MÖBIUS INVERSION

— from number theory to renormalisation

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[see arXiv:1809.00941]

- Classical : arithmetic functions, μ, $*$
- Posets : Δ, ε, $*$, ζ, μ, recursion, $\Phi_{even} - \Phi_{odd}$
- $\mathcal{P}(X)$, derangements
- Möbius categories: $(\mathbb{N}, +)$, $(\mathbb{N}^{\times}, \cdot)$ again, $\mathrm{Par}(n)$
- Abstract Möbius inversion, proof
- Renormalisation: background, $\mathcal{H}$, $\phi_{-} * \phi$

  proof with R, factorisation problem

Möbius inversion

$\mathbb{N}^\times := \{1, 2, 3, \ldots\}$

arithmetic functions : $f : \mathbb{N}^\times \longrightarrow \mathbb{C}$

associated Dirichlet series $\tilde{f}(s) = \sum_{n \geq 1} \frac{f(n)}{n^s}$

$\zeta : \mathbb{N}^\times \longrightarrow \mathbb{C}$, $n \longmapsto 1$ gives Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

<u>Thm</u> (Möbius 1832)

if $$f(n) = \sum_{d \mid n} g(d)$$

then $$g(n) = \sum_{d \mid n} f(d)\, \mu(n/d)$$

where μ is the Möbius function:

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ contains square factor} \\ (-1)^r & \text{if } n = p_1 \cdots p_r \\ & \text{product of } r \text{ distinct primes} \end{cases}$$

Convolution :
$$(f*g)(n) = \sum_{i \cdot j = n} f(i)\, g(j)$$

(corresponds to pointwise product of Dirichlet series – kind of Fourier transform)

Neutral for $*$: $\varepsilon(n) = \begin{cases} 1 & \text{for } n=1 \\ 0 & \text{else} \end{cases}$

Now the theorem says

$$f = g * \zeta \iff g = f * \mu$$

Precisely : ζ is $*$-invertible with inverse μ.

Möbius inversion for posets

(Weisner 1935, Hall 1936, Rota 1963)

P locally finite poset (i.e. all intervals finite)

$$[x,y] := \{ m \in P \mid x \leq m \leq y \}$$

C := free vector space on set of intervals

WHAT IS A COALGEBRA ?

coalgebra by

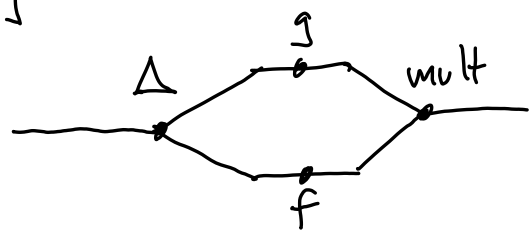
$$\Delta([x,y]) = \sum_{m \in [x,y]} [x,m] \otimes [m,y]$$

$$\varepsilon([x,y]) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{else} \end{cases}$$

Incidence algebra :

$$C^* = \text{linear dual} = \left(\mathrm{Lin}(C, \mathbb{Q}), *, \varepsilon \right)$$

$$(f*g)([x,y]) = \sum_{m \in [x,y]} f([x,m])\, g([m,y])$$



ε is unit for $*$

$\zeta \in C^*$

Thm (Rota) ζ is $*$ invertible with inverse μ

$$\mu([x,y]) = \begin{cases} 1 & \text{if } x=y \\ -\sum\limits_{\substack{m\in[x,y] \\ m\neq y}} \mu([x,m]) \end{cases}$$

(well defined recursion, since $m \neq y$)

$$= \Phi_{even}([x,y]) - \Phi_{odd}([x,y])$$

even-length chains in $[x,y]$

— # odd-length chains

Ex $\mathcal{P}(X), \subseteq$ power set

$$\mu(T \subseteq S) = (-1)^{\#(S-T)}$$

This is the inclusion-exclusion principle.

subex $\operatorname{der}(S) = \#$ derangements of S
(permutations without fixpoints)

clearly $$\operatorname{perm}(S) = \sum_{T \subseteq S} \operatorname{der}(T)$$

therefore
$$\operatorname{der}(S) = \sum_{T \subseteq S} \operatorname{perm}(T)(-1)^{\#(S-T)}$$

Möbius inversion for categories (Leroux)

Note: a poset is a category with at most one arrow between two objects

A monoid is a category with only one object.

$\mathcal{C}$ category (locally finite)

C spanned by arrows

$$\Delta(f) = \sum_{\substack{\xrightarrow{f} \\ a\searrow\nearrow b}} a \otimes b \qquad \varepsilon(f) = \begin{cases} 1 & f = id \\ 0 & \text{else} \end{cases}$$

$\zeta : C \to \mathbb{Q}$, all $f \longmapsto 1$

Under certain finiteness conditions (called Möbius category):

Thm (Leroux 1975) ζ $*$ invertible with inverse

$$\mu = \Phi_{even} - \Phi_{odd}$$

or

$$\mu(f) = -\sum_{\substack{b \circ a = f \\ b \neq id}} \mu(a)$$

$$\Phi_n(f) = \#\left\{ \begin{tikzcd} \cdot \arrow[rrrr, "f"] \arrow[dr, "a_1"'] & & & & \cdot \\ & \cdot \arrow[r, "a_2"'] & \cdot \cdots & \cdot \arrow[ur, "a_n"'] & \end{tikzcd} \;\middle|\; a_i \neq id \right\}$$

Ex $(\mathbb{N}, +)$ as a category

$$\Delta(n) = \sum_{i+j=n} i \otimes j$$

$$\mu(n) = \begin{cases} 1 & \text{for } n=0 \\ -1 & \text{for } n=1 \\ 0 & \text{else} \end{cases}$$

$$f(n) = \sum_{k \leq n} g(k) \iff g(n) = f(n) - f(n-1)$$

* μ is Euler (backward) finite difference
* ζ is "integration"

<u>Ex</u> $(\mathbb{N}^\times, \cdot) \simeq \prod_{p \text{ prime}} (\mathbb{N}, +)$

$$2^{e_2} 3^{e_3} 5^{e_5} \cdots \longleftrightarrow (e_2, e_3, e_5, \ldots)$$

Möbius function compatible with products

Get $\mu(n) = \begin{cases} 0 & \text{if contain sq.} \\ (-1)^r & \text{if } r \text{ distinct primes} \end{cases}$

<u>Rmk</u> get also Euler product expansion (1737)

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

<u>Ex</u> X random variable

moments $\phi_n = E[X^n]$

cumulants κ_n (Thiele 1889)

recursively defined

$\kappa_1 = \phi_1$

$\kappa_2 = \phi_2 - \phi_1^2$ variance

$\kappa_3 = \phi_3 - 3\phi_2\phi_1 + 2\phi_1^3$

$\vdots$

$\kappa_n(X+Y) = \kappa_n(X) + \kappa_n(Y)$ for X, Y independent

Rota: = Möbius inversion in the partitions lattice

$$\phi_n = \sum_{\pi \in Par(n)} K_\pi \quad \text{where} \quad K_\pi = \prod_{b \in \pi} K_{|b|}$$

cumulants given by Möbius inversion in the partitions lattice Par(n)

$$\mu(n) = (-1)^{n-1} (n-1)!$$

$$\boxed{\phi = K * \zeta \iff K = \phi * \mu}$$

Abstract Möbius inversion (no combinatorics)

C filtered coalgebra : $C_0 \subset C_1 \subset C_2 \cdots$

with $\Delta(C_n) \subset \sum_{p+q=n} C_p \otimes C_q$

Assume

C_0 spanned by grouplike elements $[\Delta(x) = x \otimes x]$

ex incidence coalg of Möbius cat.

consider $(\mathrm{LIN}(C, A), *, e)$ A algebra, $e = \eta \circ \varepsilon$

Thm If ϕ sends grouplike to 1, then $*$ inv

with inverse $\boxed{\psi = e - \psi * (\phi - e)}$ recursion well defined

Proof Set $\psi_0 := e$

$$\psi_{n+1} := \psi_n * (\phi - e)$$

Now $\psi_{even} := \sum_{n \text{ even}} \psi_n \qquad \psi_{odd} := \sum_{n \text{ odd}} \psi_n$

then
$$\psi_{odd} = \psi_{even} * (\phi - e)$$
$$\psi_{even} = e + \psi_{odd} * (\phi - e)$$

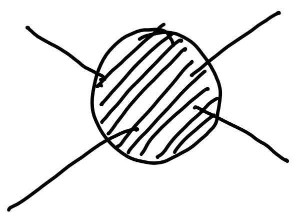
$$\psi := \psi_{even} - \psi_{odd}$$

$\Downarrow$

$$\boxed{\psi = e - \psi * (\phi - e)}$$

Perturbative renormalisation:

certain kind of interaction



$$= \sum_{\Gamma \in \mathcal{H}} \frac{\phi(\Gamma)}{|\mathrm{Aut}\,\Gamma|}$$

Sum over graphs

$\phi : \mathcal{H} \longrightarrow \mathbb{C}$ Feynman rules

Divergent for graphs with loops.

Renormalisation: extract finite values

First step : regularise : $\phi : \mathcal{H} \to \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$

Now well defined : divergencies are poles at $\varepsilon = 0$

Second step renormalise.

Naive idea: remove pole part of each $\phi(\Gamma)$.

$$\boxed{R : \mathbb{C}[\varepsilon^{-1}, \varepsilon]] \circlearrowleft \quad \text{pole part}}$$

Does not work: loses important physical features.

Need first to correct for divergent subgraphs...

Bogoliubov-Parasiuk-Hepp-Zimmerman 1955-1969

BPHZ found procedure

Kreimer 1998: all this encoded in bialgebra of graphs.

Recall: bialgebra : algebra + coalgebra + compatibility

mult: disj. union

comult: $\Delta(\Gamma) = \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma$

Counterterm:

$$\phi_-(\Gamma) = -R\Big[\sum_{\gamma \subsetneq \Gamma} \phi_-(\gamma)\,\phi(\Gamma/\gamma)\Big]$$

recursive. Well defined because of loop grading

In other words $\phi_- = e - R\left[\phi_- * (\phi - e)\right]$

Finally, renormalised Feynman rule

$$\phi_+ = \phi_- * \phi$$

Feynman rules are <u>characters</u>

The fact that ϕ_- and ϕ_+ are again characters is a consequence of R being Rota-Baxter operator:

$$R(xy) + R(x)R(y) = R\big(R(x)y + xRy\big)$$

Factorisation problem:

Given $K \subset A$ $\quad \phi : C \to A$

how to find ϕ_- such that $\phi_- * \phi \in K$

("A amplitudes, K finite amplitudes")

$$A = A_- \oplus A \qquad R : A \to A$$

$$\operatorname{Im} R = A_-$$

$$\operatorname{Ker} R = A_+ = K$$

$$\phi : C \to A$$

find ϕ_- such that $\phi_- * \phi \in A_+$

Solution: As Möbius inversion, but using R.

Define

$$\boxed{\alpha *_R \beta := R(\alpha * \beta)}$$

$$\psi_0 := e \qquad \psi_{n+1} := \psi_n *_R (\phi - e)$$

$$\psi_{even} \qquad \psi_{odd} \qquad \psi := \psi_{even} - \psi_{odd}$$

then

$$\boxed{\psi = e - \psi *_R (\phi - e)}$$

<u>Prop</u> if ϕ multiplicative and $\phi(\text{gr. like}) = 1$
then ϕ_- multiplicative

(This is where the Rota-Baxter property of R is used)

Atkinson (1966) $\quad A$ algebra

$$A = A_- \oplus A_+$$

A_- and A_+ subalgebras (non unital)

$\Updownarrow$

$R : A \to A$:= projection onto A_-

is Rota-Baxter :

$$R(xy) + R(x)R(y) = R\left[Rx \cdot y + x \cdot Ry\right]$$