

# THE MORPHISM AXIOM FOR $n$ -ANGULATED CATEGORIES

EMILIE ARENTZ-HANSEN, PETTER ANDREAS BERGH, AND MARIUS THAULE

ABSTRACT. We show that the morphism axiom for  $n$ -angulated categories is redundant.

## 1. INTRODUCTION

In [3], Geiss, Keller and Oppermann introduced  $n$ -angulated categories. These are generalizations of triangulated categories, in the sense that triangles are replaced by  $n$ -angles, that is, morphism sequences of length  $n$ . Thus a 3-angulated category is precisely a triangulated category.

There are by now numerous examples of such generalized triangulated categories. They appear for example as certain cluster tilting subcategories of triangulated categories. Recently, in [4], Jasso introduced the notion of *algebraic*  $n$ -angulated categories. Such a category is by definition equivalent to a stable category of a Frobenius  $n$ -exact category, the latter being a generalization of an exact category; exact sequences are replaced by so-called  $n$ -exact sequences. Algebraic  $n$ -angulated categories are therefore natural generalizations of algebraic triangulated categories, which are defined as categories equivalent to stable categories of Frobenius exact categories. By [4, Section 6.5], the  $n$ -angulated categories that arise as cluster tilted subcategories of algebraic triangulated categories are themselves algebraic. Actually, the only examples so far of  $n$ -angulated categories that are not algebraic are the rather exotic ones in [1]. At the time of writing, there is no notion of *topological*  $n$ -angulated categories, except in the triangulated case.

The axioms defining  $n$ -angulated categories are generalized versions of the axioms defining triangulated categories. Among these axioms, the morphism axiom and the generalized octahedral axiom are the ones that guarantee an interesting theory. Apart from the existence axiom, which states that every morphism is part of an  $n$ -angle, the other axioms are “bookkeeping axioms,” to borrow Paul Balmer’s terminology. The generalized octahedral axiom was introduced in [2], but in this paper we present it in a much more compact and readable form.

The topic of this paper is the morphism axiom, which states that a morphism between the bases of two  $n$ -angles can be extended to a morphism of  $n$ -angles. We show that this axiom is redundant; it follows from the generalized octahedral axiom and the existence axiom. For triangulated categories, this was proved by May in [5].

## 2. THE AXIOMS FOR $n$ -ANGULATED CATEGORIES

In this section, we recall the axioms for  $n$ -angulated categories. We fix an additive category  $\mathcal{C}$  with an automorphism  $\Sigma: \mathcal{C} \rightarrow \mathcal{C}$ , and an integer  $n$  greater than or equal to three.

A sequence of objects and morphisms in  $\mathcal{C}$  of the form

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} \Sigma A_1$$

---

*Date:* January 27, 2016.

*2010 Mathematics Subject Classification.* 18E30.

*Key words and phrases.* Triangulated categories,  $n$ -angulated categories, morphism axiom.

is called an  $n$ - $\Sigma$ -sequence. Whenever convenient, we shall denote such sequences by  $A_\bullet, B_\bullet$  etc. The left and right rotations of  $A_\bullet$  are the two  $n$ - $\Sigma$ -sequences

$$A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_n} \Sigma A_1 \xrightarrow{(-1)^n \Sigma \alpha_1} \Sigma A_2$$

and

$$\Sigma^{-1} A_n \xrightarrow{(-1)^n \Sigma^{-1} \alpha_n} A_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-2}} A_{n-1} \xrightarrow{\alpha_{n-1}} A_n$$

respectively, and a *trivial*  $n$ - $\Sigma$ -sequence is a sequence of the form

$$A \xrightarrow{1} A \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow \Sigma A$$

or any of its rotations.

A *morphism*  $A_\bullet \xrightarrow{\varphi} B_\bullet$  of  $n$ - $\Sigma$ -sequences is a sequence  $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$  of morphisms in  $\mathcal{C}$  such that the diagram

$$\begin{array}{ccccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & \dots & \xrightarrow{\alpha_{n-1}} & A_n & \xrightarrow{\alpha_n} & \Sigma A_1 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & & & \downarrow \varphi_n & & \downarrow \Sigma \varphi_1 \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & \dots & \xrightarrow{\beta_{n-1}} & B_n & \xrightarrow{\beta_n} & \Sigma B_1 \end{array}$$

commutes. It is an *isomorphism* if  $\varphi_1, \varphi_2, \dots, \varphi_n$  are all isomorphisms in  $\mathcal{C}$ , and a *weak isomorphism* if  $\varphi_i$  and  $\varphi_{i+1}$  are isomorphisms for some  $1 \leq i \leq n$  (with  $\varphi_{n+1} := \Sigma \varphi_1$ ). Note that the composition of two weak isomorphisms need not be a weak isomorphism. Also, note that if two  $n$ - $\Sigma$ -sequences  $A_\bullet$  and  $B_\bullet$  are weakly isomorphic through a weak isomorphism  $A_\bullet \xrightarrow{\varphi} B_\bullet$ , then there does not necessarily exist a weak isomorphism  $B_\bullet \rightarrow A_\bullet$  in the opposite direction.

Let  $\mathcal{N}$  be a collection of  $n$ - $\Sigma$ -sequences in  $\mathcal{C}$ . Then the triple  $(\mathcal{C}, \Sigma, \mathcal{N})$  is an *n-angulated category* if the following four axioms are satisfied:

- (N1) (a)  $\mathcal{N}$  is closed under direct sums, direct summands and isomorphisms of  $n$ - $\Sigma$ -sequences.  
(b) For all  $A \in \mathcal{C}$ , the trivial  $n$ - $\Sigma$ -sequence

$$A \xrightarrow{1} A \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow \Sigma A$$

belongs to  $\mathcal{N}$ .

- (c) For each morphism  $\alpha: A_1 \rightarrow A_2$  in  $\mathcal{C}$ , there exists an  $n$ - $\Sigma$ -sequence in  $\mathcal{N}$  whose first morphism is  $\alpha$ .

- (N2) An  $n$ - $\Sigma$ -sequence belongs to  $\mathcal{N}$  if and only if its left rotation belongs to  $\mathcal{N}$ .

- (N3) Given the solid part of the commutative diagram

$$\begin{array}{ccccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & \dots & \xrightarrow{\alpha_{n-1}} & A_n & \xrightarrow{\alpha_n} & \Sigma A_1 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & & & \downarrow \varphi_n & & \downarrow \Sigma \varphi_1 \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & \dots & \xrightarrow{\beta_{n-1}} & B_n & \xrightarrow{\beta_n} & \Sigma B_1 \end{array}$$

with rows in  $\mathcal{N}$ , the dotted morphisms exist and give a morphism of  $n$ - $\Sigma$ -sequences.

- (N4) Given the solid part of the diagram

$$\begin{array}{ccccccccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & \dots & \xrightarrow{\alpha_{n-2}} & A_{n-1} & \xrightarrow{\alpha_{n-1}} & A_n & \xrightarrow{\alpha_n} & \Sigma A_1 \\ \parallel & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_4 & & & & \downarrow \varphi_{n-1} & & \downarrow \varphi_n & & \parallel \\ A_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & \dots & \xrightarrow{\beta_{n-2}} & B_{n-1} & \xrightarrow{\beta_{n-1}} & B_n & \xrightarrow{\beta_n} & \Sigma A_1 \\ \downarrow \alpha_1 & & \parallel & & \downarrow \theta_3 & & \downarrow \theta_4 & & & & \downarrow \theta_{n-1} & & \downarrow \theta_n & & \downarrow \Sigma \alpha_1 \\ A_2 & \xrightarrow{\varphi_2} & B_2 & \xrightarrow{\gamma_2} & C_3 & \xrightarrow{\gamma_3} & C_4 & \xrightarrow{\gamma_4} & \dots & \xrightarrow{\gamma_{n-2}} & C_{n-1} & \xrightarrow{\gamma_{n-1}} & C_n & \xrightarrow{\gamma_n} & \Sigma A_2 \end{array}$$

with commuting squares and with rows in  $\mathcal{N}$ , the dotted morphisms exist such that each square commutes, and the  $n$ - $\Sigma$ -sequence

$$\begin{array}{c}
A_3 \xrightarrow{\begin{bmatrix} \alpha_3 \\ \varphi_3 \end{bmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{bmatrix} -\alpha_4 & 0 \\ \varphi_4 & -\beta_3 \\ \psi_4 & \theta_3 \end{bmatrix}} A_5 \oplus B_4 \oplus C_3 \xrightarrow{\begin{bmatrix} -\alpha_5 & 0 & 0 \\ -\varphi_5 & -\beta_4 & 0 \\ \psi_5 & \theta_4 & \gamma_3 \end{bmatrix}} \\
\begin{bmatrix} -\alpha_5 & 0 & 0 \\ -\varphi_5 & -\beta_4 & 0 \\ \psi_5 & \theta_4 & \gamma_3 \end{bmatrix} \xrightarrow{\quad} A_6 \oplus B_5 \oplus C_4 \xrightarrow{\begin{bmatrix} -\alpha_6 & 0 & 0 \\ \varphi_6 & -\beta_5 & 0 \\ \psi_6 & \theta_5 & \gamma_4 \end{bmatrix}} \dots \\
\dots \xrightarrow{\begin{bmatrix} -\alpha_{n-1} & 0 & 0 \\ (-1)^{n-1}\varphi_{n-1} & -\beta_{n-2} & 0 \\ \psi_{n-1} & \theta_{n-2} & \gamma_{n-3} \end{bmatrix}} A_n \oplus B_{n-1} \oplus C_{n-2} \xrightarrow{\begin{bmatrix} (-1)^n\varphi_n & -\beta_{n-1} & 0 \\ \psi_n & \theta_{n-1} & \gamma_{n-2} \end{bmatrix}} \\
\begin{bmatrix} (-1)^n\varphi_n & -\beta_{n-1} & 0 \\ \psi_n & \theta_{n-1} & \gamma_{n-2} \end{bmatrix} \xrightarrow{\quad} B_n \oplus C_{n-1} \xrightarrow{[\theta_n \ \gamma_{n-1}]} C_n \xrightarrow{\Sigma\alpha_2\circ\gamma_n} \Sigma A_3
\end{array}$$

belongs to  $\mathcal{N}$ .

In this case, the collection  $\mathcal{N}$  is an  $n$ -angulation of the category  $\mathcal{C}$  (relative to the automorphism  $\Sigma$ ), and the  $n$ - $\Sigma$ -sequences in  $\mathcal{N}$  are  $n$ -angles. If  $(\mathcal{C}, \Sigma, \mathcal{N})$  satisfies (N1), (N2) and (N3), then  $\mathcal{N}$  is a *pre- $n$ -angulation*, and the triple  $(\mathcal{C}, \Sigma, \mathcal{N})$  is a *pre- $n$ -angulated category*.

**Remark 2.1.** (1) In [3], the fourth defining axiom is the “mapping cone axiom”, which says that in the situation of (N3), the morphisms  $\varphi_3, \dots, \varphi_n$  can be chosen in such a way that the mapping cone of  $(\varphi_1, \varphi_2, \dots, \varphi_n)$  belongs to  $\mathcal{N}$ . When  $n = 3$ , that is, in the triangulated case, it was shown by Neeman in [6, 7] that this mapping cone axiom is equivalent to the octahedral axiom, when the other three axioms hold.

When  $n = 3$ , the diagram in our axiom (N4) is precisely the octahedral diagram, and the axiom is the octahedral axiom. Thus (N4) can be thought of as a higher analogue of the octahedral axiom. It was shown in [2] that when (N1), (N2) and (N3) hold, then (N4) and the mapping cone axioms are equivalent.<sup>1</sup> Thus our definition of an  $n$ -angulated category is equivalent to that of Geiss, Keller and Oppermann in [3].

(2) Let  $B$  be an object in  $\mathcal{C}$ . Any  $n$ - $\Sigma$ -sequence  $A_\bullet$  induces an infinite sequence

$$\dots \rightarrow (B, \Sigma^{-1}A_n) \xrightarrow{(\Sigma^{-1}\alpha_n)_*} (B, A_1) \xrightarrow{(\alpha_1)_*} (B, A_2) \rightarrow \dots \rightarrow (B, A_n) \xrightarrow{(\alpha_n)_*} (B, \Sigma A_1) \rightarrow \dots$$

of abelian groups and homomorphisms, where  $(B, A)$  denotes  $\text{Hom}_{\mathcal{C}}(B, A)$ . If this sequence is exact for all  $B$ , then  $A_\bullet$  is called an *exact  $n$ - $\Sigma$ -sequence*. By [3, Proposition 2.5], every  $n$ -angle in a pre- $n$ -angulated category is exact.

Now modify axiom (N1) into a new axiom (N1\*), whose parts (b) and (c) are unchanged, but with the following part (a): if  $A_\bullet \rightarrow B_\bullet$  is a weak isomorphism of exact  $n$ - $\Sigma$ -sequences with  $A_\bullet \in \mathcal{N}$ , then  $B_\bullet$  belongs to  $\mathcal{N}$ . Moreover, let (N2\*) be the following weak version of axiom (N2): the left rotation of every  $n$ - $\Sigma$ -sequence in  $\mathcal{N}$  also belongs to  $\mathcal{N}$ . Then by [2, Theorem 3.4], the following are equivalent:

- (i)  $(\mathcal{C}, \Sigma, \mathcal{N})$  satisfies (N1), (N2), (N3),
- (ii)  $(\mathcal{C}, \Sigma, \mathcal{N})$  satisfies (N1\*), (N2), (N3),
- (iii)  $(\mathcal{C}, \Sigma, \mathcal{N})$  satisfies (N1\*), (N2\*), (N3).

It is not known whether only axiom (N2) can be replaced by (N2\*), without replacing (N1) by (N1\*).

<sup>1</sup>We thank Gustavo Jasso for showing us the compact diagram we have used in axiom (N4). This axiom is not strictly the same as axiom (N4\*) in [2]. However, it follows from the proofs in [2, Section 4] that the two are equivalent.

## 3. AXIOM (N3) IS REDUNDANT

In this section we show that the morphism axiom, i.e. axiom (N3), follows from the other axioms. In light of Remark 2.1(2), it would perhaps seem natural to ask which axioms one should use to prove this. In other words, should one use axioms (N1), (N2), (N4), axioms (N1\*), (N2), (N4), or axioms (N1\*), (N2\*), (N4)? The answer is that it does not matter. In fact, axiom (N3) is a consequence of (N1)(c) and (N4).

**Theorem 3.1.** *Axiom (N3) follows from (N1)(c) and (N4).*

*Proof.* Assume that we are in the situation of axiom (N3), i.e. that we are given the solid part of the commutative diagram

$$\begin{array}{ccccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-1}} & A_n & \xrightarrow{\alpha_n} & \Sigma A_1 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & & & \downarrow \varphi_n & & \downarrow \Sigma \varphi_1 \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{n-1}} & B_n & \xrightarrow{\beta_n} & \Sigma B_1 \end{array}$$

with rows in  $\mathcal{N}$ . We must prove that dotted morphisms  $\varphi_3, \dots, \varphi_n$  exist and give a morphism  $(\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n)$  of  $n$ - $\Sigma$ -sequences.

By assumption, the equality  $\varphi_2 \circ \alpha_1 = \beta_1 \circ \varphi_1$  holds; we define  $\gamma_1$  to be this morphism, i.e.  $\gamma_1 = \varphi_2 \circ \alpha_1 = \beta_1 \circ \varphi_1$ . Now apply axiom (N1)(c) to the three morphisms

$$\gamma_1: A_1 \rightarrow B_2, \quad \varphi_2: A_2 \rightarrow B_2, \quad \varphi_1: A_1 \rightarrow B_1$$

in  $\mathcal{C}$ , and obtain three  $n$ - $\Sigma$ -sequences

$$\begin{array}{l} A_1 \xrightarrow{\gamma_1} B_2 \xrightarrow{\gamma_2} C_3 \xrightarrow{\gamma_3} \cdots \xrightarrow{\gamma_{n-1}} C_n \xrightarrow{\gamma_n} \Sigma A_1, \\ A_2 \xrightarrow{\varphi_2} B_2 \xrightarrow{\delta_2} D_3 \xrightarrow{\delta_3} \cdots \xrightarrow{\delta_{n-1}} D_n \xrightarrow{\delta_n} \Sigma A_2 \end{array}$$

and

$$A_1 \xrightarrow{\varphi_1} B_1 \xrightarrow{\epsilon_2} E_3 \xrightarrow{\epsilon_3} \cdots \xrightarrow{\epsilon_{n-1}} E_n \xrightarrow{\epsilon_n} \Sigma A_1$$

in  $\mathcal{N}$ . Next, consider the two diagrams

$$\begin{array}{ccccccccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & \cdots & \xrightarrow{\alpha_{n-2}} & A_{n-1} & \xrightarrow{\alpha_{n-1}} & A_n & \xrightarrow{\alpha_n} & \Sigma A_1 \\ \parallel & & \downarrow \varphi_2 & & \downarrow \rho_3 & & \downarrow \rho_4 & & & & \downarrow \rho_{n-1} & & \downarrow \rho_n & & \parallel \\ A_1 & \xrightarrow{\gamma_1} & B_2 & \xrightarrow{\gamma_2} & C_3 & \xrightarrow{\gamma_3} & C_4 & \xrightarrow{\gamma_4} & \cdots & \xrightarrow{\gamma_{n-2}} & C_{n-1} & \xrightarrow{\gamma_{n-1}} & C_n & \xrightarrow{\gamma_n} & \Sigma A_1 \\ \downarrow \alpha_1 & & \parallel & & \downarrow \theta_3 & & \downarrow \theta_4 & & & & \downarrow \theta_{n-1} & & \downarrow \theta_n & & \downarrow \Sigma \alpha_1 \\ A_2 & \xrightarrow{\varphi_2} & B_2 & \xrightarrow{\delta_2} & D_3 & \xrightarrow{\delta_3} & D_4 & \xrightarrow{\delta_4} & \cdots & \xrightarrow{\delta_{n-2}} & D_{n-1} & \xrightarrow{\delta_{n-1}} & D_n & \xrightarrow{\delta_n} & \Sigma A_2 \end{array}$$

and

$$\begin{array}{ccccccccccccccc} A_1 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{\epsilon_2} & E_3 & \xrightarrow{\epsilon_3} & E_4 & \xrightarrow{\epsilon_4} & \cdots & \xrightarrow{\epsilon_{n-2}} & E_{n-1} & \xrightarrow{\epsilon_{n-1}} & E_n & \xrightarrow{\epsilon_n} & \Sigma A_1 \\ \parallel & & \downarrow \beta_1 & & \downarrow \psi_3 & & \downarrow \psi_4 & & & & \downarrow \psi_{n-1} & & \downarrow \psi_n & & \parallel \\ A_1 & \xrightarrow{\gamma_1} & B_2 & \xrightarrow{\gamma_2} & C_3 & \xrightarrow{\gamma_3} & C_4 & \xrightarrow{\gamma_4} & \cdots & \xrightarrow{\gamma_{n-2}} & C_{n-1} & \xrightarrow{\gamma_{n-1}} & C_n & \xrightarrow{\gamma_n} & \Sigma A_1 \\ \downarrow \varphi_1 & & \parallel & & \downarrow \eta_3 & & \downarrow \eta_4 & & & & \downarrow \eta_{n-1} & & \downarrow \eta_n & & \downarrow \Sigma \varphi_1 \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & \cdots & \xrightarrow{\beta_{n-2}} & B_{n-1} & \xrightarrow{\beta_{n-1}} & B_n & \xrightarrow{\beta_n} & \Sigma B_1 \end{array}$$

Note that all the rows in these two diagrams are  $n$ - $\Sigma$ -sequences in  $\mathcal{N}$ . Moreover, it follows from the definition of the morphism  $\gamma_1$  that the solid parts of the diagrams commute. We may therefore apply axiom (N4); there exist dotted morphisms such that each

square commutes. Axiom (N4) also gives other morphisms, the diagonal ones, but these are not needed here.

The two diagrams share the same middle row. We may therefore combine the upper part of the first diagram and the lower part of the second, and obtain the commutative diagram

$$\begin{array}{cccccccccccccccc}
 A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & \cdots & \xrightarrow{\alpha_{n-2}} & A_{n-1} & \xrightarrow{\alpha_{n-1}} & A_n & \xrightarrow{\alpha_n} & \Sigma A_1 \\
 \parallel & & \downarrow \varphi_2 & & \downarrow \rho_3 & & \downarrow \rho_4 & & & & \downarrow \rho_{n-1} & & \downarrow \rho_n & & \parallel \\
 A_1 & \xrightarrow{\gamma_1} & B_2 & \xrightarrow{\gamma_2} & C_3 & \xrightarrow{\gamma_3} & C_4 & \xrightarrow{\gamma_4} & \cdots & \xrightarrow{\gamma_{n-2}} & C_{n-1} & \xrightarrow{\gamma_{n-1}} & C_n & \xrightarrow{\gamma_n} & \Sigma A_1 \\
 \downarrow \varphi_1 & & \parallel & & \downarrow \eta_3 & & \downarrow \eta_4 & & & & \downarrow \eta_{n-1} & & \downarrow \eta_n & & \downarrow \Sigma \varphi_1 \\
 B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & \cdots & \xrightarrow{\beta_{n-2}} & B_{n-1} & \xrightarrow{\beta_{n-1}} & B_n & \xrightarrow{\beta_n} & \Sigma B_1
 \end{array}$$

Finally, by omitting the middle row, we obtain the commutative diagram

$$\begin{array}{cccccccccccccccc}
 A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & \cdots & \xrightarrow{\alpha_{n-2}} & A_{n-1} & \xrightarrow{\alpha_{n-1}} & A_n & \xrightarrow{\alpha_n} & \Sigma A_1 \\
 \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \eta_3 \circ \rho_3 & & \downarrow \eta_4 \circ \rho_4 & & & & \downarrow \eta_{n-1} \circ \rho_{n-1} & & \downarrow \eta_n \circ \rho_n & & \downarrow \Sigma \varphi_1 \\
 B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & \cdots & \xrightarrow{\beta_{n-2}} & B_{n-1} & \xrightarrow{\beta_{n-1}} & B_n & \xrightarrow{\beta_n} & \Sigma B_1
 \end{array}$$

Now for each  $3 \leq k \leq n$ , define a morphism  $\varphi_k$  by  $\varphi_k = \eta_k \circ \rho_k$ . Then  $(\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n)$  is a morphism of  $n$ - $\Sigma$ -sequences, and this completes the proof.  $\square$

**Remark 3.2.** (1) As noted in the proof, not all of axiom (N4) is needed. The diagonal morphisms provided by the axiom play no role, and, consequently, neither does the  $n$ - $\Sigma$ -sequence involving these morphisms.

(2) Note that if  $n = 3$ , that is, in the triangulated case, the result recovers that of May in [5], but with a slightly different setup. Indeed, May's result was the inspiration for the theorem.

## REFERENCES

- [1] P.A. Bergh, G. Jasso, M. Thaule, *Higher  $n$ -angulations from local rings*, J. London Math. Soc. **93** (2016), no. 1, 123–142.
- [2] P.A. Bergh, M. Thaule, *The axioms for  $n$ -angulated categories*, Algebr. Geom. Topol. **13** (2013), no. 4, 2405–2428.
- [3] C. Geiss, B. Keller, S. Oppermann,  *$n$ -angulated categories*, J. Reine Angew. Math. **675** (2013), 101–120.
- [4] G. Jasso,  *$n$ -abelian and  $n$ -exact categories*, to appear in Math. Z.
- [5] J.P. May, *The additivity of traces in triangulated categories*, Adv. Math. **163** (2001), no. 1, 34–73.
- [6] A. Neeman, *Some new axioms for triangulated categories*, J. Algebra **139** (1991), no. 1, 221–255.
- [7] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies, **148**, Princeton University Press, Princeton, NJ, 2001.

DEPARTMENT OF MATHEMATICAL SCIENCES, NTNU, NO-7491 TRONDHEIM, NORWAY

*E-mail address:* emilieba@stud.ntnu.no

*E-mail address:* bergh@math.ntnu.no

*E-mail address:* mariusth@math.ntnu.no