Studforsk: Calculating the dispersion relation in Euler equations.

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Abstract

The system of Euler equations that describe the motion of a cross section of a flow within a background current cannot be solved ,in general, in an explicit form. However, the mentioned system of equations can be solved in a closed form for some functions for the background current and for some values for the free parameters contained in such equations.

1 Introduction and theoretical framework

The waves tipycally seen on the surface of the sea can be considered, approximately, periodic and two-dimensional. These waves can be described sufficiently by considering a cross-section of the flow that is perpendicular to the crest line. The velocity field of such flow is $(u(t,x,y), v(t,x,y))$ and homogeneity for these gravity waves is assumed.

This last assumption implies the equation of mass conservation:

$$u_x + v_y = 0$$

Another reasonable assumption for gravity waves is the inviscid flow so that equation of motion is given by Euler's equation:

$$u_t + uu_x + vu_y = -P_x \tag{1}$$

$$v_t + uv_x + vv_y = -P_y - g \tag{2}$$

Where $P(t,x,y)$ designates the pressure and g is the gravitational constant of acceleration.

The vorticity of the flow is given by:

$$v_x - u_y = \omega$$

Since the water cannot penetrate the rigid bed at y = 0, the kinematic boundary condition is given by $v = 0$ along y = 0.

Upon linearization [1] , the equations (1) and (2) become the following system of equations:

$$u_x + v_y = 0 \tag{3}$$

$$(U - c)u_x + vU_y = -p_x \tag{4}$$

$$(U - c)v_x = -\frac{p_y}{\delta^2} \tag{5}$$

Valid for $0 < y < 1$,and :

$$v = (U - c)\eta_x,$$

$$p = \eta$$

valid for y = 1.

Where δ is the shallowness parameter, U is the background current and η is the independent variable of the function for the water's free surface.

The solution and the dispersion relation to the system of equations above are known for the simplest class of vorticities i.e, when $\omega(y) = \omega$ is constant. The purpose of this project is thus to investigate the solution for such a system with more complicated functions for the vorticity.

2 Results

Looking for separable solutions we make use of the *ansatz* $\eta(x) = \cos(x)$. This operation yields the following set of expressions for $v(x, y)$, $u(x, y)$ and $p(x, y)$:

$$v(x, y) = \sin(x)\overline{V}(y)$$

$$p(x, y) = \cos(x)\overline{P}(y)$$

$$u(x, y) = \cos(x)\overline{U}(y)$$

From the equations above and equation (3), we get :

$$-\sin(x)\overline{U}(y) + \sin(x)\overline{V}'(y) = 0$$

,from where it follows the equality:

$$\overline{U}(y) = \overline{V}'(y) \tag{6}$$

Proceeding in a similar fashion for equation (4) yields the following expression for $\overline{P}(y)$:

$$-(U - c)\sin(x)\overline{U}(y) + U' \sin(x)\overline{V}(y) = \sin(x)\overline{P}(y)$$

$$\overline{P}(y) = -(U - c)\overline{U}(y) + \overline{V}(y)U'$$

Substituting the equality (6) into the expression above yields:

$$\overline{P}(y) = -(U - c)\overline{V}'(y) + U'\overline{V}(y) \tag{7}$$

For equation (5) the resulting equation is:

$$(U-c)\cos(x)\overline{V}(y) = \frac{-\cos(x)\overline{P}'(y)}{\delta^2}$$

$$\overline{P}'(y) = -\delta^2(U-c)\overline{V}(y) \tag{8}$$

Derivating equation (7) yields :

$$\overline{P}'(y) = -U\overline{V}''(y) + \overline{V}''(y)c + U''(y)\overline{V}(y)$$

Equating this last expression with equation (8) results into the following expression for $\overline{P}'(y)$:

$$\overline{P}'(y) = -U\overline{V}''(y) + \overline{V}''(y)c + U''\overline{V}(y) = -\delta^2(U-c)\overline{V}(y) \tag{9}$$

Taking all terms in the left to the right side and factoring out similar terms yields the following equation:

$$\overline{V}''(y) - \overline{V}(y)(\frac{U''}{U-c} + \delta^2) = 0 \tag{10}$$

It can be seen, by simple inspection, that for the linear case i.e., when $U = -wy + s$, equation (10) becomes independent of U and can be solved explicitly.This linear case has been studied before (as mentioned in the introduction) [1].

As for the general case(with more complicated functions for the background current U), this equation yields complicated solutions that need to be expressed in terms of hypergeometrical functions (for instance a Bessel function when $U(y) = y^2$) which do not allow explicit solutions and thus are of no interest.

However, setting the free parameter $\delta^2 = 1$,equation (10) becomes a simpler second-order differential equation which can be solved with certain functions for $U(y)$ that satisfy both this equation and the boundary conditions.Thus, we look for certain functions for the background current that allow a simpler differential equation and therefore a closed solution to it.

For both cases where U(y) = $\sinh(\lambda y)$ + c or U(y)= $\cosh(\lambda y)$ + c, equation (10) becomes:

$$\overline{V}''(y) - \overline{V}(y)(1+\lambda^2) = 0$$

The solution to the second-order differential equation above is then :

$$\overline{V}(y) = C_1 \sin\left(\sqrt{-\lambda^2-1}y\right) + C_2 \cos\left(\sqrt{-\lambda^2-1}y\right)$$

Evaluated at the boundary conditions for the case where U(y) = $\sinh(\lambda y)$, the expression for $\overline{V}(y)$ becomes:

$$\overline{V}(y)=\frac{-\sinh(\lambda)\sin\left(\sqrt{-\lambda^2-1}y\right)}{\sin\left(\sqrt{-\lambda^2-1}\right)}$$

From where it follows that the other two expressions are:

$$\overline{U}(y)=\frac{-\sqrt{\lambda^2+1}\sinh(\lambda)\cosh\left(\sqrt{\lambda^2+1}y\right)}{\sinh\left(\sqrt{\lambda^2+1}\right)}$$

$$\overline{P}(y)=\frac{\sinh(\lambda)\sinh(\lambda y)\cosh\left(\sqrt{\lambda^2+1}y\right)\sqrt{\lambda^2+1}}{\sinh\left(\sqrt{\lambda^2+1}\right)}-\frac{\sinh\left(\sqrt{\lambda^2+1}y\right)\sinh(\lambda)\lambda\cosh(\lambda y)}{\sinh\left(\sqrt{\lambda^2+1}\right)}$$

Thus the general solution for the case U(y) = $\sinh(\lambda y)$ + c is given by:

$$\begin{cases} v(x,y)=-C\sin(x)\sinh\left(\sqrt{\lambda^2+1}y\right) \\ u(x,y)=-C\cos(x)\sqrt{\lambda^2+1}\cosh\left(\sqrt{\lambda^2+1}y\right) \\ p(x,y)=C\cos(x)(\sinh(\lambda y)\cosh\left(\sqrt{\lambda^2+1}y\right)\sqrt{\lambda^2+1}-\sinh\left(\sqrt{\lambda^2+1}y\right)\lambda\cosh(\lambda y)) \end{cases}$$

where $C\equiv\frac{\sinh(\lambda)}{\sinh\left(\sqrt{\lambda^2+1}\right)}$ and λ must satisfy:

$$\sinh(\lambda)^2\coth\left(\sqrt{\lambda^2+1}\right)(\sqrt{\lambda^2+1})-\sinh(\lambda)\lambda\cosh(\lambda)=1$$

The solutions for the other case U(y) = $\cosh(\lambda y)$ + c is somewhat similar but in this case $C\equiv\frac{\cosh(\lambda)}{\sinh\left(\sqrt{\lambda^2+1}\right)}$ and the parameter λ must satisfy:

$$\cosh(\lambda)^2\coth\left(\sqrt{\lambda^2+1}\right)\sqrt{\lambda^2+1}-\sinh(\lambda)\lambda\cosh(\lambda)=1$$

For the cases where U(y) = $\sin(\lambda y)$ + c or U(y)= $\cos(\lambda y)$ + c, equation (10) becomes:

$$\overline{V}''(y)-\overline{V}(y)(1-\lambda^2)=0 \tag{11}$$

It needs to be taken into consideration the fact that the relative value of lambda (λ) to the unit leads to different equations and thus solutions. Therefore, one looks separately at each case, solves the pertinent equation and evaluates at the boundary conditions first in the case where U(y) = $\sin(\lambda y)$ + c and then when U(y)= $\cos(\lambda y)$ + c.

Case 1: $\lambda^2 >> 1$.

Equation (11) becomes $\overline{V}''(y) + \overline{V}(y)(\lambda^2) = 0$ and the solution to this particular equation is given by: $\overline{V}(y) = C_1 \sin(\lambda y) + C_2 \cos(\lambda y)$. Evaluating this solution at the boundary conditions results into:

$$\overline{V}(y) = -\sin(\lambda y)$$

$$\overline{U}(y) = -\lambda \cos(\lambda y)$$

$$\overline{P}(y) = \lambda \sin(\lambda y)\cos(\lambda y) - \lambda \cos(\lambda y)\sin(\lambda y) = 0$$

These results yield then the following expressions:

$$\begin{cases} v(x,y) = -\sin(x)\sin(\lambda y) \\ u(x,y) = -\lambda \cos(x)\cos(\lambda y) \\ p(x,y) = 0 \end{cases}$$

However, when $U(y) = \cos(\lambda y) + c$:

$$\begin{cases} v(x,y) = -\sin(x)\cot(\lambda)\sin(\lambda y) \\ u(x,y) = -\lambda \cos(x)\cos(\lambda y)\cot(\lambda) \\ p(x,y) = \cos(x)\lambda\cot(\lambda) \end{cases}$$

Case 2: $\lambda^2 = 1$.

Equation (11) becomes then $\overline{V}''(y) = 0$. The solution to this equation is given by: $\overline{V}(y) = C_1 y + C_2$. Evaluating the latter equation at the boundary conditions yield $\overline{V}(y) = -\sin(\lambda)y$.

This last expression yields the constant function $\overline{U}(y) = -\sin(\lambda)$ and therefore:

$$\overline{P}(y) = \sin(\lambda y)\sin(\lambda) - \sin(\lambda)y\cos(\lambda y)\lambda$$

Then the expressions for $v(x,y), u(x,y)$ and $p(x,y)$ are given by:

$$\begin{cases} v(x,y) = -C\sin(x)y \\ u(x,y) = -C\cos(x) \\ p(x,y) = \cos(x)C(\sin(\lambda y) - y\lambda\cos(\lambda y)) \end{cases}$$

Where $C \equiv \sin(\lambda)$ and λ must satisfy:

$$\sin(\lambda)(\sin(\lambda) - \lambda\cos(\lambda)) = 1$$

When the background current U is $U(y) = \cos(\lambda y) + c$, the expressions become:

$$\begin{cases} v(x,y) = -C\sin(x)y \\ u(x,y) = -C\cos(x) \\ p(x,y) = \cos(x)C(\cos(\lambda y) + y\lambda\sin(\lambda y)) \end{cases}$$

Where $C \equiv \cos(\lambda)$ and λ must satisfy:

$$\cos(\lambda)^2 + \cos(\lambda)\sin(\lambda)\lambda = 1$$

Case 3: $\lambda^2 << 1$.
Equation (11) becomes in this case:

$$\overline{V}''(y) - \overline{V}(y) = 0$$

The general solution to the equation above reads then :

$$\overline{V}(y) = C_1\exp(-y) + C_2\exp(y)$$

Evaluating the general solution above at the boundary conditions yields the following expressions :

$$\overline{V}(y) = \frac{\sinh(\lambda y)\sin(\lambda)}{\sinh(\lambda)}$$

$$\overline{U}(y) = \lambda\frac{\cosh(\lambda y)\sin(\lambda)}{\sinh(\lambda)}$$

$$\overline{P}(y) = -\sin(\lambda y)\frac{\sin(\lambda)\lambda\cosh(\lambda y)}{\sinh(\lambda)} + \frac{\lambda\sin(\lambda)\sinh(\lambda y)\cos(\lambda y)}{\sinh(\lambda)}$$

The solution for this case is:

$$\begin{cases} v(x,y) = C\sin(x)\sinh(\lambda y) \\ u(x,y) = C\cos(x)\lambda\cosh(\lambda y) \\ p(x,y) = \cos(x)C\lambda(-\sin(\lambda y)\cosh(\lambda y) + \sinh(\lambda y)\cos(\lambda y)) \end{cases}$$

Where $C \equiv \frac{\sin(\lambda)}{\sinh(\lambda)}$ and λ must satisfy:

$$\lambda\sin(\lambda)(\cos(\lambda) - \sin(\lambda)\coth(\lambda)) = 1$$

For the choice $U(y) = \cos(\lambda y)$ the set of expressions is given by:

$$\begin{cases} v(x,y) = C\sin(x)\sinh(\lambda y) \\ u(x,y) = C\cos(x)\lambda\cosh(\lambda y) \\ p(x,y) = -C\cos(x)(\cos(\lambda y)\lambda\cosh(\lambda y) + \sin(\lambda y)\lambda\sinh(\lambda y)) \end{cases}$$

Where $C \equiv \frac{\cos(\lambda)}{\sinh(\lambda)}$ and λ must satisfy:

$$-\lambda\cos(\lambda)^2\coth(\lambda) - \sin(\lambda)\lambda\cos(\lambda) = 1$$

3 Conclusion

The system of Euler equations in question cannot be solved in general in an explicit or closed form. Four cases yielded explicit solutions and these were when the background current U function had the values: $U(y) = \sinh(\lambda y)$, $U(y) = \cosh(\lambda y)$, $U(y) = \cos(\lambda y)$ and $U(y) = \sin(\lambda y)$.

References

[1] M.Ehrnström and G.Villari. "**Water waves with vorticity: Rotational features and particle paths**". In: *J. Differential Equations* 244 (2008), pp. 1888–1909.