DERIVATION RELATIONS AND DUALITY FOR MULTIPLE ZETA VALUES

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A *multiple zeta value* (MZV) of length k and integer weight $s = s_1 + \cdots + s_k$ is a real number defined by a k-fold iterated infinite series

$$\zeta(s_1, \ldots, s_k) := \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}. \tag{1}$$

This sum is convergent for $s_1 > 1$. The last condition can be dropped if *multiple polylogarithms* (MPL) are considered, which are multivariable functions defined for $s_i > 0$ and $|z_i| < 1$, $1 \leq i \leq k$

$$\mathsf{Li}_{s_1,\ldots,s_k}(z_1, \ldots, z_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}.$$

MZVs can be seen as special values of MPLs, i.e. $\zeta(s_1, \ldots, s_k) = \mathsf{Li}_{s_1,\ldots,s_k}(1, \ldots, 1)$. A systematic study of MZVs and MPLs, started in the early 1990s with the works of Hoffman and Zagier [4, 9]. The prehistory of these numbers can be traced back to L. Euler in the 18th century. See references [2, 10, 13] for reviews. MZVs and MPLs appear in several contexts, including number theory, algebra, combinatorics, algebraic geometry, mathematical physics, and computer sciences. On Hoffman's website[1] a comprehensive list of references can be found.

Kontsevich discovered a representation of MZVs in terms of iterated integrals with respect to the 1-forms $\omega_j\{1\} := \frac{dt_j}{1-t_j}$, $\omega_i\{0\} := \frac{dt_i}{t_i}$. Following the standard notation, where $s^{(i)} := s_1 + \cdots + s_i$ one can show that

$$\zeta(s_1, \ldots, s_k) = \int_0^1 \Big(\prod_{j=1}^{s^{(1)}-1} \omega_j\{0\} \Big) \omega_{s_1}\{1\} \cdots \Big(\prod_{j=s^{(k-1)}+1}^{s^{(k)}-1} \omega_j\{0\} \Big) \omega_s\{1\}. \tag{2}$$

It turns out that the space generated by MZVs is an algebra over $\mathbb{Q}$, since products of MZVs can be written as linear combinations of MZVs of the same weight. In fact, the two representations of MZVs in terms of iterated sums (1) and integrals (2) imply algebraic relations among MZVs. One uses the name *stuffle* (or *quasi-shuffle*) for the relations arising from the sum representation of MZVs, and *shuffle* for those arising from the integral representation of MZVs. The following simple example may elucidate this

$$\zeta(2)\zeta(2) = \sum_{n>0} \frac{1}{n^2} \sum_{m>0} \frac{1}{m^2} = \sum_{m>n>0} \frac{1}{n^2 m^2} + \sum_{n>m>0} \frac{1}{n^2 m^2} + \sum_{n>0} \frac{1}{n^4} = 2\zeta(2,2) + \zeta(4).$$

At the same time

$$\zeta(2)\zeta(2) \;=\; \int_0^1 \frac{dt}{t} \int_0^t \frac{ds}{1-s} \int_0^1 \frac{du}{u} \int_0^t \frac{dv}{1-v} = 4 \int_0^1 \frac{dt}{t} \int_0^t \frac{ds}{s} \int_0^s \frac{du}{1-u} \int_0^u \frac{dv}{1-v}$$

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[1] References on multiple zeta values and euler sums (http://www.usna.edu/Users/math/meh/biblio.html)

$$+2\int_0^1 \frac{dt}{t}\int_0^t \frac{ds}{1-s}\int_0^s \frac{du}{u}\int_0^u \frac{dv}{1-v} = 4\zeta(3,1)+2\zeta(2,2).$$

Hence, the product of two MZVs yields two different linear combinations of MZVs. Their equality in turn produces many relations among MZVs, which are known as *double shuffle relations*. For instance, comparing $\zeta(2)\zeta(2)$ as shuffle and quasi-shuffle product implies the relation $\zeta(4) = 4\zeta(3,1)$. Conjecturally, all algebraic relations among MZVs can be obtained from double shuffle structure – modulo an extension via a regularization procedure.

It turns out that a particular change of variables in the context of the integral representation (2) of MZVs gives way to a rather peculiar set of relations subsumed under the notion of *duality* for MZVs. As an example we state the well-known identity $\zeta(3) = \zeta(2,1)$ that was discovered by Euler. Another identity that follows from duality is $\zeta(5,1) = \zeta(3,1,1,1)$. See e.g. [10, Theorem 7]. A precise understanding of the mathematical relation between the notion of duality and the aforementioned double shuffle structure is part of a class of important open problems in the theory of MZVs. In the seminal paper [5], among other results, the so-called *derivation relations* for MZVs where shown to come from – regularized – double shuffle relations. Recently, several interesting results relating derivation relations and duality for MZVs came out. See e.g. [7, 8].

Generalizations of real-valued MZVs to power series in $\mathbb{Q}[[q]]$ are commonly known as q-analogues of MZVs. Several q-analogues of MZVs were shown to satisfy – regularized – double shuffle relations. The notion of duality in the context of those q-MZVs is both interesting and illuminating. See e.g. [1, 3, 11] for new results. In [6, 12] a detailed review is presented.

In [11] Zudilin presented *multiple q-zeta brackets*, which possess a natural quasi-shuffle product. The key result in [11] is an algebraic duality-type construction that permits to deduce a shuffle product for multiple q-zeta brackets from the quasi-shuffle product of the model. In a nutshell, this construction works as follows: Let (A, m) be an algebra, and $\zeta\colon (A, m) \to k$ is a multiplicative linear map from A into the ring k. The map $\tau\colon A \to A$ is a particular involution, i.e., $\tau \circ \tau = id$. Both maps ζ and τ are compatible in the sense that $\zeta \circ \tau = \zeta$. Then the *dual product* $m_\square : A \otimes A \to A$ corresponding to the original product m on A is defined by

$$m_\square := \tau \circ m \circ (\tau \otimes \tau). \tag{3}$$

It turns out that $\zeta\colon (A, m_\square) \to k$ is again a multiplicative linear map into k. In the case of multiple q-zeta brackets Zudilin proved that if (A, m) is the quasi-shuffle algebra induced by a sum representation of his model, then the dual product $m_\square$ yields a shuffle product on multiple q-zeta brackets. The reason for calling it shuffle product comes from the surprising fact that, in the limit q goes to one, the product $m_\square$ reduces to the usual shuffle product of classical MZVs. In [3] it is shown how to relate Zudilin's dual product via the construction of a Hoffman–Ohno relation to a derivation relation for MZVs.

THE AIM of this project is to further explore the aforementioned result in [3] in the context of other q-analogues of MZVs as well as classical MZVs. Beyond this aim lies the wish to develop a deeper understanding of the notion of duality for both q-MZVs as well as classical MZVs.

Prerequisites: *interest in learning about the fascinating interplay between number theory, abstract algebra, and combinatorics. The key references are* [5, 10, 11].

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