Second order approximation to particle trajectories in linear water waves subject to weak currents

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Abstract

This project concerns particle trajectories in linear periodic gravity water waves. We propose a solution to the second order approximation of the particle trajectories subject to a weak background current. The accuracy of the solution is unknown.

1 Summary of the project

1.1 Description

Linear water waves have small amplitudes, which makes it possible to linearize the Euler equations of the problem. To the first order in the small amplitude parameter, and no background current, it is easy to show that the particle trajectories are closed ellipses. However, it can be proven that there are indeed no closed particle trajectories, and that the particles experience an overall forward drift per period [1].

Our project was to seek a solution to the second order approximation of the particle trajectories, and, if possible, find an analytical solution in the case of an additional background current.

1.2 Result

The general solution to the particle trajectories is given by the differential equations

$$\frac{dx}{dt} = M\cosh(ky)\cos(kx - wt) + U(y) \tag{1}$$

and

$$\frac{dy}{dt} = M\sinh(ky)\sin(kx - wt), \tag{2}$$

which are derived by linearizing the Euler equations of the problem[1]. M is a parameter dependent on ratio between the wave amplitude and depth, which for a linear water wave is small. $U(y)$ is a background current, free to variate as a function of the depth of the water. These equations are not

[1] We refer to [1] for details on the linearization of the Euler equations.

analytically solvable, hence we use perturbation theory in order to find some analytical expression for the particle trajectories.

We start by expanding the solutions in the small parameter M:

$$x(t) = x_0(t) + Mx_1(t) + M^2x_2(t) + O(M^3), \tag{3}$$

$$y(t) = y_0(t) + My_1(t) + M^2y_2(t) + O(M^3). \tag{4}$$

To the first order in M and no background current, the particle trajectories can be shown to be closed ellipses, which disagree with theory. Our goal was therefore to solve the equations to the second order in M. Assuming that $[x_0^*, y_0^*]$ is the starting point of the trajectory, and that the current $U(y)$ is of the order M, we managed to obtain the following solutions for $x(t)$ and $y(t)$:

$$x_0(t) = x_0^*, \tag{5}$$

$$x_1(t) = \sin(kx_0^* - wt)C_1 + tC_2 + C_3, \tag{6}$$

$$\begin{aligned} x_2(t) = {} & \sin(kx_0^* - wt)C_4 + \cos(kx_0^* - wt)C_5 + t\cos(kx_0^* - wt)C_6 \\ & + \sin(kx_0^* - wt)\cos(kx_0^* - wt)C_7 + tC_8 + C_9, \end{aligned} \tag{7}$$

and

$$y_0(t) = y_0^*, \tag{8}$$

$$y_1(t) = K_1\cos(kx_0^* - wt) + K_2, \tag{9}$$

$$\begin{aligned} y_2(t) = {} & K_3\sin(kx_0^* - wt) + K_4\cos(kx_0^* - wt) \\ & + K_5\sin(kx_0^* - wt)^2 + K_6t\sin(kx_0^* - wt) + K_7. \end{aligned} \tag{10}$$

The coefficients C_i and K_i are shown in figure 3 and 4 further down.

Figure 1 shows a plot of a numerical solution to the exact equations and a plot of the perturbed solution, in the case of no background current $U(y)$. The starting point is $[0, 0.95h]$, where h is the depth of the water and arbitrarily set to 10m. Figure 2 shows the numerical and perturbed solution for some different velocity profiles for $U(y)$. For further details about the mathematics and numerical work, see section 2 and 3. The Matlab code used to produce the plots can be found in Appendix B.

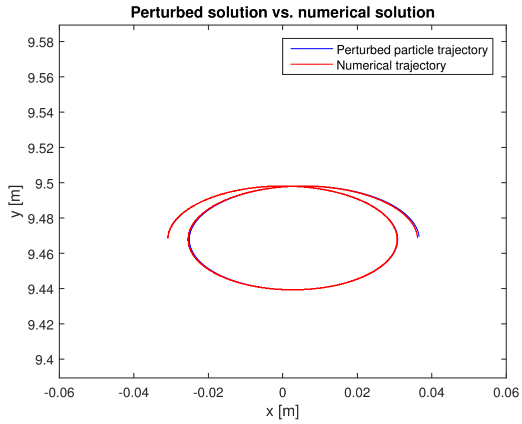
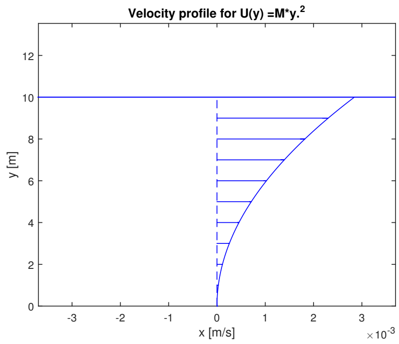
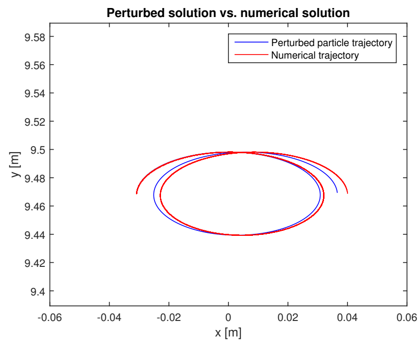


Figure 1: Plot of the numerical and perturbed solution for the case of no background current.

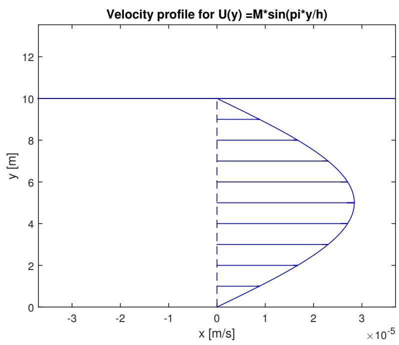
Figure 2: Plots of the numerical and perturbed solution for some different currents $U(y)$.



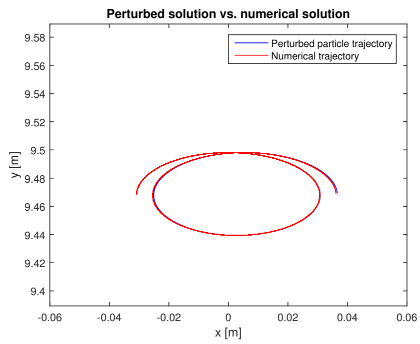
(a) Velocity profile for $U(y) = My^2$.



(b) Plot of the numerical and the perturbed solution for $U(y) = My^2$.



(c) Velocity profile for $U(y) = M\sin(\frac{\pi y}{h})$.



(d) Plot of the numerical and the perturbed solution for $M\sin(\frac{\pi y}{h})$.

1.3 Final comments

Although we are satisfied with the outcome of our efforts, we feel in many regards that we leave the project unfinished. Mainly, we don't know anything about the accuracy of the solutions. As the graphs in figure 1 and 2 show, the solution seems reasonable for short time periods. However, it is expected that the accuracy falls pretty drastically as the displacement grows larger. Neither do we know much about validity of the solutions for different background currents, as the condition $U(y) \approx O(M)$ is a somewhat vague statement which preferably should be stricter. An example of this is Figure 2b, where the y^2 factor grows large enough to make the solution inaccurate.

Additionally, part of the task was to assess the drift induced by different currents, which we have not had the time for. The logical steps forward would be to find estimates of the error and areas of applicability of the solution. Finally, in order to make the solutions as general as possible, we have needed to make some approximations during the calculations[2]. It should however be possible to find exact solutions, in the domain of second order perturbation theory, for specific currents. This would offer a nice test to the accuracy of our solution for arbitrary weak currents.

[2]See Section 3 for details.

Figure 3: **Table of constants** C_i

$$
\begin{aligned}
C_1 &= -\frac{\sinh(ky_0^*)}{w^2}U'(y_0^*) - \frac{\cosh(ky_0^*)}{w}\\
C_2 &= U_1(y_0^*) - \frac{\sinh(ky_0^*)\cos(kx_0^*)}{w}U_1'(y_0^*)\\
C_3 &= \frac{\sin(kx_0^*)}{w}\{U'(y_0^*)\frac{\sinh(ky_0^*)}{w} + \cosh(ky_0^*)\}\\
C_4 &= \frac{k}{w^2}\{\frac{\sinh(ky_0^*)\cosh(ky_0^*)\cos(kx_0^*)}{w}U'(y_0^*) - \cosh(ky_0^*)U(y_0^*) + \sinh(ky_0^*)^2\cos(kx_0^*)\}\\
C_5 &= \frac{k\cosh(ky_0^*)\sin(kx_0^*)}{w^2}\{-\cosh(ky_0^*) - U'(y_0^*)\frac{\sinh(ky_0^*)}{w}\}\\
C_6 &= \frac{k\cosh(ky_0^*)}{w}\{U'(y_0^*)\frac{\sinh(ky_0^*)\cos(kx_0^*)}{w} - U(y_0^*)\}\\
C_7 &= \frac{k}{2w^2}\{1 + \frac{\sinh(ky_0^*)\cosh(ky_0^*)}{w}U'(y_0^*)\}\\
C_8 &= \frac{k}{2w}\{\sinh(ky_0^*)^2 + \cosh(ky_0^*)^2\}\\
C_9 &= \frac{k\sin(kx_0^*)\cosh(ky_0^*)}{w^2}\{U(y_0^*) - \frac{\sinh(ky_0^*)\cos(kx_0^*)}{2w}U'(y_0^*)\} + \frac{k\sin(kx_0^*)\cos(kx_0^*)}{2w^2}
\end{aligned}
$$

Figure 4: **Table of constants** K_i

$$
\begin{aligned}
K_1 &= \frac{\sinh(ky_0^*)}{w}\\
K_2 &= -\frac{\sinh(ky_0^*)}{w}\cos(kx_0^*)\\
K_3 &= \frac{k\sinh(ky_0^*)\sin(kx_0^*)}{w^2}\{-U'(y_0^*)\frac{\sinh(ky_0^*)}{w} - \cosh(ky_0^*)\}\\
K_4 &= \frac{k\sinh(ky_0^*)\cos(kx_0^*)}{w^2}\{-U'(y_0^*)\frac{\sinh(ky_0^*)}{w} + U(y_0^*) - \cosh(ky_0^*)\}\\
K_5 &= U'(y_0^*)\frac{k\sinh(ky_0^*)^2}{2w^3}\\
K_6 &= U'(y_0^*)\frac{k\sinh(ky_0^*)^2\cos(kx_0^*)}{w^2} - U(y_0^*)\frac{k\sinh(ky_0^*)}{w}\\
K_7 &= U'(y_0^*)\frac{k\sinh(ky_0^*)^2}{w^3}\{\frac{1}{2}\sin(kx_0^*)^2 + \cos(kx_0^*)^2\} - \frac{k\sinh(ky_0^*)}{w^2}\{\cos(kx_0^*)U_1(y_0^*) + \cosh(ky_0^*)\}
\end{aligned}
$$

2 Details on the numerics

To produce a numerical solution of the Euler equations we used a fourth order explicit Runge Kutta method for the system

$$\frac{dx}{dx} = f(t, x, y),$$
$$\frac{dy}{dx} = g(t, x, y).$$

Hence x_{n+1} and y_{n+1} are given by

$$x_{n+1} = x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

$$y_{n+1} = y_n + \frac{h}{6}(l_1 + 2l_2 + 2l_3 + l_4),$$

where

$$\begin{aligned}
k_1 &= f(t_n, x_n, y_n),\\
l_1 &= g(t_n, x_n, y_n),\\
k_2 &= f(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hk_1, y_n + \frac{1}{2}hl_1),\\
l_2 &= g(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hk_1, y_n + \frac{1}{2}hl_1),\\
k_3 &= f(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hk_2, y_n + \frac{1}{2}hl_2),\\
l_3 &= g(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hk_2, y_n + \frac{1}{2}hl_2),\\
k_4 &= f(t_n + h, x_n + hk_3, y_n + hl_3),\\
l_4 &= g(t_n + h, x_n + hk_3, y_n + hl_3),
\end{aligned}$$

and h is the step size. For an intuitive explanation of the 4th order Runge Kutta method we refer to [2].

3 Details on the analytical work

This section explains the analytical approach in greater detail.

The equations that need to be solved are:

$$\frac{dx}{dt} = M\cosh(ky)\cos(kx - wt) + U(y) \tag{11}$$

and

$$\frac{dy}{dt} = M\sinh(ky)\sin(kx - wt). \tag{12}$$

We start by assuming that we can expand $x(t)$ in the small parameter M:

$$x(t) = x_0(t) + Mx_1(t) + M^2x_2(t) + O(M^3) \tag{13}$$

$$y(t) = y_0(t) + My_1(t) + M^2y_2(t) + O(M^3) \tag{14}$$

M is a parameter dependent on the amplitude a, frequency w, wave number k and depth h_0, through the equation

$$M = \frac{aw}{\sinh(kh_0)}.$$

For linear water waves a is small compared to h_0, and hence M is small. More precisely,

$$\frac{a}{h_0} \leq \frac{\tanh(kh_0)}{kh_0}$$

is the condition that needs to be fulfilled in order to consider the wave a linear water wave and M as small[3].

Since we're looking for the second order solution to 11 and 12, we neglect all orders of M^3 and above throughout the calculations. Also, we will make the assumption that $U(y) \approx O(M)$, and use it's taylor expansion around y_0^*.

$$\begin{aligned} U(y) &\approx MU_1(y_0^*) + MU'(y_0^*)(y(t) - y_0^*) \\ &= MU_1(y_0^*) + MU'(y_0^*)(My_1(t)) + O(M^3) \end{aligned}$$

3.1 Initial values

Denoting (x_0^*, y_0^*) as the coordinates at $t = 0$, we get the initial conditions

$$x(0) = x_0^* \implies x_0(0) = x_0^*,\ x_1(0) = x_2(0) = 0$$

$$y(0) = y_0^* \implies y_0(0) = y_0^*,\ y_1(0) = y_2(0) = 0$$

Inserting 13 into 11 and 14 into 12 we get

$$\begin{aligned} \frac{dx_0}{dt} + M\frac{dx_1}{dt} + M^2\frac{dx_2}{dt} = &M\cosh(k(y_0(t) + My_1(t) + M^2y_2(t)) \\ &\times \cos(k(x_0(t) + Mx_1(t) + M^2x_2(t)) - wt) \\ &+ MU_1(y(t)) \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{dy_0}{dt} + M\frac{dy_1}{dt} + M^2\frac{dy_2}{dt} = &M\sinh(k(y_0(t) + My_1(t) + M^2y_2(t))) \\ &\times \sin(k(x_0(t) + Mx_1(t) + M^2x_2(t)) - wt). \end{aligned} \tag{16}$$

The equations must match power by power in M. We seek first the solution to the unperturbed equation by looking at M^0. This determines $x_0(t)$ and $y_0(t)$

$$\frac{dx_0}{dt} = 0 \implies x_0(t) = x_0^*$$

$$\frac{dy_0}{dt} = 0 \implies y_0(t) = y_0^*$$

[3]This condition is derived in [1]

3.2 Power series expansion

Seeking equations for $x_1(t)$, $y_1(t)$, $x_2(t)$ and $y_2(t)$, we use standard trigonometric and hyperbolic identities[4] to expand all the hyperbolic and trigonometric functions in equations 15

$$
\begin{aligned}
M\frac{dx_1}{dt} + M^2\frac{dx_2}{dt} &= M\{\cosh(ky_0^*)\cosh(kMy_1 + kM^2y_2) + \sinh(ky_0^*)\sinh(kMy_1 + kM^2y_2)\} \\
&\times \{\cos(kx_0 - wt)\cos(kMx_1 + kM^2x_2) - \sin(kx_0 - wt)\sin(kMx_1 + kM^2x_2)\} \\
&+ MU_1(y),
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
M\frac{dx_1}{dt} + M^2\frac{dx_2}{dt} =& M\{\cosh(ky_0^*) + \sinh(ky_0^*)(kMy_1)\} \\
&\times \{\cos(kx_0 - wt) - \sin(kx_0 - wt)(kMx_1)\} \\
&+ MU_1(y) + O(M^3).
\end{aligned}
$$

Doing the same for equation 16

$$
\begin{aligned}
M\frac{dy_1}{dt} + M^2\frac{dy_2}{dt} &= M\{\sinh(ky_0^*)\cosh(kMy_1 + kM^2y_2) + \cosh(ky_0^*)\sinh(kMy_1 + kM^2y_2)\} \\
&\times \{\sin(kx_0 - wt)\cos(kMx_1 + kM^2x_2) + \cos(kx_0 - wt)\sin(kMx_1 + kM^2x_2)\},
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
M\frac{dy_1}{dt} + M^2\frac{dy_2}{dt} &= M\{\sinh(ky_0^*) + \cosh(ky_0^*)(kMy_1)\} \\
&\times \{\sin(kx_0 - wt) + \cos(kx_0 - wt)(kMx_1)\} + O(M^3).
\end{aligned}
$$

3.3 New differential equations

Finally, matching powers of M and M^2, we get the following equations for $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$

$$\frac{dx_1}{dt} = U_1(y_0^*) + U_1'(y_0^*)y_1(t) + \cosh(ky_0^*)\cos(kx_0^* - wt)$$

$$\frac{dx_2}{dt} = \sinh(ky_0^*)\cos(kx_0 - wt)ky_1(t) - \cosh(ky_0^*)\sin(kx_0 - wt)kx_1(t)$$

$$\frac{dy_1}{dt} = \sinh(ky_0^*)\sin(kx_0 - wt)$$

$$\frac{dy_2}{dt} = \sinh(ky_0^*)\cos(kx_0 - wt)kx_1(t) + \cosh(ky_0^*)\sin(kx_0 - wt)ky_1(t)$$

In order to solve this set we first need to find $y_1(t)$ and $x_1(t)$. From there on the solution is found by solving a large number of well defined integrals. Due to the sheer amount of these integral, we will simply state our solutions below.

[4]See Appendix A for a summary of the formulaes used

3.4 Solution for $y_1(t)$

$$y_1(t) = \frac{\sinh(ky_0^*)}{w}\{\cos(kx_0^* - wt) - \cos(kx_0^*)\}$$

3.5 Solution for $x_1(t)$

$$x_1(t) = U_1(y_0^*)t + \frac{\sinh(ky_0^*)}{w}\{-U_1'(y_0^*)\cos(kx_0^*)t + U'(y_0^*)\frac{\sin(kx_0^*)}{w} - U'(y_0^*)\frac{\sin(kx_0^* - wt)}{w}\}$$
$$- \frac{\cosh(ky_0^*)}{w}\{\sin(kx_0^* - wt) - \sin(kx_0^*)\}$$

3.6 Solution for $y_2(t)$

$$\begin{aligned}
y_2(t) = & \frac{k\sinh(ky_0^*)\cosh(ky_0^*)}{2w^2}\{\sin(kx_0^*)^2 - \sin(kx_0^* - wt)^2\} \\
& + \frac{k\sinh(ky_0^*)\cosh(ky_0^*)\cos(kx_0^*)}{w^2}\{\cos(kx_0^*) - \cos(kx_0^* - wt)\} \\
& + \frac{k\sinh(ky_0^*)^2\sin(kx_0^*)}{w^3}U'(y_0^*)\{\sin(kx_0^*) - \sin(kx_0^* - wt)\} \\
& + \frac{k\sinh(ky_0^*)^2\cos(kx_0^*)}{w^3}U'(y_0^*)\{\cos(kx_0^*) - \cos(kx_0^* - wt) + wt\sin(kx_0^* - wt)\} \\
& + \frac{k\sinh(ky_0^*)^2}{2w^3}U'(y_0^*)\{\sin(kx_0^* - wt)^2 - \sin(kx_0^*)^2\} \\
& + \frac{k\sinh(ky_0^*)\cosh(ky_0^*)}{2w^2}\{\sin(kx_0^* - wt)^2 - \sin(kx_0^*)^2\} \\
& + \frac{k\sinh(ky_0^*)\cosh(ky_0^*)\sin(kx_0^*)}{w^2}\{\sin(kx_0^*) - \sin(kx_0^* - wt)\} \\
& + \frac{k\sinh(ky_0^*)}{w^2}U_1(y_0^*)\{\cos(kx_0^* - wt) - wt\sin(kx_0^* - wt) - \cos(kx_0^*)\}
\end{aligned}$$

3.7 Solution for $x_2(t)$

$$
\begin{aligned}
x_2(t) = \; & \frac{k\sinh(ky_0^*)\cosh(kx_0^*)\cos(kx_0^*)}{w^3}U'(y_0^*)\{\sin(kx_0^*-wt)+wt\cos(kx_0^*-wt)-\sin(kx_0^*)\} \\
& -\frac{k\cosh(kx_0^*)}{w^2}U(y_0^*)\{\sin(kx_0^*-wt)+wt\cos(kx_0^*-wt)-\sin(kx_0^*)\} \\
& -\frac{k\cosh(ky_0^*)^2\sin(kx_0^*)}{w^2}\{\cos(kx_0^*-wt)-\cos(kx_0^*)\} \\
& +\frac{k\sinh(ky_0^*)\cosh(ky_0^*)\sin(kx_0^*)}{w^3}U'(y_0^*)\{\cos(kx_0^*-wt)-\cos(kx_0^*)\} \\
& +\frac{k\cosh(ky_0^*)^2}{4w^2}\{2wt+2\sin(kx_0^*-wt)\cos(kx_0^*-wt)-2\sin(kx_0^*)\cos(kx_0^*)\} \\
& +\frac{k\sinh(ky_0^*)\cosh(ky_0^*)}{4w^3}U'(y_0^*)\{2wt+2\sin(kx_0^*-wt)\cos(kx_0^*-wt)-2\sin(kx_0^*)\cos(kx_0^*)\} \\
& +\frac{k\sinh(ky_0^*)^2}{4w^2}\{2wt-2\sin(kx_0^*-wt)\cos(kx_0^*-wt)+2\sin(kx_0^*)\cos(kx_0^*)\} \\
& +\frac{k\sinh(ky_0^*)^2\cos(kx_0^*)}{w^2}\{\sin(kx_0^*-wt)-\sin(kx_0^*)\}
\end{aligned}
$$

4 Appendix A

4.1 Useful formulaes

An overview of the applied trigonometric and hyperbolic identitites, as well as their power series expansion.

$$
\begin{aligned}
\sin(x\pm y) &= \sin(x)\cos(y)\pm\sin(y)\cos(x) \\
\cos(x\pm y) &= \cos(x)\cos(y)\mp\sin(y)\sin(x) \\
\sinh(x\pm y) &= \sinh(x)\sinh(y)\mp\sinh(y)\cosh(x) \\
\cosh(x\pm y) &= \cosh(x)\cosh(y)\pm\sinh(y)\sinh(x)
\end{aligned}
$$

$$
\begin{aligned}
\sin(x) &\approx x+O(x^3) \\
\cos(x) &\approx 1+O(x^2) \\
\sinh(x) &\approx x+O(x^3) \\
\cosh(x) &\approx 1+O(x^2)
\end{aligned}
$$

5 Appendix B

5.1 Matlab code

```
%Second order perturbation vs numerical particle trajectory

%Initializing constants
k = 1; % Wavenumber, arbitrarily chosen
```

```
h = 10; % Depth, arbitrarily chosen
g = 9.81; % gravitational acceleration
e = 0.05*tanh(k*h)/(k*h); % amplitude parameter. Scaling factor 0.05 chosen arbitrarily
c1 = sqrt(g*tanh(k*h)/k);
M = e*h*c1*k/sinh(k*h);
w = k*c1;

%starting coordinates
x0 = 0;
y0 = h*0.95;

%simplifying notation
a = sinh(k*y0);
b = cosh(k*y0);
c = sin(k*x0);
d = cos(k*x0);
%%%

U = @(y) M*sin(pi*y/h); %0; %M*y^2;
Udiv = @(y) M*(pi/h)*cos(pi*y/h);% 0;% 2*M*y;

C1 = -a*Udiv(y0)/w^2 - b/w;
C2 = U(y0) - a*d*Udiv(y0)/w;
C3 = (c/w)*(a*Udiv(y0)/w + b);
C4 = (k/w^2)*(a*b*d*Udiv(y0)/w -b*U(y0) + d*a^2);
C5 = (k*b*c/w^2)*(-b-Udiv(y0)*a/w);
C6 = (k*b/w)*(Udiv(y0)*a*d/w - U(y0));
C7 = (0.5*k/w^2)*(1+a*b*Udiv(y0)/w);
C8 = (0.5*k/w)*(a^2 + b^2);
C9 = (k*c*b/w^2)*(U(y0) - Udiv(y0)*a*d*0.5/w) + k*0.5*c*d/w^2;

K1 = a/w;
K2 = -a*d/w;
K3 = (k*a*c/w^2)*(-Udiv(y0)*a/w - b);
K4 = (k*a*d/w^2)*(-Udiv(y0)*a/w + U(y0) - b);
K5 = Udiv(y0)*0.5*k*(a^2)/w^3;
K6 = Udiv(y0)*k*(a^2)*d/w^2 - U(y0)*k*a/w;
K7 = (Udiv(y0)*k*(a^2)/w^3)*(0.5*c^2 + d^2) - (k*a/w^2)*(d*U(y0) + b);

x1 = @(t) sin(k*x0 - w*t)*C1 + t*C2 + C3;
x2 = @(t) sin(k*x0 - w*t)*C4 + cos(k*x0 - w*t)*C5 + t*cos(k*x0 - w*t)*C6 ...
    + sin(k*x0 - w*t)*cos(k*x0 - w*t)*C7 + t*C8 + C9;
y1 = @(t) cos(k*x0 - w*t)*K1 + K2;
y2 = @(t) sin(k*x0 - w*t)*K3 + cos(k*x0 - w*t)*K4 ...
    + K5*sin(k*x0 - w*t)^2 + t*sin(k*x0 - w*t)*K6 + K7;
x = @(t) x0 + M*x1(t) + (M^2)*x2(t);
y = @(t) y0 + M*y1(t) + (M^2)*y2(t);
```

```
num = 1; %number of periods
p = num*2*pi/w; %time of num-periods

dt = 0.01; %timestep
i = 1;
t = -p/4;
while (t < p + p/4);
    xplot(i) = x(t);
    yplot(i) = y(t);
    t = t+dt;
    figure(1)
    plot(xplot(1,1:i),yplot(1,1:i),'b')
    ylim([y0-0.1, y0 + 0.1]);
    xlim([-0.06,0.06]);
    pause(0.1)
    i = i+1;
end


%%%%%%%%%%%%%%%
%Numerical solution

f = @(t,x,y) M*cosh(k*y)*cos(k*x - k*c1*t) + U(y);
g = @(t,x,y) M*sinh(k*y)*sin(k*x - k*c1*t);


%Runge-Kutta method
dt = 0.01; %timestep

xnum(1) = xplot(1);
ynum(1) = yplot(1);
i = 1;
t = -p/4;
while (t<p+p/4);
    k1 = f(t,xnum(i),ynum(i));
    j1 = g(t,xnum(i),ynum(i));
    k2 = f(t + dt/2, xnum(i) + (dt/2)*k1, ynum(i) + (dt/2)*j1);
    j2 = g(t + dt/2, xnum(i) + (dt/2)*k1, ynum(i) + dt*j1/2);
    k3 = f(t + dt/2, xnum(i) + dt*k2/2, ynum(i) + dt*j2/2);
    j3 = g(t + dt/2, xnum(i) + dt*k2/2, ynum(i) + dt*j2/2);
    k4 = f(t + dt, xnum(i)+dt*k3, ynum(i)+dt*j3);
    j4 = g(t + dt, xnum(i)+dt*k3, ynum(i)+dt*j3);

    i = i + 1;
    xnum(i) = xnum(i-1) + (dt/6)*(k1 + 2*k2 + 2*k3 + k4);
    ynum(i) = ynum(i-1) + (dt/6)*(j1 + 2*j2 + 2*j3 + j4);
    t = t + dt;
    figure(1)
```

```
    hold on
    plot(xnum(1,1:i),ynum(1,1:i),'r')
    pause(0.05)
end

hold on
xlabel('x [m]')
ylabel('y [m]')
ylim([y(1)-0.05, y(1) + 0.15]);
xlim([-0.06,0.06]);
title('Perturbed solution vs. numerical solution')
legend('Perturbed particle trajectory', 'Numerical trajectory')
```

References

[1] A. Constantin and G. Villari, Particle Trajectories in Linear Water Waves, Birkhäuser Verlag, Basel, 2006

[2] Fourth order explicit Runge Kutta method http://lpsa.swarthmore.edu/NumInt/NumIntFourth.html