Abelian integrals and the genesis of Abel's greatest discovery

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March 20, 2020

$$
\begin{aligned}
R(x) &= \frac{P(x)}{Q(x)} \\
&= \frac{3x^7 - 18x^6 + 46x^5 - 84x^4 + 127x^3 - 122x^2 + 84x - 56}{(x-2)^3(x+i)(x-i)} \\
&= 3x^2 + 7 + \frac{4x^4 - 12x^3 + 7x^2 - x + 1}{(x-2)^3(x+i)(x-i)} \\
&= 3x^2 + 7 + \frac{(-1)}{(x-2)^3} + \frac{3}{(x-2)^2} + \frac{4}{x-2} + \frac{i/2}{x+i} + \frac{(-i/2)}{x-i}
\end{aligned}
$$

$$
\begin{aligned}
\int R(x)\,dx &= x^3 + 7x + \frac{1/2}{(x-2)^2} + \frac{-3}{x-2} + 4\log(x-2) \\
&\qquad + \frac{i}{2}\log(x+i) + \frac{-i}{2}\log(x-i) \\
&= \widetilde{R}(x) + 4\log(x-2) + \frac{i}{2}\log(x+i) + \frac{-i}{2}\log(x-i)
\end{aligned}
$$

The integral of a rational function $R(x)$ is a rational function $\widetilde{R}(x)$ plus a sum of logarithms of the form $\log(x+a)$, $a \in \mathbb{C}$, multiplied with constants.

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \arcsin x = \frac{1}{i} \log\left(ix + \sqrt{1 - x^2}\right)$$

Trigonometric functions and the inverse arcus-functions can be expressed by (complex) exponential and logarithmic functions.

A rational function $R(x, y)$ of x and y is of the form

$$R(x, y) = \frac{P(x, y)}{Q(x, y)},$$

where P and Q are polynomials in x and y. So,

$$R(x, y) = \frac{\sum a_{ij} x^i y^j}{\sum b_{kl} x^k y^l}$$

Similarly we define a rational function of $x_1, x_2, \ldots, x_n$.

An algebraic function $y = y(x)$ is (implicitly) defined by

$$y^n + R_{n-1}(x)y^{n-1} + \cdots + R_1(x)y + R_0(x) = 0$$

where $R_i(x)$ is a rational function for $i = 0, 1, 2, \ldots, n-1$.

Similarly we define y to be an algebraic function of $x_1, x_2, \ldots, x_n$.

Examples

(i) $y + R_0(x) = 0$, i.e. $y = y(x) = -R_0(x)$ is a rational function.

(ii) $y^n + R_0(x) = 0$, i.e. $y = y(x) = \sqrt[n]{-R_0(x)}$.

(iii) $y^4 - 4xy^2 - 4xy - x = 0$, i.e. $y = y(x) = \sqrt{x} + \sqrt{x + \sqrt{x}}$.

(iv) $y^5 - y - x = 0$. Here $y = y(x)$ can not be given an explicit presentation, i.e. in terms of root extractions.

Example (ii): $y^n + R_0(x) = 0$. Choose $n = 2$ and

$$R_0(x) = -\frac{P(x)^2}{Q(x)^2}\left(ax^2 + bx + c\right),$$

where $P(x)$ and $Q(x)$ are polynomials in x. Then

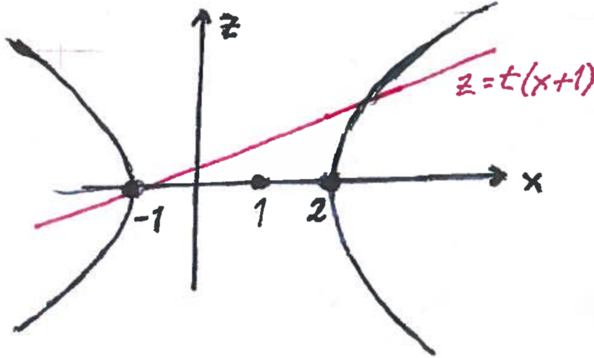
$$y = y(x) = \frac{P(x)\sqrt{ax^2 + bx + c}}{Q(x)} = \frac{P(x)z}{Q(x)},$$

where $z = \sqrt{ax^2 + bx + c}$.

Remark If $n = 2$ and $R_0(x) = -\frac{P(x)^2}{Q(x)^2} f(x)$, where $f(x)$ is a polynomial of degree 3 or 4, $\int y\, dx$ is an *elliptic* integral. If $\mathrm{degree}(f(x)) > 4$, the integral $\int y\, dx$ is a *hyperelliptic* integral.

Concrete example

$$y = y(x) = \frac{\sqrt{x^2 - x - 2}}{(x+1)^2} = \frac{z}{(x+1)^2},$$



where $z = \sqrt{x^2 - x - 2}$.

$$z^2 = x^2 - x - 2 = (x+1)(x-2) \qquad z^2 = t^2(x+1)^2 = (x+1)(x-2)$$

$$x = \frac{t^2+2}{1-t^2}, \quad z = \frac{3t}{1-t^2}, \quad dx = \frac{6t}{(1-t^2)^2}$$

Remark The curve defined by $z^2 - x^2 + x + 2 = 0$ is intersected by the curve $z = t(x+1) = 0$, where t is a (variable) parameter.

$$\begin{aligned}
\int \frac{\sqrt{x^2 - x - 2}}{(x+1)^2} dx &= 2 \int \frac{t^2}{1 - t^2} dt \\
&= -2t - \log(1 - t) + \log(1 + t) \\
&= \frac{2z}{x+1} - \log\left(\frac{x - z + 1}{x+1}\right) + \log\left(\frac{x + z + 1}{x+1}\right)
\end{aligned}$$

Abelian integral: $W = \int y\,dx$ is called an *abelian integral*, whenever $y = y(x)$ is an algebraic function. In other words, $W = W(x)$ is an abelian integral if it is a solution to the differential equation $\frac{dW}{dx} = y$.

Comment $\int R(x, y)\,dx$, where $y = y(x)$ is an algebraic function and $R(x, y)$ is a rational function in x and y, is an abelian integral. In fact, one can show that $R(x, y) = R(x, y(x))$ is an algebraic function of x.

Example

$$\int \frac{x + \sqrt[3]{1 + x^7}}{(x-2)\sqrt[3]{1+x^7}}\, dx = \int R(x,y)\, dx,$$

where $y = \sqrt[3]{1+x^7}$, $R(x,y) = \dfrac{x+y}{(x-2)y}$. Now

$$u = R(x,y) = \frac{x + \sqrt[3]{1 + x^7}}{(x-2)\sqrt[3]{1+x^7}}$$

is an algebraic function and we can write the integral $\int R(x,y)\, dx$ as $\int u\, dx$.

Elementary functions. These are functions that can be "built" from algebraic functions and logarithmic functions and their inverses. Note that the inverse of an algebraic function is again an algebraic function. In fact, if $y = y(x)$ is defined by the equation

$$P_n(x)y^n + P_{n-1}y^{n-1} + \cdots + P_1(x)y + P_0(x) = 0$$

where the $P_i(x)$'s are polynomials, then the inverse $x = x(y)$ is defined by a similar equation, now written as

$$Q_m(y)x^m + Q_{m-1}(y)x^{m-1} + \cdots + Q_1(y) + Q_0(y) = 0$$

where the $Q_j(y)$'s are polynomials.

The hierarchy of elementary functions

Order 0:	algebraic functions
Order 1:	algebraic functions of exp or log of functions of order 0
Order 2:	algebraic functions of exp or log of functions of order 1.
etc.	

Examples

$e^{x^2} + e^x\sqrt{\log x}$ (order 1)

y defined by $y^5 - y - e^x \log x = 0$ (order 1)

e^{e^x}, $\log\log x$ (order 2)

Analogy with classification of radicals (over $\mathbb{Q}$):
2, $\sqrt[3]{7}$, $\sqrt[5]{3+\sqrt{5}}$ (orders 0, 1, 2, respectively).

Observe that taking the the derivative of an elementary function of order n yields an elementary function of order $\leq n$.

Theorem 1 (*Abel (1829), Liouville (1833)*) If the abelian integral $\int y\,dx$ is an elementary function then it must have the form

$$\int y\,dx = t + A\log u + B\log v + \cdots + F\log w$$

where $t, u, v, \ldots, w$ are algebraic functions of x and $A, B, \ldots, F$ are constants.

Theorem 2 (*Abel's Theorem; Précis (1829)*) The functions $t, u, v, \ldots, w$ in Theorem 1 are *rational* functions of x and y.

(Footnote in Précis: "I have founded on this theorem a new theory of integration of algebraic differentials, but circumstances have made it impossible for me to publish this yet…")

Cela posé, imaginons une fonction algébrique θ telle qu'on puisse exprimer toutes les fonctions

$$(62) \qquad u, v_1, v_2, \ldots v_\nu; \; t_1, t_2, \ldots t_n, \; \Delta_1(t_1), \; \Delta_2(t_2), \ldots \Delta_n(t_n)$$

rationnellement en

$$(63) \qquad \theta, x_1, x_2, x_3, \ldots x_\mu, y_1, y_2, y_3, \ldots y_\mu.$$

Il existe une infinité de fonctions θ qui jouissent de cette propriété. Une telle fonction sera par exemple la somme de toutes les fonctions (62), multipliées chacune par un coefficient indéterminé et constant. C'est ce qui est facile à démontrer par la théorie des équations algébriques. La quantité θ, étant une fonction algébrique des variables $x_1, x_2, \ldots$, pourra donc satisfaire à une équation algébrique, dans laquelle tous les coefficiens sont des fonctions *rationnelles* de $x_1, x_2, \ldots$. Or au lieu de supposer ces coefficiens rationnels en $x_1, x_2, \ldots$, nous les supposerons rationnels en

$$(64) \qquad x_1, x_2, x_3, \ldots x_\mu, y_1, y_2, y_3, \ldots y_\mu;$$

car cette supposition permise simplifiera beaucoup le raisonnement. Soit donc

$$(65) \qquad V = 0$$

l'équation en θ; désignons son degré par δ et supposons, ce qui est permis, qu'il soit impossible que la fonction θ puisse être racine d'une autre équation de la même forme, mais dont le degré soit moindre que δ.

Remarks concerning Abel's Theorem and its proof in Précis.

Galois cited this particular result in Précis, and Liouville included in his paper the complete proof given in Précis of Theorem 2.

In 1916 Hardy and Littlewood (erroneously) claimed that Abel's proof was wrong!

Using Abel's Theorem one can show that "most" elliptic integrals, for example

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

are not elementary functions.

However, as an example of the subtlety involved consider

$$\int \frac{x\,dx}{\sqrt{x^4+10x^2-96x-71}} = -\frac{1}{8}\log\left[\begin{array}{l}\left(x^6+15x^4-80x^3+27x^2-528x+781\right)\cdot\sqrt{x^4+10x^2-96x-71} \\ -\left(x^8+20x^6-128x^5+54x^4-1408x^3+3124x^2+10001\right)\end{array}\right]$$

But

$$\int \frac{x\,dx}{\sqrt{x^4+10x^2-96x-72}}$$

is *not* an elementary function.

1802 Born Aug. 5

1815

1816

1817

1818 New math teacher (Bernt Michael Holmboe)

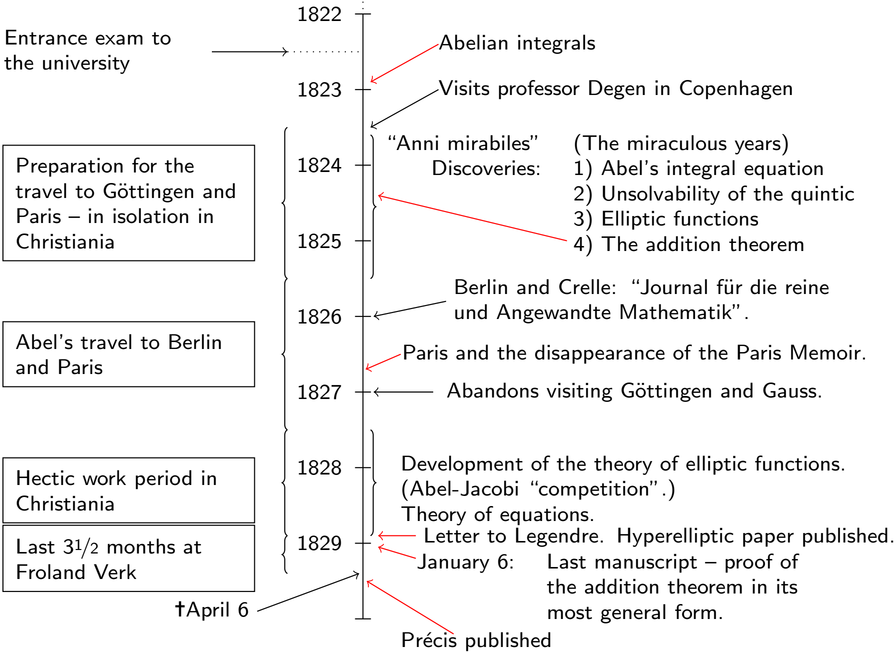
Pupil at the Cathedral School in Christiania (later Oslo)

1819

1820 Death of Abel's father

1821 "Proof" of solvability (sic!) of the quintic

Examen Artium


1822
Entrance exam to the university
Abelian integrals
1823
Visits professor Degen in Copenhagen
Preparation for the travel to Göttingen and Paris – in isolation in Christiania
1824
"Anni mirabiles" (The miraculous years)
Discoveries: 1) Abel's integral equation
2) Unsolvability of the quintic
3) Elliptic functions
4) The addition theorem
1825
Berlin and Crelle: "Journal für die reine und Angewandte Mathematik".
1826
Abel's travel to Berlin and Paris
Paris and the disappearance of the Paris Memoir.
1827
Abandons visiting Göttingen and Gauss.
1828
Hectic work period in Christiania
Development of the theory of elliptic functions.
(Abel-Jacobi "competition".)
Theory of equations.
Letter to Legendre. Hyperelliptic paper published.
1829
Last 3½ months at Froland Verk
January 6: Last manuscript – proof of the addition theorem in its most general form.
†April 6
Précis published

Atiyah in his acceptance speech in Oslo 2004 on the occasion of receiving the Abel Prize:

Abel was really the first modern mathematician. His whole approach, with its generality, its insight and its elegance set the tone for the next two centuries. (...) Had Abel lived longer, he would have been the natural successor to the great Gauss.

Yuri Manin being interviewed in 2009:

Take for example the first volume of Crelle's Journal *(*Journal of Pure and Applied Mathematics*), which appeared first time in 1826. Abel's article appeared there, on the unsolvability in radicals of the general equation of degree higher than four. A wonderful article! As a member of the editorial board of Crelle, I would accept it even today with great pleasure.*

Given polynomials $R = R(x)$ and $\rho = \rho(x)$. Abel (1826) proved that $\int \frac{\rho\, dx}{\sqrt{R}}$ can be expressed by logarithms if and only if $\int \frac{\rho\, dx}{\sqrt{R}}$ is of the form

$$\int \frac{\rho\, dx}{\sqrt{R}} = A \log \left(\frac{p + q\sqrt{R}}{p - q\sqrt{R}} \right)$$

where p and q are polynomials in x, A is a constant.

Furthermore, he showed that there exists ρ such that

$$\int \frac{\rho\, dx}{\sqrt{R}} = \log \left(\frac{p + \sqrt{R}}{p - \sqrt{R}} \right)$$

if and only if $\sqrt{R}$ has a periodic continued fraction expansion.

$$\sqrt{R} = r + \cfrac{1}{2\mu + \cfrac{1}{2\mu_1 + \cfrac{1}{\ddots + \cfrac{1}{2\mu_1 + \cfrac{1}{2\mu + \cfrac{1}{2r + \cfrac{1}{2\mu + \cfrac{1}{2\mu_1 + \cfrac{1}{\ddots}}}}}}}}}$$

period

$$p = r + \cfrac{1}{2\mu + \cfrac{1}{2\mu_1 + \cfrac{1}{\ddots + \cfrac{1}{2\mu + \cfrac{1}{2r}}}}}$$

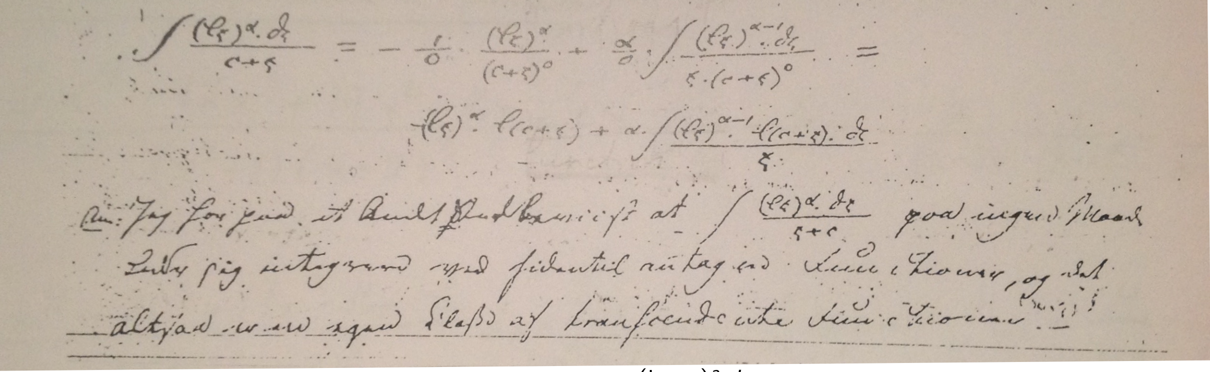
The proof of this result is obtained by Abel by putting it in connection with Pell-like equations

$$F^2 - G^2 R = a$$

where F and G are polynomials in x, and a is a constant.

"The ramparts are raised all around but, enclosed in its last redoubt, the problem defends itself desperately. Who will be the fortunate genius who will lead the assault upon it or force it to capitulate?"

Jean Étienne Montucla (1725-1799)
Histoire des Mathématiques (Tome III)



$$\int \frac{(lx)^{\alpha}\,dx}{c+x} = -\frac{1}{0}\cdot\frac{(lx)^{\alpha}}{(c+x)^{0}} + \frac{\alpha}{0}\int\frac{(lx)^{\alpha-1}\,dx}{x\cdot(c+x)^{0}} =$$

$$(lx)^{\alpha}\,l(c+x) + \alpha\int\frac{(lx)^{\alpha-1}\,l(c+x)\,dx}{x}$$

Anm: Jeg har paa et andet Sted beviist at $\int \frac{(lx)^{\alpha}\,dx}{x+c}$ paa ingen Maade lader sig integrere ved de hidentil antagne Functioner, og at det altsaa er en egen Classe af transcendente Functioner.

Jeg har paa et andet Sted beviist at $\int \frac{(\log x)^a\,dx}{c+x}$ *paa ingen Maade lader sig integrere ved de hidentil antagne Functioner, og at det altsaa er en egen Classe af transcendente Functioner.*

[*I have proved another place that* $\int \frac{(\log x)^a\,dx}{c+x}$ *in no way whatsoever can be integrated in terms of the up to now familiar functions, and hence this belongs to a separate class of transcendent functions.*]

“Abel told me”, Holmboe said, “that already during his stay in Paris autumn 1826 he had finished the essential part of the principles that he wanted to put forward regarding these functions. He would have preferred to postpone the publication of his discoveries until he was able to present them as a unified theory, if not in the meantime Jacobi appeared on the scene, which had shaken up his plans.”

Let $\mathbb{C}(x)$ denote the field of rational functions over $\mathbb{C}$. Let $y = y(x)$ be an algebraic function given by

$$\text{(i)} \qquad y^n + R_{n-1}(x)y^{n-1} + \cdots + R_1(x)y + R_0(x) = 0.$$

Take the derivative of (i):

$$ny^{n-1}y' + (R'_{n-1}y^{n-1} + (n-1)R_{n-1}y^{n-2}y') + \cdots + (R'_1 y + R_1 y') + R'_0 = 0.$$

<u>Conclusion</u> y' is a rational function of x and y, and hence y' is an algebraic function.

Let $a = a(x, y)$ be a rational function of x and y, where y is as above. Then

$$(a' =)\frac{da}{dx} = \frac{\partial a}{\partial x} + \frac{\partial a}{\partial y}\frac{dy}{dx} \qquad \text{is a rational function of } x \text{ and } y.$$

Theorem (*very special case of Abel's Theorem*) Let $y = y(x)$ be an algebraic function. Assume the abelian integral $u = \int y\,dx$ is an algebraic function. Then u is a *rational* function of x and y.

Proof: Let $\mathbb{C}(x)$ be the field of rational functions of x. Then $\mathbb{C}(x, y)$ ($\equiv$ rational functions of x and y) is a finite-dimensional field extension of $\mathbb{C}(x)$. Since u is algebraic over $\mathbb{C}(x)$, it is obviously algebraic over $\mathbb{C}(x, y)$.
(Abel: "...car cette supposition permise simplifiera beaucoup le raisonnement.")
Now u is a root of an *irreducible* polynomial

$$f(z) = z^k + a_{k-1}(x, y)z^{k-1} + \cdots + a_1(x, y) + a_0(x, y)$$

over $\mathbb{C}(x, y)$ (the minimal polynomial of u over K).

$$(*) \qquad u^k + a_{k-1}u^{k-1} + \cdots + a_1 u + a_0 = 0$$

Take the derivative and use that $u' = y$:

$(**)$

$$ku^{k-1}y + \left(a'_{k-1}u^{k-1} + (k-1)a_{k-1}u^{k-2}y\right) + \cdots + \left(a'_1 u + a_1 y\right) + a'_0 = 0$$

Now each a'_i is a rational function of x and y, and so $a'_i \in \mathbb{C}(x, y)$. From $(**)$ we get

$$(***) \qquad (ky + a'_{k-1})\, u^{k-1} + \cdots + (2a_2 y + a'_1)u + (a_1 y + a'_0) = 0.$$

Hence $ky + a'_{k-1} = 0$, and so

$$u = \int y\, dx = -\frac{1}{k}\int a'_{k-1}\, dx = -\frac{a_{k-1}}{k}, \qquad \text{q.e.d.}$$

R. H. Risch showed in 1969 that there exists an algorithm to decide whether the indefinite integral of an elementary function is elementary or not.

The crucial theorem that Risch's result is based upon is an analogue of Theorem 1 (due to Abel and Liouville).

23. December 1751: “Die Geburtstag der elliptischen Funktionen” (Jacobi).

Euler (23. December 1751 —).

$$\frac{dx}{\sqrt{A+2Bx+Cx^2+2Dx^3+Ex^4}}+\frac{dy}{\sqrt{A+2By+Cy^2+2Dy^3+Ey^4}}=0$$

$$\left(\frac{\sqrt{A+2Bx+\cdots}-\sqrt{A+2By+\cdots}}{x-y}\right)^2=2D(x+y)+E(x+y)^2+F.$$

$$\int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} + \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

$$z = \frac{x\sqrt{(1-y^2)(1-k^2y^2)} + y\sqrt{(1-x^2)(1-k^2y^2)}}{1-k^2x^2y^2}$$

"Potius tentando vel divinando."

(Abel's letter to Degen in Copenhagen dated March 2, 1824.)

Jeg er af en Hændelse kommen dertil, at jeg kan udtrykke en Egenskab af alle transcendente Functioner af Formen $\int \phi(z)dz$, hvor $\phi(z)$ betegner en hvilkensomhelst algebraisk irrational Function af z, ved en saadan Ligning, og det mellem saa mange variable Størrelser som man vil; nemlig dersom man betegner $\int \phi(z)dz = \psi(z)$ saa kan man altid finde en Ligning af Formen

$$\psi(z_1) + \psi(z_2) + \psi(z_3) + \cdots + \psi(z_n) = \psi(\alpha_1) + \psi(\alpha_2) + \cdots + \psi(\alpha_n) + p$$

hvor z_1, z_2 etc. ere algebraiske Functioner af et hvilketsomhelst Antal variable Størrelser (n er afhængig av dette Antal og i Alm. meget større; α_1, α_2 etc. ere constante Størrelser og p en algebraisk og logarithmisk Function; den er i mange Tilfælde liig Nul). Dette Theorem og en Afhandling derom har jeg tænkt at sende til det franske Institut, da jeg synes det vil udbrede Lys over de transcendente Functioner i det Hele."

"... I have come across a remarkable discovery: I can express a property of all transcendental functions of the form $\int \phi(z)dz$, where $\phi(z)$ is an arbitrary algebraic function of z, by an equation of the following form (denoting $\int \phi(z)dz = \psi(z)$):

$$\psi(z_1) + \psi(z_2) + \psi(z_3) + \cdots + \psi(z_n) = \psi(\alpha_1) + \psi(\alpha_2) + \cdots + \psi(\alpha_n) + p$$

where $z_1, z_2, \ldots$ are algebraic functions of an arbitrary number of variables (n depends on this number and is in general much larger; α_1, α_2 are constant entries and p is an algebraic/logarithmic function, which in many cases is zero). This theorem and a memoir based on it I expect to send to the French Institute, for I believe it will throw light over the whole theory of transcendental functions ..."

Abel's addition theorem in the hyperelliptic case, generalizing Euler's addition theorem in the elliptic case. (Published in Crelle, Volume 3, December 3, 1828).

Let n be any natural number and let $x_1, x_2, \ldots, x_n$ be any complex numbers. Then,

$$\int_0^{x_1} \frac{dx}{\sqrt{f(x)}} + \int_0^{x_2} \frac{dx}{\sqrt{f(x)}} + \cdots + \int_0^{x_n} \frac{dx}{\sqrt{f(x)}} = \sum_{k=1}^{g} \int_0^{z_k} \frac{dx}{\sqrt{f(x)}}$$

where $f(x)$ is a polynomial of degree $2g + 2$ or $2g + 1$ with no multiple roots and $z_1, z_2, \ldots, z_g$ are algebraic functions of $x_1, x_2, \ldots, x_n$.

In a letter to Legendre dated March 14, 1829, Jacobi wrote:

“Quelle découverte de M. Abel que cette généralisation de l’intégrale d’Euler! A-t-on jamais vu pareille chose! Mais comment s’est-il fait que cette découverte, peut-être la plus importante de ce qu’a fait dans les mathématiques le siècle dans lequel nous vivons, étant communiquée à votre Académie il y a deux ans, elle ait pu échapper à l’attention de vous et de vos confrères?”

“What a discovery by Abel, this generalization of Euler’s integral! Has anything like it ever been seen? But how is it possible that this discovery, perhaps the most important in our century, could have avoided the attention of yourself and your colleagues after having been communicated to the Academy more than two years ago?”

Démonstration d'une propriété générale d'une certaine classe de fonctions transcendentes.

(Par Mr. *N. H. Abel.*)

Théorème. Soit y une fonction de x qui satisfait à une équation quelconque irréductible de la forme:

$$1. \quad 0 = p_0 + p_1 . y + p_2 . y^2 + \ldots + p_{n-1} . y^{n-1} + y^n,$$

où $p_0, p_1, p_2, \ldots p_{n-1}$ sont des fonctions entières de la variable x. Soit

$$2. \quad 0 = q_0 + q_1 . y + q_2 . y^2 + \ldots + q_{n-1} . y^{n-1},$$

une équation semblable, $q_0, q_1, q_2, \ldots q_{n-1}$ étant également des fonctions entières de x, et supposons variables les coëfficiens des diverses puissances de x dans ces fonctions. Nous désignerons ces coëfficiens par a, a', a'', ... En vertu des deux équations (1.) et (2.) x sera fonction de $a, a', a'', \ldots$ et on en déterminera les valeurs en éliminant la quantité y. Désignons par:

$$3. \quad \varrho = 0$$

le résultat de l'élimination, en sorte que ϱ ne contiendra que les variables $x, a, a', a'', \ldots$. Soit μ le degré de cette équation par rapport à x, et désignons par

$$4. \quad x_1, x_2, x_3, \ldots x_\mu$$

ses μ racines, qui seront autant de fonctions de $a, a', a'', \ldots$. Cela posé, si l'on fait

$$5. \quad \psi x = \int f(x, y) . \partial x$$

où $f(x, y)$ désigne une fonction **rationnelle** quelconque de x et de y, je dis, que la fonction transcendente ψx jouira de la propriété générale exprimée par l'équation suivante:

$$6. \quad \psi x_1 + \psi x_2 + \ldots + \psi x_\mu = u + k_1 \log v_1 + k_2 \log v_2 + \ldots + k_n \log v_n,$$

$u, v_1, v_2, \ldots v_n$ étant des fonctions rationnelles de $a, a', a'', \ldots$, et $k_1, k_2, \ldots k_n$ des constantes.

Démonstration. Pour prouver ce théorème il suffit d'exprimer la différentielle du premier membre de l'équation (6.) en fonction de $a, a', a'', \ldots$; car il se réduira par là à une différentielle rationnelle, comme on le verra. D'abord les deux équations (1.) et (2.) donneront y en fonction rationnelle de $x, a, a', a'', \ldots$. De même l'équation (3.): $\varrho = 0$ donnera pour ∂x une expression de la forme:

$$\partial x = \alpha . \partial a + \alpha' . \partial a' + \alpha'' . \partial a'' + \ldots,$$

où $\alpha, \alpha', \alpha'', \ldots$ sont des fonctions rationnelles de $x, a, a', a'', \ldots$. De là il suit que la différentielle $f(x, y) . \partial x$ pourra être mise sous la forme:

$$f(x, y) \partial x = \varphi x . \partial a + \varphi_1 x . \partial a' + \varphi_2 x . \partial a'' + \ldots,$$

où $\varphi x, \varphi_1 x, \ldots$ sont des fonctions rationnelles de $x, a, a', a'', \ldots$. En intégrant, il viendra:

$$\psi x = \int \{\varphi x . \partial a + \varphi_1 x . \partial a' + \ldots\}$$

et de là on tire, en remarquant que cette équation aura lieu en mettant pour x les μ valeurs de cette quantité:

$$7. \quad \psi x_1 + \psi x_2 + \ldots + \psi x_\mu$$
$$= \int [\{\varphi x_1 + \varphi x_2 + \ldots + \varphi x_\mu\} \partial a + \{\varphi_1 x_1 + \varphi_1 x_2 + \ldots + \varphi_1 x_\mu\} \partial a' + \ldots]$$

Dans cette équation les coëfficiens des différentielles $\partial a, \partial a', \ldots$ sont des fonctions rationnelles de $a, a', a'', \ldots$ et de $x_1, x_2, \ldots x_\mu$, mais d'ailleurs ils sont symétriques par rapport à $x_1, x_2, \ldots x_\mu$; donc, en vertu d'un théorème connu, on pourra exprimer ces fonctions rationnellement par $a, a', a'', \ldots$ et par les coëfficiens de l'équation $\varrho = 0$; mais ceux-ci sont eux-mêmes des fonctions rationnelles des variables $a, a', a'', \ldots$, donc enfin les coëfficiens de $\partial a, \partial a', \partial a'', \ldots$ de l'équation (7.) le seront également. Donc, en intégrant, on aura une équation de la forme (6.).

Je me propose de développer dans une autre occasion des nombreuses applications de ce théorème, qui jetteront un grand jour sur la nature des fonctions transcendentes dont il s'agit.

Christiania, le 6. Janvier 1829.

The Paris Memoir starts with the remark that a sum of abelian integrals $(*)\ \omega(x_1) + \omega(x_2) + \cdots + \omega(x_n)$, where $\omega(x) = \int_a^x u\, dx$, can be expressed in terms of algebraic/logarithmic functions of $x_1, x_2, \ldots, x_n$ provided there exist certain algebraic relations between $x_1, x_2, \ldots, x_n$. Abel then attacks the following problem: Determine the minimal number γ such that an arbitrary sum $(*)$ can be expressed in terms of γ summands of the form $\omega(z) = \int_a^z u\, dx$ where the z's are algebraic functions of $x_1, x_2, \ldots, x_n$ (plus algebraic/logarithmic terms). He then establishes that this question is intimately related to when the algebraic/logarithmic terms vanish.

This is the first time in the history of mathematics that the all important concept of the genus of an algebraic curve is introduced and given "life" and significance.

Dirichlet (1st July 1852): Gedächtnisrede auf Jacobi:

"...das Eulersche Theorem bildete damals auf dem Gebiete, dem es angehört, die Grenze der Wissenschaft, über welche hinauszugehen Euler selbst, Lagrange und andere Vorgänger Abels sich vergebens bemüth hatte. Welche Bewunderung musste daher eine Entdeckung hervorrufen, welche, die Integrale aller algebraischen Functionen umfassend, die Grundeigenschaft derselben enthüllte."

"...Euler's Theorem represented then – within the field to which it belonged – the limit of mathematical science and beyond which Euler himself, Lagrange and other predecessors of Abel had endeavoured in vain to surmount. What awe must therefore a discovery have brought forth which unveiled the essential properties of the integrals of *all* algebraic functions."

Brill-Noether (Jahresbericht der Deutschen Mathematiker-Vereinigung – 1892/93)
"Die Entwicklung der Theorie der algebraischen Functionen in älterer und neuerer Zeit."

"First of all, it is Abels's glory to have brought integrals of higher radicals (i.e. hyperelliptic integrals) within the accessible reach of the mathematical learning of his time. Confronted with these even Euler's acuity proved to be insufficient. Furthermore, he put these integrals in such close relation to the elliptic functions that their complete understanding was only a question of time. However, this was for Abel just a transitional step on the way to a type of integrals that nobody before him had ever thought of."

"$\ldots y$ is by means of the polynomial equation $\Theta(x, y) = 0$ defined as the general algebraic function of x. This concept is here for the first time included in a theorem, and attains thereby life and significance. In this sense Abel is the founder of the theory of algebraic functions."

Picard (1899): "Le théorème parait tout à fait élémentaire, et il n'y a peut-être pas, dans l'histoire de la Science, de proposition aussi importante obtenue à l'aide de considérations aussi simples."

["The theorem appears as completely elementary, and perhaps there has never occurred in the history of science a proposition so important which is obtained by so simple considerations."]

Kronecker to Mittag-Leffler (1874): "The whole enormous edifice of modern mathematics rests on the shoulder of this Scandinavian giant."

Atle Selberg:

Det har alltid stått for meg som den rene magi. Hverken Gauss eller Riemann, eller noen annen, har noe som riktig kan måle seg med dette.

[*For me this has always appeared as pure magic. Neither Gauss nor Riemann nor anyone else have anything that really measures up to this.*]

What was the “Hændelse” (“epiphany”) that Abel refers to in his letter to Degen, and which gave him the ingenious idea how to generalize Euler’s addition theorem for elliptic integrals to any abelian integral? Nobody knows. (The crucial mathematical notebooks that he had at that time are lost.) However, a clue may be to take a look at the proof that Abel presents in the hyperelliptic case, and specialize that proof to the elliptic case.

The Euler addition theorem for elliptic integrals (of the first kind) can be formulated in the following manner, which for the general theory of algebraic integrals is of fundamental importance: Given three pairs

$$\left(x_1, \sqrt{f(x_1)}\right), \left(x_2, \sqrt{f(x_2)}\right), \left(x_3, \sqrt{f(x_3)}\right)$$

where $f(x)$ is a polynomial of third or fourth degree. Assume two of these pairs are given. Then one can in an *algebraic way* determine the third pair so that the differential equation

$$\frac{dx_1}{\sqrt{f(x_1)}} + \frac{dx_2}{\sqrt{f(x_2)}} + \frac{dx_3}{\sqrt{f(x_3)}} = 0$$

is satisfied. (This is the genus one case. Abel's addition theorem generalizes this to the higher genus cases.)

Abel's proof is based on the following lemma:

Let $F(x)$ be a polynomial of degree n with n distinct roots $x_1, x_2, \dots, x_n$, and let $F'(x)$ be the derivative of $F(x)$. Let $\phi(x)$ be a polynomial of degree less or equal to $n-2$. Then

$$\frac{\phi(x_1)}{F'(x_1)} + \frac{\phi(x_2)}{F'(x_2)} + \cdots + \frac{\phi(x_n)}{F'(x_n)} = 0.$$

Proof: The partial fraction decomposition of the rational function $\frac{x\phi(x)}{F(x)}$ is equal to

$$\frac{x_1\phi(x_1)}{F'(x_1)(x-x_1)} + \frac{x_2\phi(x_2)}{F'(x_2)(x-x_2)} + \cdots + \frac{x_n\phi(x_n)}{F'(x_n)(x-x_n)}$$

Setting $x = 0$, yields the result.

Let $f(x)$ be a polynomial of degree 3 or 4. Define

$$P(x) = a + bx + cx^2$$

where a, b, c are (variable) parameters. Define

$$(*) \qquad F(x) = P(x)^2 - f(x) = \left(P(x) + \sqrt{f(x)}\right)\left(P(x) - \sqrt{f(x)}\right)$$

Let x_1, x_2, x_3, x_4 be the four roots of $F(x) = 0$, and consider these roots as functions of the parameters a, b, c. Take the differential of $(*)$. Then we get by a simple computation:

$$\frac{(\delta P)(x_i)}{F'(x_i)} = \frac{dx_i}{2\sqrt{f(x_i)}}; \quad i = 1, 2, 3, 4$$

where δ denotes the differential with respect to the parameters a, b, c.

By the lemma the sum on the left side is 0, and so we get:

$$\sum_{i=1}^{4} \frac{dx_i}{\sqrt{f(x_i)}} = 0.$$

Eliminating the parameters a, b, c from

$$F(x_i) = (a + bx_i + cx_i^2)^2 - f(x_i) = 0, \quad i = 1, 2, 3, 4$$

we get Euler's addition theorem for elliptic integrals.

<u>Remark</u> What is going on is the following: The curve $y^2 - f(x) = 0$ is intersected by the "movable" curve (Clebsch: "Bewegliche" Curve) $y - (a + bx + cx^2) = 0$, where a, b, c are (variable) parameters.

(Sylow, 1902)
I sin egentlige forfattertid, kun tre aar omtrent, havde han i grunden alle sine emner samtidig under behandling. Han holdt dem i forbindelse med hinanden og beskjæftigde sig i korte perioder snart med den ene, snart med det andet; mest tid synes den saa rige teori for de elliptiske funktioner at have krævet. Hans arbeidsomhed var overordentlig; foruden at han i disse tre aar offentliggjorde alle sine store opdagelser, har han ogsaa forberedt arbeider, som han ikke fik tid til at fuldende, og det er ingenlunde sikkert, at vi kjender alle hans planer for fremtiden. Den endelige redaktion af hans arbeider kom ofte aar efter opdagelsen af de nye resultater, som de indeholt, og de færdige afhandlinger har derpaa ofte maattet vente temmelig længe, inden deres tur til trykning kom.

Hvad der især udmerker Abel, foruden hans iderigdom, er hans stræben efter den fulde stringens, samt den store almindelighed, hvori problemerne stilles, og deres udtømmende behandling. En eiendommelighed er det ogsaa, at han i sin fremstilling kun anvender saa simple midler; af selve den vedvalgte problemstilling synes alt at flyde lige til.