NEHARI'S THEOREM FOR MULTIPLICATIVE HANKEL MATRICES

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1. Introduction

Let $M_\rho : \ell^2 \to \ell^2$ be the operator defined by the additive Hankel matrix

$$M_\rho = (\rho_{m+n})_{m,n\geq 0},$$

for some sequence $\rho \in \ell^2$. The operator can then be realised as a Hankel form $H_\varphi = \langle fg, \varphi\rangle$ on the Hardy space, $H^2(\mathbb{T})$, as defined in Section 2.

Nehari [7] proved in 1957 that M_ρ is a bounded operator on ℓ^2 if and only if it can be generated by a bounded symbol

$$\psi(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \rho_n e^{in\theta}, \quad \psi \in L^\infty(\mathbb{T}),$$

such that

$$H_\varphi = H_\psi, \quad \|M_\rho\| = \|\psi\|_{L^\infty(\mathbb{T})}.$$

The Lebesgue space, $L^q(\mathbb{T})$, will be defined in Section 2.

However, in the case defined by the multiplicative Hankel matrix

$$\mathcal{M}_\rho = (\rho_{mn})_{m,n\geq 1}$$

with the corresponding multiplicative Hankel form in $H^2(\mathbb{T}^\infty)$, the theorem of Nehari is not generally valid [8]. Here, the infinite polytorus, $\mathbb{T}^\infty$, is the infinite cartesian product $\mathbb{T}\times\mathbb{T}\times\cdots$.

Let $\|H_\varphi\|_{\mathcal{S}_r}$ be the Schatten-norm of H_φ, equal to the ℓ^r-norm of the singular values of the matrix corresponding to the form H_φ. The form is in the Schatten class $\mathcal{S}_r$ if the $\mathcal{S}_r$-norm is finite. We also have $\mathcal{S}_x \subseteq \mathcal{S}_y$ if $x \leq y$

Helson [6] showed that any form belonging to the Schatten class $\mathcal{S}_2$ has a bounded symbol, while Brevig and Perfekt [4] proved that for for every $r > r_0 = (1 - \ln\pi/\ln 4)^{-1}$ there exist multiplicative Hankel forms in $\mathcal{S}_r$ without a bounded symbol. The interesting range is therefore $2 < r \leq r_0$.

This report consists of two parts. In the first part we will present an improvement of the main result in [8] in section 2 by considering L^q symbols instead of L^∞ symbols. It is proven that Nehari's theorem fails for multiplicative Hankel forms whenever $q > 2$. It trivially holds for $q = 2$.

The main part considers a method of looking for counter examples in one dimension with the ambition of finding a lower bound than r_0. The method is described in Section 3. In order to justify the method, properties of the operators connecting the Hardy space $H^p(\mathbb{T}^d)$ and the Bergman space $A^p_d(\mathbb{D})$ are investigated. The main result here is that $E_d : A^2_d(\mathbb{D}) \to H^2(\mathbb{T}^d)$, the adjoint of the diagonalisation operator $D_d : H^2(\mathbb{T}^d) \to A^2_d(\mathbb{D})$, is a contraction if and only if $p = 2$. This demonstrates that

Date: May 15, 2019.

one specific approach to the problem does not work. In addition, we will present numerical results which indicate that r_0 may be difficult to decrease.

2. Hankel forms on $\mathbb{T}^d$

Let the torus $\mathbb{T}^d$, where d can be infinite, be defined as

$$\mathbb{T}^d = \mathbb{T} \times \mathbb{T} \times \cdots \times \mathbb{T} = \{z \in \mathbb{C}^d : |z_j| = 1 \text{ for } 1 \leq j \leq d\}.$$

Define the Lebesgue space $L^p(\mathbb{T}^d)$ as the Fourier series

$$f(z) = \sum_{\alpha \in \mathbb{Z}^d} a_\alpha z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}$$

with finite norm

$$\|f\|_{L^p(\mathbb{T}^d)}^p = \int_{\mathbb{T}^d} |f(z)|^p dm_d(z) = \int_0^{2\pi} \cdots \int_0^{2\pi} |f(e^{i\theta_1}, \cdots, e^{i\theta_d})|^p \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi}.$$

Here $dm_d(z)$ is the product of the normalised Lebesgue arc length measure on $\mathbb{T}$. Also, the Hardy space $H^p(\mathbb{T}^d)$ is defined as the subspace of $L^p(\mathbb{T}^d)$ consisting of the functions

$$f(z) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}.$$

Recall that a Hankel form is defined on $H^2(\mathbb{T}^d)$ by $H_\varphi = \langle fg, \varphi \rangle_{H^2(\mathbb{T}^d)}$ for $f, g \in H^2(\mathbb{T}^d)$. We say that H_φ has a *bounded symbol* if there exist $\psi \in L^\infty(\mathbb{T}^d)$ such that $H_\varphi = H_\psi$. Let us also introduce the notion L^q *symbol* and say H_φ has a L^q *symbol* if there exist $\psi \in L^q(\mathbb{T}^d)$ such that $H_\varphi = H_\psi$. Clearly, the definition of a L^q symbol coincides with that of a bounded symbol if we let $q \to \infty$. Also, $L^x(\mathbb{T}^d) \subseteq L^y(\mathbb{T}^d)$ for all $x \geq y$. Therefore, decreasing q means allowing symbols from a larger space.

If every H_φ in $\mathcal{S}_r(H^2(\mathbb{T}^\infty) \times H^2(\mathbb{T}^\infty))$ has a L^q symbol, then the argument of [4] yields that every H_φ in $\mathcal{S}_r(H^2(\mathbb{T}^d) \times H^2(\mathbb{T}^d))$ has a symbol L^q symbol ψ with

$$\|\psi\|_{L^q(\mathbb{T}^d)} \leq \|H_\varphi\|_{\mathcal{S}_r(H^2(\mathbb{T}^d) \times H^2(\mathbb{T}^d))}$$

for every $1 \leq d \leq \infty$.

From the Hahn-Banach theorem we know that such ψ exist if and only if

$$\varphi \in (H^p(\mathbb{T}^d))^* \tag{1}$$

with

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad \|\psi\|_{L^q(\mathbb{T}^d)} = \|\varphi\|_{(H^p(\mathbb{T}^d))^*}. \tag{2}$$

Here, $(H^p(\mathbb{T}^d))^*$ denotes the dual space consisting of functionals on $H^p(\mathbb{T}^\infty)$ with norm

$$\|\varphi\|_{(H^p(\mathbb{T}^d))^*} := \sup_{f \in H^p(\mathbb{T}^d)} \frac{|\langle f, \varphi \rangle_{H^2(\mathbb{T}^d)}|}{\|f\|_{H^p(\mathbb{T}^d)}}.$$

In conclusion, if there exist polynomials f and φ, both depending on the first d complex variables $z_1, z_2, ..., z_d$, such that

$$\frac{\|\varphi\|_{(H^p(\mathbb{T}^d))^*}}{\|H_\varphi\|_{\mathcal{S}_r(H^2(\mathbb{T}^d) \times H^2(\mathbb{T}^d))}} \geq \frac{|\langle f, \varphi \rangle_{H^2(\mathbb{T}^d)}|}{\|H_\varphi\|_{\mathcal{S}_r(H^2(\mathbb{T}^d) \times H^2(\mathbb{T}^d))} \|f\|_{H^p(\mathbb{T}^d)}} > 1, \tag{3}$$

we prove the existence of a multiplicative Hankel form $H_\varphi \in \mathcal{S}_r(H^2(\mathbb{T}^\infty) \times H^2(\mathbb{T}^\infty))$ without a bounded symbol.

2.1. **Bounded forms without a symbol in $L^q(\mathbb{T}^\infty)$.** In [4] the multiplicative Hankel forms have been in the Schatten classes $\mathcal{S}_r(H^2(\mathbb{T}^d)\times H^2(\mathbb{T}^d))$ and the bounded symbols in $L^\infty(\mathbb{T}^d)$. It has proven difficult to find an example to decrease the value of r_0 obtained by $\varphi = z_1+\cdots+z_d$ in [4]. Another approach is to allow symbols from a larger space. Therefore, we will now allow all bounded forms, that is $H_\varphi \in \mathcal{S}_\infty(H^2(\mathbb{T}^d)\times H^2(\mathbb{T}^d))$ since the $\mathcal{S}_\infty$-norm is the standard operator norm, and look for symbols in the space $L^q(\mathbb{T}^d)$.

If $q=2$ and $H_\varphi \in \mathcal{S}_\infty(H^2(\mathbb{T}^d)\times H^2(\mathbb{T}^d))$, then clearly for $1\le d\le\infty$

$$\begin{aligned}\|H_\varphi\|_{\mathcal{S}_\infty(H^2(\mathbb{T}^d)\times H^2(\mathbb{T}^d))} = \|H_\varphi\| &= \sup_{f,g\in H^2(\mathbb{T}^d)} \frac{|\langle fg,\varphi\rangle_{H^2(\mathbb{T}^d)}|}{\|f\|_{H^2(\mathbb{T}^d)}\,\|g\|_{H^2(\mathbb{T}^d)}}\\ &\ge \sup_{f\in H^2(\mathbb{T}^d)} \frac{|\langle f,\varphi\rangle_{H^2(\mathbb{T}^d)}|}{\|f\|_{H^2(\mathbb{T}^d)}}\\ &= \|\varphi\|_{(H^2(\mathbb{T}^d))^*},\end{aligned}$$

by choosing $g(z)=1$. Therefore any form in $\mathcal{S}_\infty(H^2(\mathbb{T}^d)\times H^2(\mathbb{T}^d))$ has a symbol in $H^q(\mathbb{T}^d)\subseteq L^q(\mathbb{T}^d)\subseteq L^2(\mathbb{T}^d)$ for $q\le 2$.

Theorem 1. *Fix $q>2$. Then there exist multiplicative Hankel forms $H_\varphi\in\mathcal{S}_\infty(H^2(\mathbb{T}^\infty)\times H^2(\mathbb{T}^\infty))$ without a symbol in $L^q(\mathbb{T}^\infty)$.*

Proof. Define the function $f(z_1,z_2)=z_1+z_2$, and let $\varphi=f$.

We get $|\langle f,\varphi\rangle|_{H^2(\mathbb{T}^d)}=2$ and $\|H_\varphi\|_{\mathcal{S}_\infty(H^2(\mathbb{T}^d)\times H^2(\mathbb{T}^d))}=\sqrt{2}$ [4]. By (2) if $q>2$ then $p<2$. Thus, we need only to prove that $\|f\|^p_{H^p(\mathbb{T}^2)}$ is less than $2^{p/2}$ to have the contradiction (3).

$$\begin{aligned}\|f\|^p_{H^p(\mathbb{T}^2)} &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}|e^{i\theta_1}+e^{i\theta_2}|^p d\theta_1 d\theta_2\\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}|1+e^{i(\theta_2-\theta_1)}|^p d\theta_1 d\theta_2\\ &= \frac{1}{2\pi}\int_0^{2\pi}\big((1+\cos\vartheta)^2+\sin^2\vartheta\big)^{p/2}d\vartheta\\ &= \frac{2^p}{2\pi}\int_0^{2\pi}\big|\cos\big(\frac{\vartheta}{2}\big)\big|^p d\vartheta = \frac{2^{2x}\,\Gamma(x+\frac12)}{\sqrt{\pi}\,\Gamma(x+1)},\end{aligned}\tag{4}$$

where Γ is the gamma function and $p=2x$. It is equivalent to prove that

$$g(x) := x\ln 2+\ln\Gamma(x+\frac12)-\ln\Gamma(x+1)-\ln\sqrt{\pi}<0,\tag{5}$$

for $\frac12\le x<1$. We have $g(1)=0$, and with $\psi(x):=\frac{d}{dx}\ln\Gamma(x)$, the digamma function, the derivative becomes

$$\begin{aligned}g'(x)=\ln 2+\psi\Big(x+\frac12\Big)-\psi(x+1) &= \ln 2+\sum_{k=1}^{\infty}\left(\frac{1}{k+x}-\frac{1}{k+x+\frac12}\right)\\ &= \ln 2-\frac12\sum_{k=1}^{\infty}\frac{1}{(k+x)(k+x-\frac12)},\end{aligned}$$

which is clearly an increasing function. Inserting $x=\frac12$ gives

$$g'\Big(\frac12\Big)=\ln 2+\psi(1)-\psi\Big(1+\frac12\Big)=3\ln 2-2>0 \quad\Rightarrow\quad g'(x)>0,\quad \frac12\le x<1,$$

by use of the integer and half integer argument relations for the digamma function. Therefore $g(x)$ is evidently strictly increasing for $\frac{1}{2} \leq x < 1$, and we have proved the strict inequality (5). □

This result constitutes an improvement on the main result in [8].

3. Hankel forms in $\mathbb{D}$

Let the Bergman space $A^p_d(\mathbb{D})$ be the closure of polynomials g, in the norm

$$\|g\|^p_{A^p_d(\mathbb{D})} := \frac{1}{\pi} \int_{\mathbb{D}} |g(w)|^p (d-1)(1-|w|)^{d-2} dA(w),$$

where $dA(w)$ is the Lebesgue area measure, $d \geq 2$. A computation shows that

$$\|g\|^2_{A^2_d(\mathbb{D})} = \sum_{j=0}^{\infty} \frac{|a_j|^2}{c_d(j)}, \qquad c_d(j) := \binom{j+d-1}{j},$$

and a_j are the coefficients of g.

Introduce the operators defined as

$$D_d : H^p(\mathbb{T}^d) \to A^p_d(\mathbb{D}), \quad (D_d f)(w) = f(w, w, ..., w),$$

$$E_d : A^p_d(\mathbb{D}) \to H^p(\mathbb{T}^d), \quad (E_d g)(z_1, ..., z_d) = \sum_{j=0}^{\infty} \frac{a_j}{c_d(j)} \sum_{\alpha_1 + ... + \alpha_d = j} z_1^{\alpha_1} z_2^{\alpha_2} ... z_d^{\alpha_d}.$$

Then D_d and E_d are bounded operators [5].

Consider $\varphi_d(w) = wd$. We want to calculate the dual norm

$$\|\varphi_d\|_{(A^1_d(\mathbb{D}))^*} = \sup_{f \in A^1_d(\mathbb{D})} \frac{|\langle f, \varphi_d \rangle_{A^2_d(\mathbb{D})}|}{\|f\|_{A^1_d(\mathbb{D})}}.$$

Calculation with $f(w) = \sum_{j \geq 0} a_j w^j$ gives, by orthogonality, that

$$\langle f, \varphi_d \rangle_{A^2_d(\mathbb{D})} = a_1 d \int_{\mathbb{D}} |w|^2 (d-1)(1-|w|^2)^{d-2} \frac{dA(w)}{\pi} = a_1,$$

since $c_d(1) = d$. From the triangle inequality we get

$$a_1 r = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{i\theta} d\theta \quad \Rightarrow \quad |a_1| r \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta.$$

Now, multiplication with $2r(d-1)(1-|w|^2)^{d-2}$ and integration from 0 to 1 yields

$$|a_1| \int_0^1 2r^2 (d-1)(1-|w|^2)^{d-2} dr \leq \|f\|_{A^1_d(\mathbb{D})}.$$

The dual norm becomes

$$\|\varphi_d\|_{(A^1_d(\mathbb{D}))^*} = \sup_{f \in A^1_d(\mathbb{D})} \frac{|\langle f, \varphi_d \rangle_{A^2_d(\mathbb{D})}|}{\|f\|_{A^1_d(\mathbb{D})}} = \sup_{f \in A^1_d(\mathbb{D})} \frac{|a_1|}{\|f\|_{A^1_d(\mathbb{D})}} \leq \frac{\Gamma(d+1/2)}{\Gamma(3/2)\Gamma(d)},$$

and by choosing $f = a_1 w$ we obtain equality, proving that the norm is indeed the upper bound on the right hand side.

We find that

$$\|H_{\varphi_d}\|_{\mathcal{S}_r(A^2_d(\mathbb{D}) \times A^2_d(\mathbb{D}))} = 2^{1/r} \sqrt{d}.$$

By Stirling's formula we get

$$\lim_{d\to\infty} \frac{\|\varphi_d\|_{(A_d^1(\mathbb{D}))^*}}{\|H_{\varphi_d}\|_{\mathcal{S}_r(A_d^2(\mathbb{D})\times A_d^2(\mathbb{D}))}} = \frac{2}{2^{1/r}\sqrt{\pi}} \tag{6}$$

and rearranging shows that this is greater than 1 for $r > r_0 = (1 - \ln\pi/\ln 4)^{-1}$.

This is the exact same value r_0 obtained from the counterexample in two dimensions by Brevig and Perfekt [4]. It is also clear that their example $\varphi(z) = z_1 + \cdots + z_d$ is equal to $(E_d\varphi_d)(z)$. This observation leads us to investigate this relationship.

We know (from [3], Lemma 5) that

$$\|E_d\varphi_d\|_{(H^1(\mathbb{T}^d))^*} = \frac{|\langle E_d\varphi_d, E_d\varphi_d\rangle_{H^2(\mathbb{T}^d)}|}{\|E_d\varphi_d\|_{H^1(\mathbb{T}^d)}} = \frac{d}{\|E_d\varphi_d\|_{H^1(\mathbb{T}^d)}}.$$

For $d = 2$ we get from equation (4) that

$$\|E_2\varphi_2\|_{(H^1(\mathbb{T}^2))^*} = \frac{\pi}{2} > \frac{3}{2} = \|\varphi_2\|_{(A_2^1(\mathbb{D}))^*}.$$

For $d = 3$ we get by a result in [2] that

$$\|E_3\varphi_3\|_{(H^1(\mathbb{T}^3))^*} = 3\Big[\frac{3}{16}\frac{\sqrt[3]{2}}{\pi^4}\Gamma^6\Big(\frac{1}{3}\Big) + \frac{27}{4}\frac{\sqrt[3]{4}}{\pi^4}\Gamma^6\Big(\frac{2}{3}\Big)\Big]^{-1} > \frac{15}{8} = \|\varphi_3\|_{(A_3^1(\mathbb{D}))^*}.$$

In addition the central limit theorem [4] gives us

$$\lim_{d\to\infty} \frac{\|E_d\varphi_d\|_{(H^1(\mathbb{T}^d))^*}}{\sqrt{d}} = \lim_{d\to\infty} \frac{\sqrt{d}}{\|E_d\varphi_d\|_{H^1(\mathbb{T}^d)}} = \frac{2}{\sqrt{\pi}},$$

so by (6), $\|E_d\varphi_d\|_{(H^1(\mathbb{T}^d))^*}$ and $\|\varphi_d\|_{(A_d^1(\mathbb{D}))^*}$ are equal in the limit $d \to \infty$. It is therefore fair to assume that

$$\|E_d\varphi\|_{(H^1(\mathbb{T}^d))^*} \geq \|\varphi\|_{(A_d^1(\mathbb{D}))^*} \tag{7}$$

holds for all φ_d. Whether it holds for all φ is interesting and would have desirable consequences for those looking for $H_\varphi \in \mathcal{S}_r(H^2(\mathbb{T}^\infty)\times H^2(\mathbb{T}^\infty))$ without a bounded symbol for $r < r_0$.

The following is known.

Lemma 1. $\|H_\varphi\|_{S_r(A_d^2(\mathbb{D})\times A_d^2(\mathbb{D}))} = \|H_{E_d\varphi}\|_{S_r(H^2(\mathbb{T}^d)\times H^2(\mathbb{T}^d))}$.

Proof. From Lemma 19 in [1] we have that all singular values of H_φ and $H_{E_d\varphi}$ are equal, and the statement of the lemma follows immediately. □

Therefore, if we assume (7) for all φ with $H_\varphi \in \mathcal{S}_r(A_d^2(\mathbb{D}) \times A_d^2(\mathbb{D}))$ we get

$$\frac{\|E_d\varphi\|_{(H^1(\mathbb{T}^d))^*}}{\|H_{E_d\varphi}\|_{\mathcal{S}_r(H^2(\mathbb{T}^d)\times H^2(\mathbb{T}^d))}} \geq \frac{\|\varphi\|_{(A_d^1(\mathbb{D}))^*}}{\|H_\varphi\|_{\mathcal{S}_r(A_d^2(\mathbb{D})\times(A_d^2(\mathbb{D}))}}.$$

Thus if we can find such φ and $f \in A_d^1(\mathbb{D})$ satisfying

$$\frac{|\langle f, \varphi\rangle_{A_d^2(\mathbb{D})}|}{\|H_\varphi\|_{\mathcal{S}_r(A_d^2(\mathbb{D})\times(A_d^2(\mathbb{D}))}\|f\|_{A_d^1(\mathbb{D})}} > 1 \tag{8}$$

we would get the contradiction (3), proving that there are $H_{E_d\varphi} \in \mathcal{S}_r(H^2(\mathbb{T}^d) \times H^2(\mathbb{T}^d))$ without a bounded symbol. In this case looking for counterexamples in one dimension would make sense, especially because it is significantly easier.

3.1. **Numerical estimates.** Even though (7) is not proven we can look for one dimensional polynomials φ and f satisfying (8) in order to guess polynomials to test in (3). This makes sense since calculations are significantly simpler in one dimension.

However, numerical estimation with $\varphi(w) = \sum_{j=0}^{J} \rho_j w^j$ and $f(w) = \sum_{j=0}^{2J-1} a_j w^j$ for $J < 6$, random coefficients and $2 \leq d < 100$ gave no immediate results with the inequality (8) satisfied for $r < r_0$. This, despite the fact that the estimate for $\|\varphi\|_{(A_d^1(\mathbb{D}))^*}$ done by

$$\|\varphi\|_{(A_d^1(\mathbb{D}))^*} \geq \frac{|\langle f, \varphi \rangle_{A_d^2(\mathbb{D})}|}{\|f\|_{A_d^1(\mathbb{D})}}$$

for random polynomials f, quickly converged to a value, which can fairly be assumed close to the actual value.

In conclusion, it proved difficult, with the method used, to find other polynomials than $f(w) = \varphi(w) = w$ satisfying the inequality (8) giving the sought contradiction $r < r_0$. This might indicate that if such polynomials in fact exist, they might in some sense be special. A numerical optimisation approach could be interesting and definitely possible because of the smoothness of the norms.

3.2. **Operator norms.** Let $\|E_d\|_p$ be the norm of $E_d : A_d^p(\mathbb{D}) \to H^p(\mathbb{T}^d)$ and similarly for $\|D_d\|_p$. Deciding the norms of the operators D_d and E_d might help to establish (7). That D_d has norm 1 when $p = 2$ is easily proved by use of the Cauchy-Schwarz inequality [1]. It follows that D_d is a contraction for all even integers p, and that E_d has norm 1 for $p = 2$. It is also known that for $p < 2$, $\|D_d\|_p > 1$ [1].

It is easily proven that D_d and E_d are adjoint operators for $p = 2$ and that $D_d \circ E_d$ is the identity operator. Therefore we get

$$\begin{aligned} \|E_d\varphi\|_{(H^1(\mathbb{T}^d))^*} &= \sup_{f \in H^1(\mathbb{T}^d)} \frac{|\langle f, E_d\varphi \rangle_{H^2(\mathbb{T}^d)}|}{\|f\|_{H^1(\mathbb{T}^d)}} \geq \sup_{g \in A_d^1(\mathbb{D})} \frac{|\langle E_d g, E_d\varphi \rangle_{H^2(\mathbb{T}^d)}|}{\|E_d g\|_{H^1(\mathbb{T}^d)}} \\ &= \sup_{g \in A_d^1(\mathbb{D})} \frac{|\langle g, \varphi \rangle_{A_d^2(\mathbb{D})}|}{\|E_d g\|_{A_d^1(\mathbb{D})}} \geq \frac{1}{\|E_d\|_1} \|\varphi\|_{(A_d^1(\mathbb{D}))^*}. \end{aligned}$$

Thus, if $\|E_d\|_1 = 1$ we have proven (7). This observation leads us to attempt to determine the operator norms of D_d and E_d.

Numerical estimation of the norms, with up to fifth degree polynomials for $g(w)$ and $(D_d f)(w)$, suggests the following conjectures

$$\|D_d\|_p = 1, \quad p \geq 2 \quad \text{and} \quad \|E_d\|_p > 1, \quad p \neq 2.$$

The last conjecture is proven in the following Theorem.

Theorem 2. *If $p \neq 2$ then $\|E_d\|_p > 1$.*

Proof. We will make use of the following binomial expansion, valid for $|z| < 1$,

$$(1+z)^{p/2} = \sum_{j=0}^{\infty} \binom{p/2}{j} z^j. \tag{9}$$

Consider the function $g(w) = 1 + \lambda w$, with $\lambda < 1$. We get (with (9)) that

$$\begin{aligned}
\|g\|^p_{A^p_d(\mathbb{D})} &= \frac{1}{\pi}\int_0^1\int_0^{2\pi} |1+\lambda r e^{i\theta}|^p (d-1)(1-r^2)^{d-2} r\,d\theta\,dr \\
&= \sum_{j=0}^{\infty} \binom{p/2}{j}^2 \lambda^{2j} \cdot 2(d-1)\int_0^1 r^{2j+1}(1-r^2)^{d-2}dr \\
&= \sum_{j=0}^{\infty} \binom{p/2}{j}^2 \frac{\Gamma(d)\Gamma(j+1)}{\Gamma(d+j)}\lambda^{2j} \\
&= 1 + \Big(\frac{p}{2}\Big)^2 \frac{1}{d}\lambda^2 + \left(\frac{p/2(p/2-1)}{2}\right)^2 \frac{2}{d(d+1)}\lambda^4 + O(\lambda^6).
\end{aligned}$$

Note that $p/2 - 1 = 0$ if and only if $p = 2$. The other norm becomes (by orthogonality)

$$\begin{aligned}
\|E_d g\|^p_{H^p(\mathbb{T}^d)} &= \int_{\mathbb{T}^d} \left|1 + \lambda d^{-1}\sum_{k=1}^d z_k\right|^p dm_d(z) \\
&= \sum_{j=0}^{\infty} \binom{p/2}{j}^2 \lambda^{2j} d^{-2j} \int_{\mathbb{T}^d} \left|\Big(\sum_{k=1}^d z_k\Big)^j\right|^2 dm_d(z) \\
&= \sum_{j=0}^{\infty} \binom{p/2}{j}^2 \lambda^{2j} d^{-2j} \left\|\Big(\sum_{k=1}^d z_k\Big)^j\right\|^2_{H^2(\mathbb{T}^d)}.
\end{aligned}$$

The H^2-norm reduces, again by orthogonality, to the sum of the squared coefficients, so for $j = 2$ we get

$$\begin{aligned}
\left\|\Big(\sum_{k=1}^d z_k\Big)^2\right\|^2_{H^2(\mathbb{T}^d)} &= \left\|\sum_{k=1}^d z_k^2 + 2\sum_{k=1}^d\sum_{l=k+1}^d z_k z_l\right\|^2_{H^2(\mathbb{T}^d)} \\
&= d(1^2) + \frac{(d-1)d}{2}(2^2) = d(2d-1).
\end{aligned}$$

The norm then becomes

$$\|E_d g\|^p_{H^p(\mathbb{T}^d)} = 1 + \frac{p}{2}^2 d^{-2} d\lambda^2 + \left(\frac{p/2(p/2-1)}{2}\right)^2 d^{-4} d(2d-1)\lambda^4 + O(\lambda^6).$$

Since

$$\frac{2d-1}{d^3} > \frac{2}{d^2+d}, \quad \text{when} \quad d > 1,$$

by letting $\lambda \to 0$, we can conclude that E_d is a contraction only for $p = 2$. □

This result does not disprove (7), but for now we are unable to prove that inequality.

Defining the operator

$$T_d := E_d \circ D_d$$

we see that $T_d T_d = T_d$ and T_d is self-adjoint for $p = 2$ so $\|T_d\|_2 = 1$. We will also look into the properties of this norm for various p.

Numerical computation indicates that

$$\|T\|_p = 1, \quad p \geq 2 \quad \text{and} \quad \|T\|_p > 1, \quad p < 2.$$

Applying T to the function

$$f(z_1, z_2) = 1 + \lambda_1(z_1 + z_2) + \lambda_2 z_1 z_2$$

should, according to the numerical calculations, provide an example proving $\|T\|_p > 1$ for $p < 2$. This should also be the case for coefficients such that $2\lambda_1 + \lambda_2 < 1$, allowing the use of (9). Still, the exact calculation is tedious and, as far as we know, of little significance. It is therefore not performed.

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