Εὐκλείδης

$p_1 p_2 \cdots p_n + 1$

EULER 1737

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{p_k}}$$

$$\sum \frac{1}{p} = \log(\log(\infty))$$

$$\boxed{\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod \frac{1}{1-\frac{1}{p^s}} \quad (s>1)}$$

$$1+1+1+\cdots = -\frac{1}{2} \qquad \zeta(0)$$

$$1+2+3+\cdots = -1/12 \qquad \zeta(-1)$$

$$1^2+2^2+3^2+\cdots = 0 \qquad \zeta(-2)$$

$$1^3+2^3+3^3+\cdots = \frac{1}{120} \qquad \zeta(-3)$$

$$\pi(x) \sim \frac{x}{\log(x)}$$ EULER 1762/63

$$\pi(x) \sim \int_2^x \frac{dy}{\log(y)}$$ GAUSS 1791

$$\pi(x) \sim \frac{x}{\log(x) - 1}$$ LEGENDRE 1798

1,08

TSCHEBYSCHEF 1851

$$0{,}92129 \frac{x}{\log(x)} < \pi(x) < 1{,}10555 \frac{x}{\log(x)}$$

J. HADAMARD 1896

de la VALLEÈ - POUSSIN 1896

$$li(x) = \int_0^x \frac{dy}{\log(y)}$$

$\sqrt{x} \log(x)$

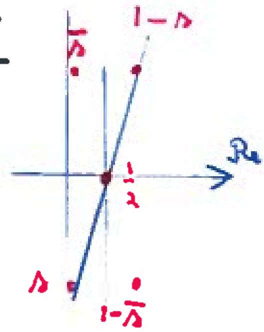
$$\boxed{\zeta(s) = \frac{1}{2} + \frac{1}{s-1} - s\int_1^\infty \frac{y-[y]-\frac{1}{2}}{y^{1+s}}\,dy}$$

$s = \sigma + it$

- When $\sigma > 1$ this yields $$\sum_{n=1}^{\infty} n^{-s}$$
- Valid also when $\sigma > 0$ (except at $s = 1$).

"... so lange der reelle Theil von s größer als 1 ist; es lässt sich indess leicht EIN IMMER GÜLTIG BLEIBENDER AUSDRUCK der Function finden." 1859

$$\frac{\Gamma\left(\frac{s}{2}\right)\zeta(s)}{\pi^{s/2}} = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{1-s}{2}} + x^{\frac{s}{2}}\right) \sum_{n=1}^{\infty} e^{-n^2\pi x}\,\frac{dx}{x}$$



THE FUNCTIONAL EQUATION

$$\boxed{\frac{\Gamma\left(\frac{s}{2}\right)\zeta(s)}{\pi^{s/2}} = \frac{\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)}{\pi^{(1-s)/2}}}$$

TAKES REAL VALUES WHEN $s = \frac{1}{2} + it$

$$\xi(s) = \frac{1}{2}\, s(s-1)\, \pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$\boxed{\xi(s) = \xi(1-s)}$$

$$\max_{|s|=R} |\xi(s)| \sim R^{R/2} \qquad (R \to \infty)$$

POISSON $$\sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{-\frac{n^2\pi}{x}}$$

$$\log \zeta(s) = -\sum \log\left(1 - \frac{1}{p^s}\right)$$

PRIMES SPIRITED AWAY !

$$= -\sum_{n=2}^{\infty} \left(\pi(n) - \pi(n-1)\right) \log\left(1 - \frac{1}{n^s}\right)$$

Rearrangement

$$= -\sum_{n=2}^{\infty} \pi(n) \left\{ \log\left(1 - \frac{1}{n^{-s}}\right) - \log\left(1 - \frac{1}{(n+1)^s}\right) \right\}$$

Newton – Leibnitz

$$= + \sum_{n=2}^{\infty} \pi(n) \int_{n}^{n+1} \frac{d}{dx}\left(\log\left(1 - \frac{1}{x^s}\right)\right) dx$$

$\pi(x) = \pi(n), \quad n \le x < n+1$

$$= \sum_{n=2}^{\infty} \int_{n}^{n+1} \pi(x) \frac{d}{dx}\left(\log\left(1 - \frac{1}{x^s}\right)\right) dx$$

Differentiate !

$$= \sum_{n=2}^{\infty} \int_{n}^{n+1} \frac{s\,\pi(x)}{x(x^s - 1)}\, dx$$

$$= s \int_{2}^{\infty} \frac{\pi(x)\, dx}{x(x^s - 1)}$$

$$\boxed{\frac{\log \zeta(s)}{s} = \int_0^\infty \frac{\pi(x)\,dx}{x(x^s-1)}}$$

(!)

$s = \sigma + it\,, \quad \sigma > 1.$

FOURIER
MELLIN

Inversion:

$$\begin{cases} F(s) = \displaystyle\int_0^\infty \frac{f(x)\,dx}{x^{s+1}} \\[2ex] f(x) = \displaystyle\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^s F(s)\,ds \end{cases}$$

$c >> 1$

$$\frac{\log \zeta(s)}{s} = \int_{\cancel{2}\ 0}^{\infty} \frac{\pi(x)\,dx}{x(x^s-1)}$$

$$\frac{x^{-s}}{1-x^{-s}}$$

Geometric Series.

$$= \cdots\cdots$$

$$= \int_0^{\infty} \frac{\sum_{n=1}^{\infty} \frac{1}{n}\pi\left(y^{\frac{1}{n}}\right)}{y^{s+1}}\,dy$$

By INVERSION

$$\Pi(x) \overset{\text{DEF.}}{=} \sum_{n=1}^{\infty} \frac{1}{n}\pi\left(\sqrt[n]{x}\right)$$

INVERSION

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^s \frac{\log \zeta(s)}{s}\,ds$$

$$\Pi(x) = \pi(x) + \underbrace{O\left(\frac{\sqrt{x}\,\cancel{\log}\log\log(x)}{\log(x)}\right)}_{\text{Smaller than the "error term } \sqrt{x}\log(x)\text{"}}$$

$$\boxed{\begin{aligned} &\pi(x) + \frac{1}{2}\pi(\sqrt{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + \cdots \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^s \frac{\log \zeta(s)}{s}\, ds \qquad \cancel{s} > 1 \\ &\qquad\qquad \sigma = \operatorname{Re}(s) > 1 \end{aligned}}$$

$$X^s = X^c X^{it} \qquad (s = c + it)$$

$$|X^s| = X^c, \quad c > 1$$

TOO BIG !

J. HADAMARD ZEROS EXHIBITED!

$$(s-1)\,\zeta(s) = \frac{1}{2}\left(\frac{2\pi}{e}\right)^{s} \prod_{n=1}^{\infty}\left(1+\frac{s}{2n}\right)e^{-\frac{s}{2n}} \times \prod_{k=1}^{\infty}\left(1-\frac{s}{\rho_k}\right)e^{+\frac{s}{\rho_k}}$$

ZEROS

$$\lim_{T\to\infty} \sum_{|\gamma_k|<T} \frac{1}{\rho_k} = 1+\frac{\gamma}{2}-\frac{\log(4\pi)}{2}$$

$\rho_k = \beta_k + i\gamma_k$

$$= 0{,}02309\ldots$$

Riemann gave 20 decimal places

Remark: $\sum_k \frac{1}{|\rho_k|} = +\infty$

$$\Pi(x) = \int_0^x \frac{du}{\log(u)}$$

ZEROS

$$- \sum_k li\left(x^{\rho_k}\right) - \log(2)$$

$$+ \int_x^\infty \frac{du}{u(u^2-1)\log(u)}$$

$$|li(x^\rho)| \sim \frac{x^{Re(\rho)}}{|\rho|\log(x)} \stackrel{?}{=} \frac{\sqrt{x}}{|\rho|\log(x)}$$

HYPOTHESIS

$$li(x^\rho) = \int_0^{x^\rho} \frac{du}{\log(u)}$$

$$\boxed{\begin{aligned}&(s-1)\,\zeta(s)\\ &= \frac{1}{2}\left(\frac{2\pi}{e}\right)^{s}\prod_{n=1}^{\infty}\left(1+\frac{s}{2n}\right)e^{-\frac{s}{2n}}\\ &\qquad\times\prod_{k}\left(1-\frac{s}{\rho_k}\right)e^{+\frac{s}{\rho_k}}\end{aligned}}$$

ZEROS IN THE CRITICAL STRIP

[von Mangoldt]

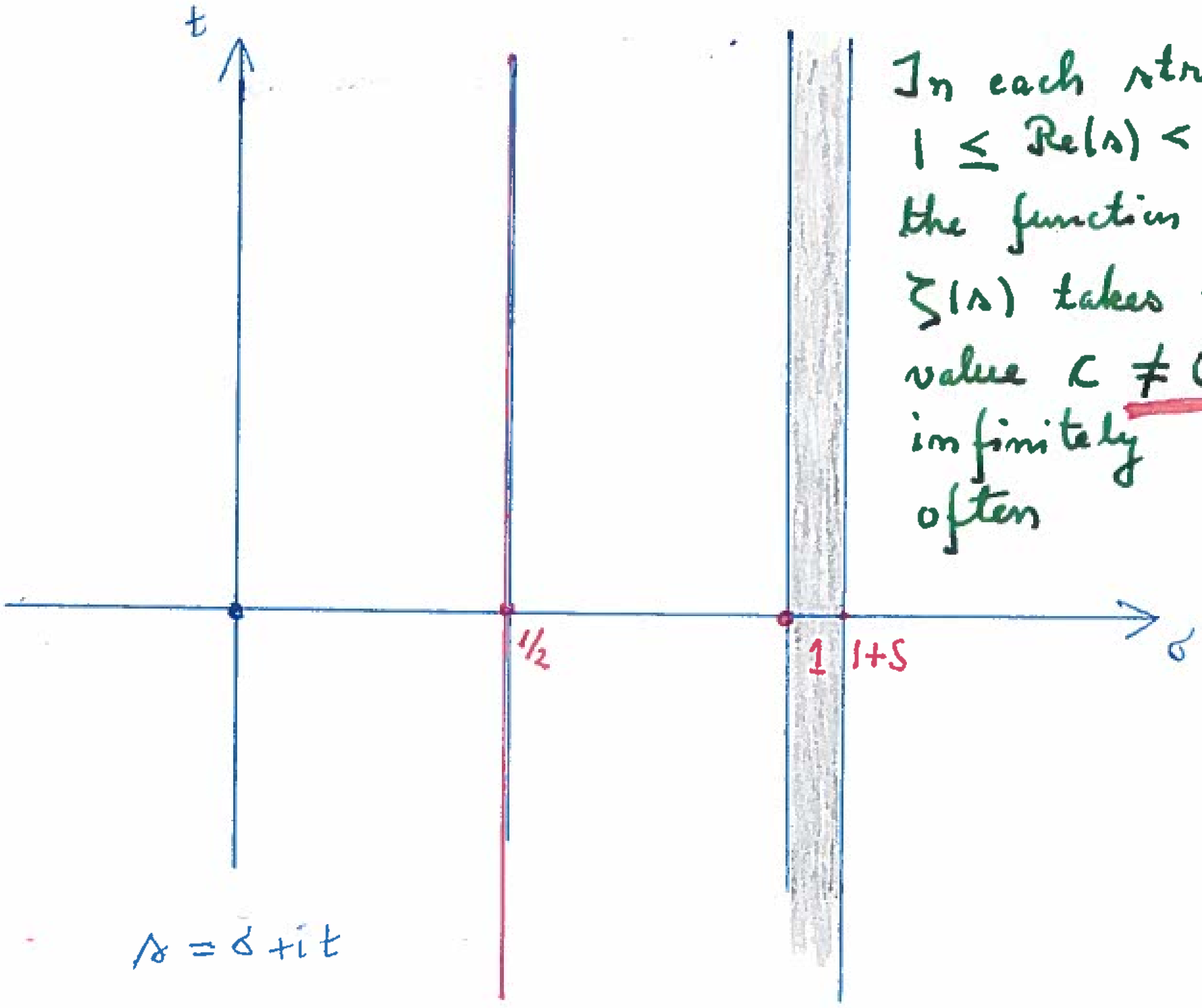
$$\psi(x) = \sum\sum_{p^m \le x} \log(p) \qquad |x^s| = x^c$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds$$

$\underline{c > 1}$

$$= x - \sum_k \frac{x^{\rho_k}}{\rho_k} - \log(2\pi)$$

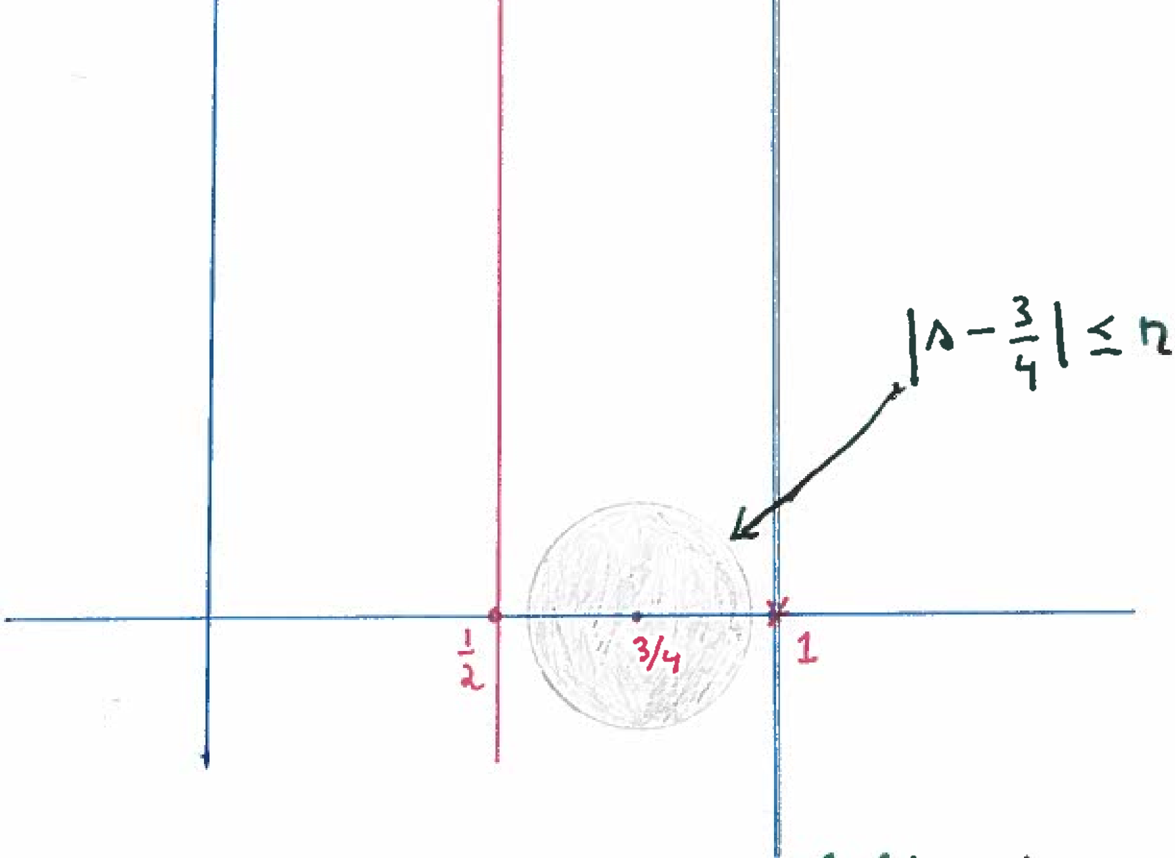
$$- \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right)$$



In each strip $1 \leq \mathrm{Re}(s) < 1 + \delta$ the function $\zeta(s)$ takes every value $c \neq 0$ infinitely often

VORONIN'S UNIVERSALITY THEOREM

1975



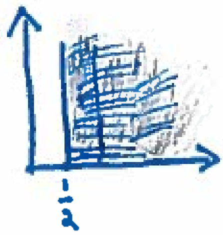
If $f(s) \neq 0$ and analytic in the disk $|s - \frac{3}{4}| \le r < \frac{1}{4}$, then, given $\varepsilon > 0$,

$$\max_{|s-\frac{3}{4}| \le r} \left| f(s) - \zeta(s + it_\varepsilon) \right| < \varepsilon$$

UNIFORM APPROXIMATION

for some $t_\varepsilon >> 1$.

$$\int_0^\infty \frac{1-12t^2}{(1+4t^2)^3}\int_{1/2}^\infty \log|\zeta(\sigma+it)|\,d\sigma\,dt$$



$$\stackrel{?}{=} \frac{\pi(3-\gamma)}{32}$$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$ conv. when

$Re(s) > \frac{1}{2} \iff$ The RIEMANN HYP.

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\log|\zeta(s)|}{|s|^s}\,ds = 0 \iff$$ RIEMANN HYP.

3	Gram 1895 (Riemann 1859?)
15	Gram 1903
79	Backlund 1914
1041	Titchmarsh 1936
1104	A. Turing 1953
∞	Hardy 1914
cT	Hardy – Littlewood 1921
$cT \log T$	Selberg 1942
33%	Levinson 1974
40%	Conrey 1989

$$\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}$$

$0 \le t \le T$

$$\int_0^x \frac{du}{\log(u)} = li(x) > \pi(x)$$

at least for $x \le 10^{23}$.

Sign change before 10^{361}

LITTLEWOOD 1914

∞'ly many sign changes

'97

Bohr – Landau 1914:

ONLY AN "INFINITESIMAL PORTION OF THE ZEROS" LIE OUTSIDE THE STRIP

$$\left|\frac{1}{2} - \mathrm{Re}(s)\right| < \varepsilon.$$

$$\begin{array}{r} 7005{,}101 \\ 7005{,}063 \\ \hline 0{,}038 \end{array}$$