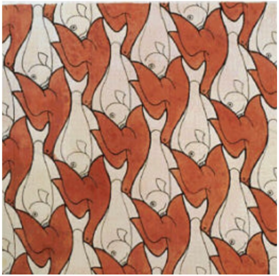




# 3 Elements: Honeycombs - Spirals - Symmetries





# Symmetries - Mattesirkelen

EMNE FOR VIDEREGÅENDEELEVER

## Matematikksirkelen

Går du på videregående og er ekstra interessert i matematikk? Da har vi et flott tilbud til deg. Kurset er på universitetsnivå og gir 10 studiepoeng.

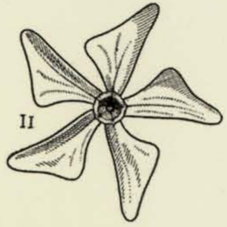
Professor Hans Munthe-Kaas er kursansvarlig for universitetskurset for VGS-elever. Foto: Antonella Zanna Munthe-Kaas Opphavsrett: Antonella Zanna Munthe-Kaas

# Discretre Planar Symmetries - Basic Isometries:

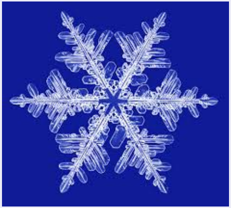
- Translation: 
- Rotation: 
- Reflection: 
- Glide reflection: 

# Discrete Planar Symmetries - Classification

## Rosettes (finite symmetries) - Leonardos Theorem:

Cyclic $C_n$ : 



Dihedral $D_n$ : 

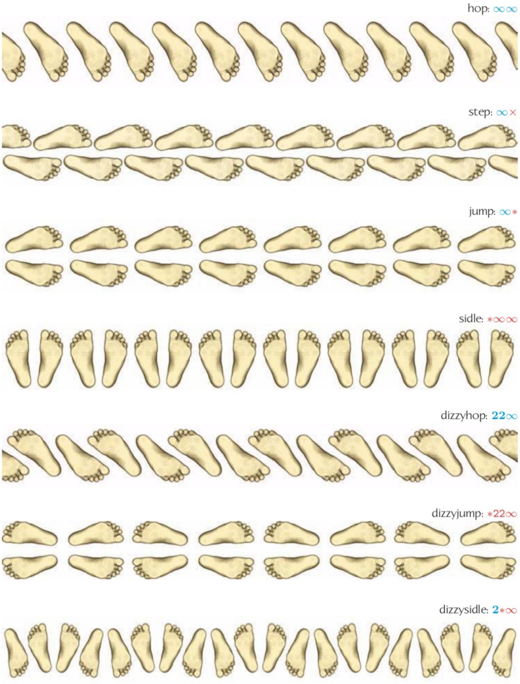
## Friezes (translations 1 dir.)

7 different frieze groups.

## Wallpapers (translations 2 dir.) - Fedorov & Schoenflies 1891:

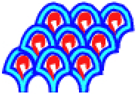
17 different wallpaper groups.

# 7 Frieze groups:

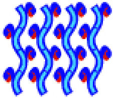


# 17 Wallpaper groups:

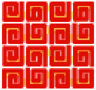
*p1*



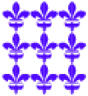
*pg*



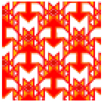
*pgg*



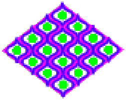
*pm*



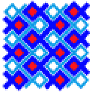
*cm*



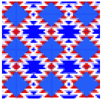
*cmm*



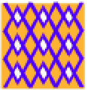
*pmg*



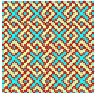
*pmm*



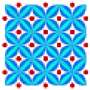
*p2*



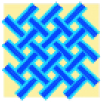
*p4*



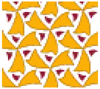
*p4m*



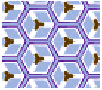
*p4g*



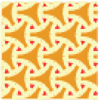
*p3*



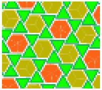
*p3m1*



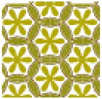
*p31m*



*p6*



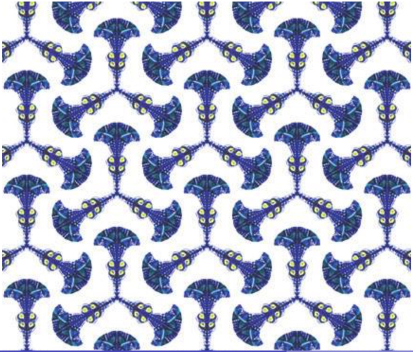
*p6m*



# Orbifold notation (Conway - Thurston)

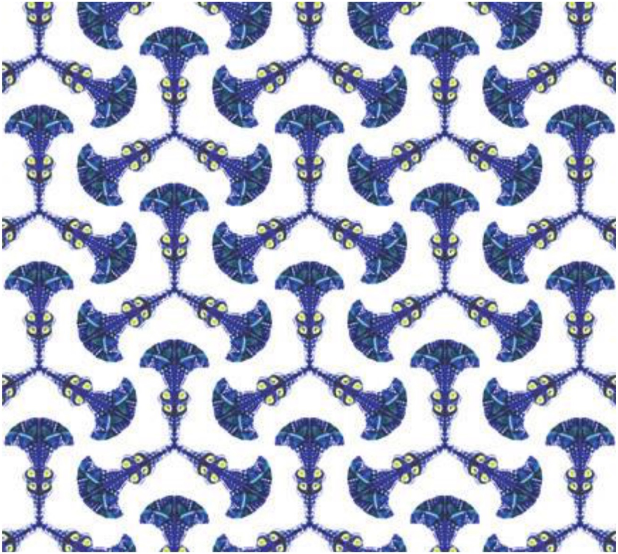
## Elementary symmetries:

- Kaleidoscope: $*N$ (N-fold rotation on mirror)
- Gyration: $N$ (N-fold rotation not on mirror)
- Miracle: $\times$ (glide reflection)
- Wonderring: $\circ$ (only translations in 2-directions)



3*3

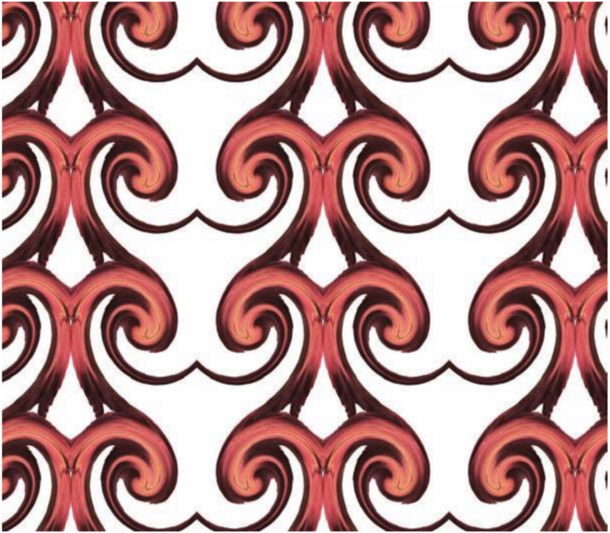
# Example: Signature $3{*}3$

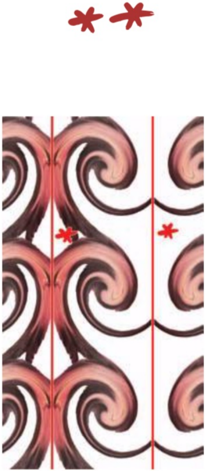




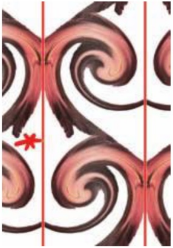
3*3

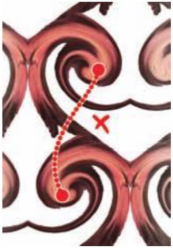
# Example: Signature **

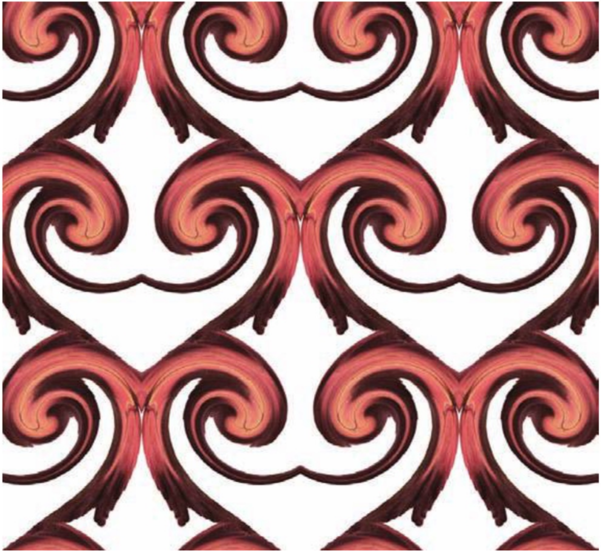




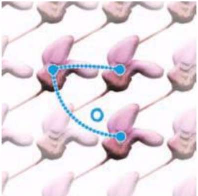
# Example: Signature $*\times$



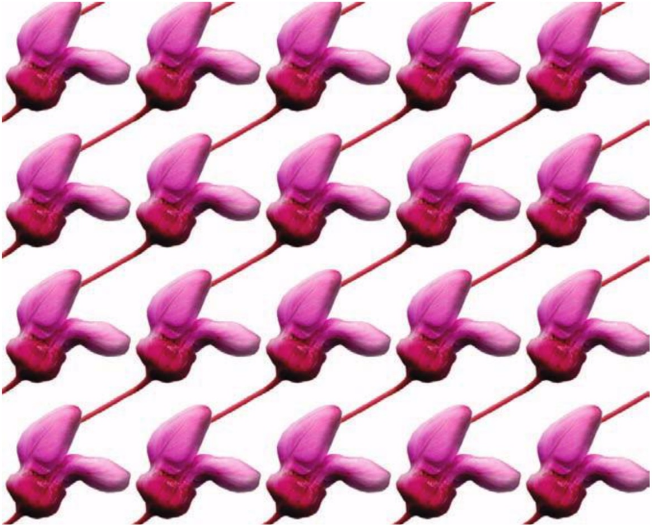




# Example: Signature ∘




o



# The Orbifold Shop - The cost of a signature

## Shop prices:

| | Cost ($) |
|---|---|
| $N$ | $\frac{N-1}{N}$ |
| $N$ | $\frac{N-1}{2N}$ |
| $*$ | 1 |
| $\times$ | 1 |
| $\circ$ | 2 |

## Examples:

$$\text{Cost}(3{*}3) = \frac{2}{3} + 1 + \frac{2}{6} = 2$$
$$\text{Cost}({*}{*}) = 1 + 1 = 2$$
$$\text{Cost}({*}\times) = 1 + 1 = 2$$

# The Magic Theorem

**Theorem (Conway)**

- *All planar symmetries cost 2$.*
- *Spherical symmetries cost less than 2$.*
- *Hyperbolic symmetries cost more than 2$.*

*(Almost) all signatures represent uniquely a symmetry group.*

Proof: Later!

# The 17 wallpaper groups:

| | | | | |
|---|---|---|---|---|
| *632 | *442 | *333 | *2222 | ** |
| | | | 2*22 | *× |
| | 4*2 | 3*3 | 22* | ×× |
| | | | 22× | |
| 632 | 442 | 333 | 2222 | ○ |

# The 17 wallpaper groups:

| | | | | |
|---|---|---|---|---|
| *632 | *442 | *333 | *2222 | ** |
| | | | 2*22 | *× |
| | 4*2 | 3*3 | 22* | ×× |
| | | | 22× | |
| 632 | 442 | 333 | 2222 | ○ |

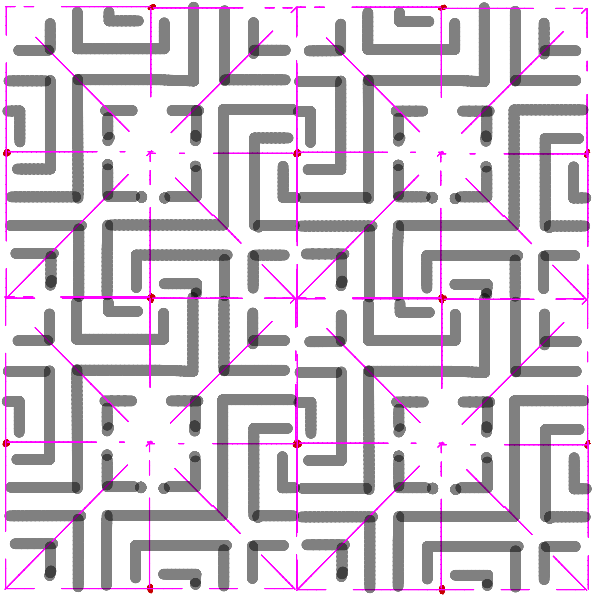
## Arkimedes' Labyrint

We want:

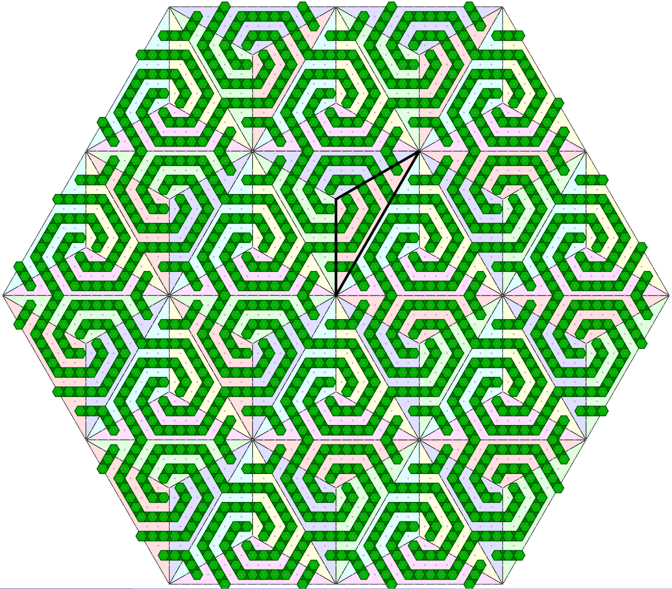
1. Spirals of symmetry at least 3 in both directions ($N \geq 3$ and $*$).
2. Wallpaper symmetry.
3. Hexagonal underlying lattice.

The only possibility: 3*3.

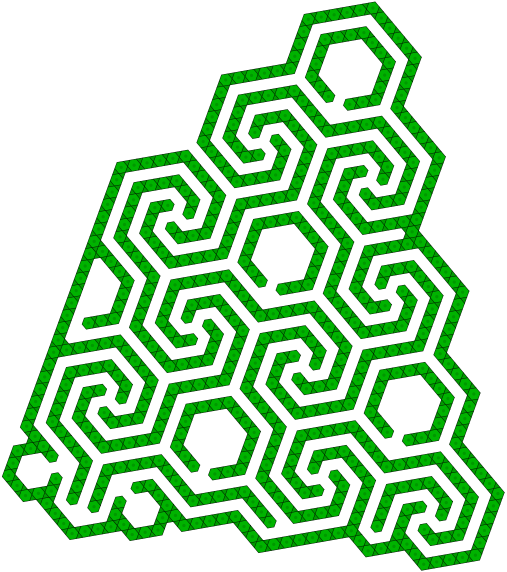
# A 4∗2 maze:



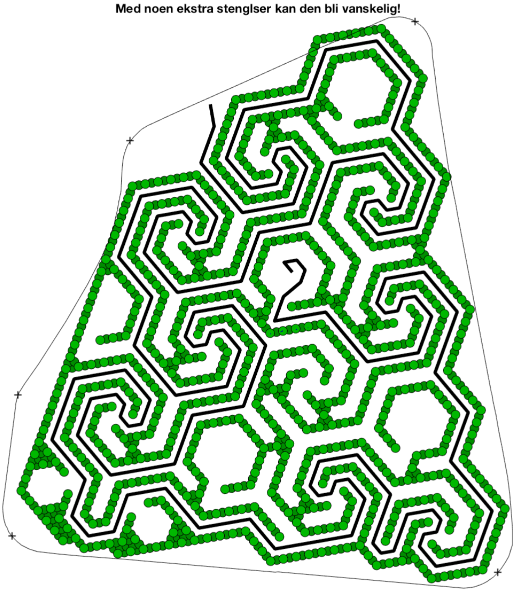
# Archimedes infinite version 3∗3 (Pure Math):



# Adaption to boundaries (Applied Math):



# More difficult versions:



# Spherical Symmetries

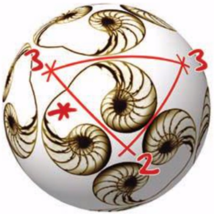
## Signatures, less than 2$:

| | | | | |
|---|---|---|---|---|
| ∗532 | ∗432 | ∗332 | ∗22N | ∗MN |
| | | 3∗2 | 2∗N | N∗<br>N× |
| 532 | 432 | 332 | 22N | MN |

NOTE: $M \neq N$ is not allowed (geometry).

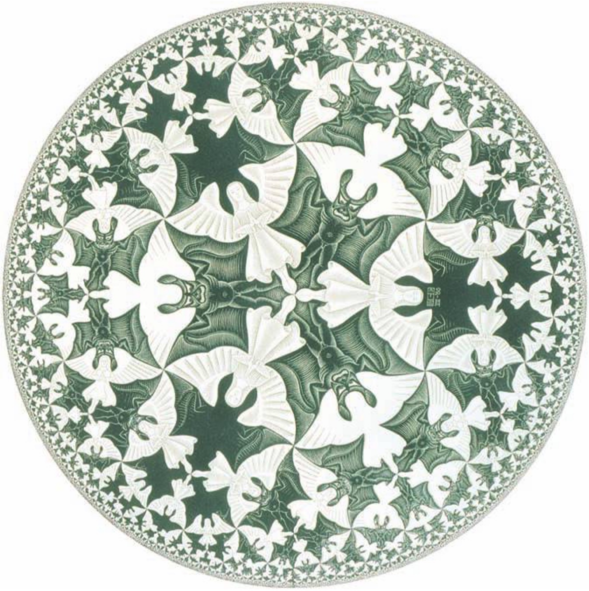

Signature ∗432


Signature ∗532




Signature ∗332

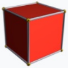
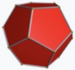
# Hyperbolic symmetries: $4{*}3$ or ${*}3333$?



# Proof of Magic Therem (sketch):

## Euler Characteristic of Surface:

$$\chi = V - E + F$$

| Name | Image | Vertices $V$ | Edges $E$ | Faces $F$ | Euler characteristic: $V - E + F$ |
|---|---|---|---|---|---|
| Tetrahedron |  | 4 | 6 | 4 | **2** |
| Hexahedron or cube |  | 8 | 12 | 6 | **2** |
| Octahedron |  | 6 | 12 | 8 | **2** |
| Dodecahedron |  | 20 | 30 | 12 | **2** |
| Icosahedron |  | 12 | 30 | 20 | **2** |

# Proof of Magic Therem (sketch):

**Euler Characteristic of Surface:**

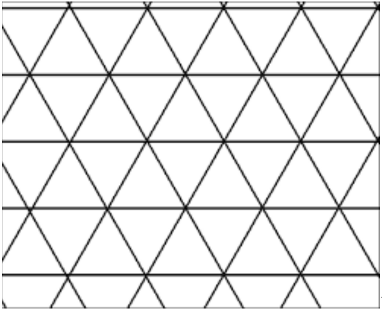
$$\chi = V - E + F$$

- Sphere: $\chi = 2$.
- Disc: $\chi = 1$.
- Torus: $\chi = 0$.
- Double torus: $\chi = -2$.

# Proof of Magic Therem (sketch):

**Euler Characteristic of Orbifold (divide by symmetries):**

$$\chi_o = V/N - E/2 + F$$

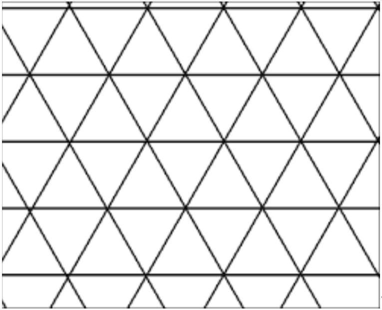


$\chi_o = 3/6 - 3/2 + 1 = 0.$

# Proof of Magic Therem (sketch):

## Euler Characteristic of Orbifold (divide by symmetries):

$$\chi_o = V/N - E/2 + F$$



$\chi_o = 3/6 - 3/2 + 1 = 0.$

## Theorem

$$\chi_o(signature) = 2 - Cost(signature)$$

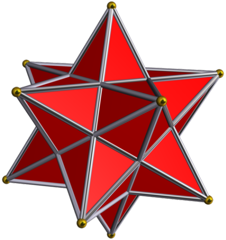
*(this is explained by eamples in the talk)*

# Proof of Magic Therem (sketch):

We finally argue that $\chi_o = 0$ for all planar symmetries and $\chi_o < 0$ for the hyperbolic case by scaling argument.
In spherical cases we can use the cost of the signature to compute the size of the symmetry group, e.g. for $*532$, we have

$$chi_o(*532) = 2 - 1 - 4/10 - 2/6 - 1/4 = 1/60.$$

Since the Euler number for the whole sphere is 2, we have
$|G| = 2/(1/60) = 120$.
A spericial caleidoscope of type $*532$ projects the entrance window onto a small stellated dodecahedron, with a mirror line in the middle of



each of the 60 triangles.
(I didn't have time to explain this properly in the talk).

# And Beyond ...

**An interest of mine (for 30 years, on and off):**

Caleidoscopic groups (affine Weyl groups) and multivariate Chebyshev polynomials in approximation theory.

- Computational aspects.
- Quadrature rules,
- Group Fourier Transforms,
- Spectral Approximations on Simplexes.

(Probably not time for that, sorry!)

# Conway, Burgiel, Goodman-Strauss : *'The Symmetry of Things'*, Taylor and Francis 2008.

`http://hans.munthe-kaas.no/protect/Conway`
User: abel
Password: lie

Thank you!