Peakon solutions to the Camassa–Holm equation I.

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1 Introduction

The Camassa–Holm equation

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad \kappa \geq 0$$

models propagation of unidirectional gravitational waves in shallow water. In the equation u represents the fluid velocity. Some of the interesting properties of the Camasa–Holm equation are that it is completely integrable and experiences wave breaking in finite time for a large class of initial data [1]. In the special case when $\kappa = 0$, that is,

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1}$$

the equation admits solutions of the form

$$u(t, x) = \sum_{i=1}^{n} p_i(t) e^{-|x - q_i(t)|}, \tag{2}$$

which are called multipeakons [2]. When looking for solutions of this form, one should expect to find only local solutions. In general, a solution is no longer unique after a wave break occurs.

The goal of this project was to investigate four-peakon solutions of (1), that is, solutions of the form

$$u(t, x) = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|} + p_3(t)e^{-|x-q_3(t)|} + p_4(t)e^{-|x-q_4(t)|},$$
$$q_1(t) < q_2(t) < q_3(t) < q_4(t), \quad t \geq 0,$$

satisfying the following restriction on the initial condition

$$u(0, x) \begin{cases} = 0, & x \in [q_2(0), q_3(0)] \\ \neq 0, & x \notin [q_2(0), q_3(0)] \end{cases} \tag{3}$$

Even though condition (3) is very specific, it resulted in computations too demanding for the scope of this project. In order to simplify the problem further, another assumption was added to the initial condition: that it must be antisymmetric.

$$u(0,x) = u(0,-x), \quad \text{for all } x. \tag{4}$$

This assumption leads to

$$\frac{d}{dt}u(0,x) = -\frac{d}{dt}u(0,x).$$

Assuming that $p_i(0) \neq 0$ for $i = 1,2,3,4$, we have that $\frac{d}{dt}u(0,x)$ is undefined if and only if $x \in \{q_1(0), q_2(0), q_3(0), q_4(0)\}$. Using our assumption that $q_1(0) < q_2(0) < q_3(0) < q_4(0)$ we must have that $q_1(0) = -q_4(0)$ and $q_2(0) = -q_3(0)$. Inserting this into (4) we also get that $p_1(0) = -p_4(0)$ and $p_2(0) = -p_3(0)$. In summary we get

$$\begin{aligned} q_1(0) &= -q_4(0) \qquad & p_1(0) &= -p_4(0) \\ q_2(0) &= -q_3(0) \qquad & p_2(0) &= -p_3(0) \end{aligned} \tag{5}$$

This assumption simplifies the computations significantly. It can also be shown that an antisymmetric initial condition always results in a solution which is antisymmetric for all values of t.

2 Computations

2.1 Initial condition

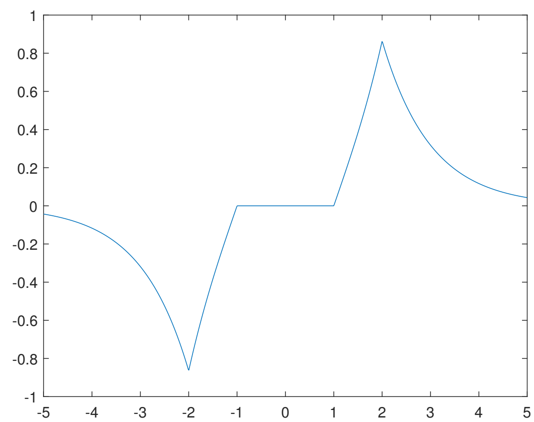

Figure 1: Example of an initial condition which satisfies (3) and (5).

We are looking for the set of all initial conditions that satisfy (3) and (5). Let $c =$

$u(0, q_1(0))$ be the height of the leftmost peak. Then we have

$$u(0, q_1(0)) = p_1(0) + p_2(0)e^{q_1(0)-q_2(0)} - p_2(0)e^{q_1(0)+q_2(0)} - p_1(0)e^{2q_1(t)} = c,$$
$$u(0, q_2(0)) = p_1(0)e^{q_1(0)-q_2(0)} + p_2(0) - p_2(0)e^{2q_2(0)} - p_1(0)e^{q_1(0)+q_2(0)} = 0.$$

This can be rewritten as

$$p_1(0)e^{q_1(0)}(e^{-q_1(0)} - e^{q_1(0)}) + p_2(0)e^{q_1(0)}(e^{-q_2(0)} - e^{q_2(0)}) = c, \tag{6}$$
$$p_1(0)e^{q_1(0)}(e^{-q_2(0)} - e^{q_2(0)}) + p_2(0)e^{q_2(0)}(e^{-q_2(0)} - e^{q_2(0)}) = 0. \tag{7}$$

First we determine $p_1(0)$. Therefore multiply (6) with $e^{q_2(0)}$ and (7) with $e^{q_1(0)}$. Taking the difference yields

$$\begin{aligned} &e^{q_2(0)} \cdot (6) - e^{q_1(0)} \cdot (7) \\ &= p_1(0)e^{q_1(0)+q_2(0)}(e^{-q_1(0)} - e^{q_1(0)}) - p_1(0)e^{2q_1(0)}(e^{-q_2(0)} - e^{q_2(0)}) \\ &= ce^{q_2(0)}. \end{aligned}$$

After solving for $p_1(0)$ and simplifying we get

$$p_1(0) = ce^{q_2(0)}\big(e^{q_2(0)} - e^{2q_1(0)-q_2(0)}\big)^{-1}. \tag{8}$$

Similarly, we determine $p_2(0)$. Multiply (6) with $(e^{-q_2(0)} - e^{q_2(0)})$ and (7) with $(e^{-q_1(0)} - e^{q_1(0)})$. Taking the difference yields

$$\begin{aligned} &(e^{-q_2(0)} - e^{q_2(0)}) \cdot (6) - (e^{-q_1(0)} - e^{q_1(0)}) \cdot (7) \\ &= p_2(0)e^{q_1(0)}(e^{-q_2(0)} - e^{q_2(0)})^2 - p_2(0)e^{q_2(0)}(e^{-q_2(0)} - e^{q_2(0)})(e^{-q_1(0)} - e^{q_1(0)}) \\ &= c(e^{-q_2(0)} - e^{q_2(0)}). \end{aligned}$$

After solving for $p_2(0)$ and simplifying we get

$$p_2(0) = c\big(e^{q_1(0)-q_2(0)} - e^{q_2(0)-q_1(0)}\big)^{-1}. \tag{9}$$

Now we introduce the restriction given in (3). Suppose $q_2(0) \leq x \leq -q_2(0)$. Then we have that

$$u(0, x) = p_1(0)e^{q_1(0)-x} + p_2(0)e^{q_2(0)-x} - p_2(0)e^{q_2(0)+x} - p_1(0)e^{q_1(0)+x} = 0,$$

which implies that

$$\frac{d}{dx}u(0, x) = \underbrace{-p_1(0)e^{q_1(0)-x} - p_2(0)e^{q_2(0)-x}}_{=0} - \underbrace{p_2(0)e^{q_2(0)+x} - p_1(0)e^{q_1(0)+x}}_{=0} = 0,$$

(since the first part and the second part are linearly independent, they must both be zero). After multiplying the first part by e^x we obtain

$$p_1(0)e^{q_1(0)} + p_2(0)e^{q_2(0)} = 0. \tag{10}$$

Now we insert conditions (8) and (9) into (10) and obtain

$$
\begin{aligned}
0 &= ce^{q_1(0)+q_2(0)}\left(e^{q_2(0)} - e^{2q_1(0)-q_2(0)}\right)^{-1} + ce^{q_2(0)}\left(e^{q_1(0)-q_2(0)} - e^{q_2(0)-q_1(0)}\right)^{-1} \\
&= \frac{ce^{q_2(0)}}{e^{q_2(0)-q_1(0)} - e^{q_1(0)-q_2(0)}} - \frac{ce^{q_2(0)}}{e^{q_2(0)-q_1(0)} - e^{q_1(0)-q_2(0)}}.
\end{aligned}
$$

This means that once c is fixed, then $q_1(0) < q_2(0)$ can be chosen freely, however $p_1(0)$ and $p_2(0)$ are fully determined by this choice.

2.2 System of equations

It can be shown that any multipeakon solution must satisfy the following system of ODEs [5]:

$$
\begin{aligned}
\dot{q}_i &= \sum_{j=1}^{n} p_j e^{-|q_i-q_j|}, \\
\dot{p}_i &= \sum_{j=1}^{n} p_i p_j \operatorname{sgn}(i-j) e^{-|q_i-q_j|}
\end{aligned}
$$

In our case, the above system reduces to

$$\dot{q}_1 = p_1 + p_2 e^{-|q_1-q_2|} - p_2 e^{-|q_1+q_2|} - p_1 e^{-|2q_1|} \tag{11}$$

$$\dot{q}_2 = p_1 e^{-|q_2-q_1|} + p_2 - p_2 e^{-|2q_2|} - p_1 e^{-|q_1+q_2|} \tag{12}$$

$$\dot{p}_1 = -p_1 p_2 e^{-|q_2-q_1|} + p_1 p_2 e^{-|q_1+q_2|} + p_1^2 e^{-|2q_1|} \tag{13}$$

$$\dot{p}_2 = p_1 p_2 e^{-|q_1-q_2|} + p_2^2 e^{-|2q_2|} + p_1 p_2 e^{-|q_1+q_2|} \tag{14}$$

From this point on we will be working with this system.

2.3 Outer peakons

In this section we investigate the behaviour of the two outer peakons in our initial condition (positioned at $q_1(t)$ and $q_4(t)$ respectively). In more precise terms, we will be looking at the relationship bewteen p_1 and q_1. We start by computing

$$
\begin{aligned}
\dot{q}_1 p_1 &= p_1^2 + \underbrace{p_1 p_2 e^{-|q_1-q_2|} - p_1 p_2 e^{-|q_1+q_2|} - p_1^2 e^{-|2q_1|}}_{-\dot{p}_1} \\
&= p_1^2 - \dot{p}_1.
\end{aligned}
$$

Regrouping the terms we get

$$\frac{\dot{p}_1}{p_1^2} + \frac{\dot{q}_1}{p_1} = 1.$$

Introducing $v = 1/p_1$, we obtain

$$\dot{v} - \dot{q}_1 v = -1. \tag{15}$$

After multiplying (15) by e^{-q_1} we get,

$$\begin{aligned} \dot{v}e^{-q_1} - \dot{q}_1 v e^{-q_1} &= -e^{-q_1}, \\ \frac{d}{dt}\left(ve^{-q_1}\right) &= -e^{-q_1}, \\ ve^{-q_1} = \frac{1}{p_1}e^{-q_1} &= -\int_0^t e^{-q_1(\tau)}d\tau + E, \end{aligned}$$

where E is some constant. Solving for p_1 we conclude with

$$p_1(t) = \frac{e^{-q_1(t)}}{-\int_0^t e^{-q_1(\tau)}d\tau + E}. \tag{16}$$

2.4 Inner peakons

In this section we consider the two inner peakons (positioned at $q_2(t)$ and $q_3(t)$ respectively). We start by rewriting (11) to the form

$$p_2(e^{-q_2} - e^{q_2}) = \dot{q}_1 e^{-q_1} + p_1(e^{q_1} - e^{-q_1}). \tag{17}$$

We compute the derivative of the left hand side, and using (14) we obtain

$$\frac{d}{dt}\big(p_2(e^{-q_2} - e^{q_2})\big) = -p_2^2(e^{-q_2} - e^{q_2}).$$

Using (17) we conclude that

$$p_2 = -\frac{\frac{d}{dt}\big(p_2(e^{-q_2} - e^{q_2})\big)}{p_2(e^{-q_2} - e^{q_2})} = -\frac{\frac{d}{dt}\big(\dot{q}_1 e^{-q_1} + p_1(e^{q_1} - e^{-q_1})\big)}{\dot{q}_1 e^{-q_1} + p_1(e^{q_1} - e^{-q_1})}$$

and

$$q_2 = \sinh^{-1}\left(-\frac{\dot{q}_1 e^{-q_1} + p_1(e^{q_1} - e^{-q_1})}{2p_2}\right).$$

2.5 Ansatz

Notice that at this point we have that p_1, p_2 and q_2 are all expressed in terms of q_1. This means that having a suitable ansatz for q_1, the problem would be solved. From [3] one knows that in the antisymmetric case $q_1(t)$ takes the form

$$q_1(t) = -\ln\left(f(t)\right) \quad \text{where} \quad f(t) = Ae^{at} + Be^{bt} + Ce^{ct} + De^{dt}. \tag{18}$$

Unfortunately the above ansatz contains 8 unknowns which should be determined by 4 equations uniquely if $p_1(0)$, $p_2(0)$, $q_1(0)$ and $q_2(0)$ are given. This is an impossible task. Thus one has to find a possibility to reduce the number of unknowns by 4 by having an in depth look at [3].

The setting in [3]. The equation considered in [3] is given by

$$v_t - \frac{1}{4}v_{txx} + 3vv_x - \frac{1}{2}v_x v_{xx} - \frac{1}{4}vv_{xxx} = 0, \tag{19}$$

which is quite similar to the CH equation. Denote by $v(t,x)$ solutions to (19) and by $u(t,x)$ solutions of the CH equation. Then the relation between those is given by

$$u(t,x) = \sqrt{2}v\left(\frac{1}{\sqrt{2}}t, \frac{1}{2}x\right). \tag{20}$$

In [3] solutions of the following form are constructed

$$v(t,x) = \frac{1}{2}\sum_{j=1}^{m} m_j(t)e^{-2|x-x_j(t)|}.$$

According to (20) the multipeakons of the CH equation are then given by

$$u(t,x) = \frac{1}{\sqrt{2}}\sum_{j=1}^{n} m_j\left(\frac{1}{\sqrt{2}}t\right)e^{-2|\frac{1}{2}x - x_j(\frac{1}{\sqrt{2}}t)|}.$$

On the other hand, we represented them by

$$u(t,x) = \sum_{j=1}^{n} p_j(t)e^{-|x-q_j(t)|}.$$

Hence we must have

$$p_j(t) = \frac{1}{\sqrt{2}}m_j\left(\frac{1}{\sqrt{2}}t\right) \quad \text{and} \quad q_j(t) = 2x_j\left(\frac{1}{\sqrt{2}}t\right).$$

The antisymmetric 2 peakon case. In this case we must have that $q_1(t) = -q_2(t)$ and $p_1(t) = -p_2(t)$. So let's have a quick look at what this means in terms of the functions in [3]:

$$\begin{aligned} p_1(t) = -p_2(t) \quad &\Leftrightarrow \quad m_1(t) = -m_2(t) \\ q_1(t) = -q_2(t) \quad &\Leftrightarrow \quad x_1(t) = -x_2(t) \end{aligned}$$

The hard part is to determine how these functions are defined. Given λ_1 and λ_2 one defines

$$a_j(t) = a_j(0)e^{-\frac{2t}{\lambda_j}} = a_j e^{-\frac{2t}{\lambda_j}} \quad \text{where } a_0 = \frac{1}{2} \text{ and } \lambda_0 = 0$$

and subsequently

$$A_k = \sum_{j=0}^{2}(-\lambda_j)^k a_j. \tag{21}$$

In addition,

$$\Delta_k^l \ldots \text{ determinant of the } k \times k \text{ matrix whose } (i,j) \text{ entry is } A_{l+i+j-2}$$

and one defines $\Delta_0^l = 1$. Then one can represent $x_1(t)$ and $x_2(t)$ as follows

$$x_1(t) = \frac{1}{2} \ln \left(\frac{2\tilde{\Delta}_2^0}{\Delta_1^2} \right) \quad \text{and} \quad x_2(t) = \frac{1}{2} \ln \left(\frac{2\tilde{\Delta}_1^0}{\Delta_0^2} \right). \tag{22}$$

Here $\tilde{\Delta}_k^0 = \Delta_k^0 - \frac{1}{2}\Delta_{k-1}^2$. (We don't have to worry about the factor $\frac{1}{2}$ in front since it is going to cancel out.) So lets compute these terms

- $\Delta_0^2 = 1$ according to the definition
- $\Delta_1^2 = A_2$ since the corresponding 1×1 matrix is given by (A_2)
- $\Delta_1^1 = A_1$ since the corresponding 1×1 matrix is given by (A_1)
- $\Delta_2^1 = A_1 A_3 - A_2^2$ since the corresponding 2×2 matrix is given by $\begin{pmatrix} A_1 & A_2 \\ A_2 & A_3 \end{pmatrix}$
- $\Delta_2^0 = A_0 A_2 - A_1^2$ since the corresponding 2×2 matrix is given by $\begin{pmatrix} A_0 & A_1 \\ A_1 & A_2 \end{pmatrix}$
- $\tilde{\Delta}_2^0 = \Delta_2^0 - \frac{1}{2}\Delta_1^2 = A_0 A_2 - A_1^2 - \frac{1}{2}A_2$

Combining (25) and (22) leads to

$$x_1(t) = \frac{1}{2} \ln \left(\frac{2(\lambda_1 - \lambda_2)^2 a_1(t) a_2(t)}{\lambda_1^2 a_1(t) + \lambda_2^2 a_2(t)} \right) \quad \text{and} \quad x_2(t) = \frac{1}{2} \ln \left(2(a_1(t) + a_2(t)) \right). \tag{23}$$

Similar considerations yield that

$$m_1(t) = \frac{2\tilde{\Delta}_2^0 \Delta_1^2}{\Delta_2^1 \Delta_1^1} = -2 \frac{\lambda_1^2 a_1(t) + \lambda_2^2 a_2(t)}{\lambda_1 \lambda_2 (\lambda_1 a_1(t) + \lambda_2 a_2(t))}$$

and

$$m_2(t) = \frac{2\tilde{\Delta}_1^0 \Delta_0^2}{\Delta_1^1 \Delta_0^1} = -2 \frac{a_1(t) + a_2(t)}{\lambda_1 a_1(t) + \lambda_2 a_2(t)}.$$

Recall that $\lambda_1 \neq \lambda_2$ and that $a_j(t) = a_j e^{-\frac{2t}{\lambda_j}}$. Then $m_1(t) = -m_2(t)$ is equivalent with

$$\frac{1}{\lambda_1 a_1(t) + \lambda_2(t)} \left(\lambda_1 \lambda_2 (a_1(t) + a_2(t)) + \lambda_1^2 a_1(t) - \lambda_2^2 a_2(t) \right) = 0$$

If this should be true for all t, then this means that the second term on the left hand side must equal zero, which yields

$$\lambda_1(\lambda_1 + \lambda_2) a_1(t) = -\lambda_2(\lambda_1 + \lambda_2) a_2(t).$$

Due to the fact that $\lambda_1 \neq \lambda_2$ and the definitions of $a_1(t)$ and $a_2(t)$, this can only be true for all t if

$$\lambda_1 = -\lambda_2.$$

Reducing the number of unknowns from 4 to 3. The final step is to reduce from 3 to 2 unknowns. Now $x_1(t) - x_2(t)$ is equivalent with

$$16a_1(t)a_2(t) = 1 = 16a_1a_2$$

and hence

$$a_2 = \frac{1}{16a_1}.$$

Thus we have for $x_1(t)$ that

$$\begin{aligned}
x_1(t) &= -\frac{1}{2}\ln\left(2(a_1(t) + a_2(t)\right) \\
&= -\frac{1}{2}\ln(2(a_1e^{-\frac{2t}{\lambda_1}} + a_2e^{\frac{2t}{\lambda_1}}) \\
&= -\frac{1}{2}\ln(2(a_1e^{-\frac{2t}{\lambda_1}} + \frac{1}{16a_1}e^{\frac{2t}{\lambda_1}}) \\
&= -\frac{1}{2}\ln(\frac{1}{2}(4a_1e^{-\frac{2t}{\lambda_1}} + \frac{1}{4a_1}e^{\frac{2t}{\lambda_1}}).
\end{aligned}$$

Recalling that $q_1(t) = 2x_1(\frac{1}{\sqrt{2}}t)$ finally yields that in this case, one would have to choose that

$$f(t) = -\ln(\frac{1}{2}(ce^{-dt} + \frac{c}{e}^{dt}).$$

The antisymmetric 4 peakon case. In this case we must have

$$\begin{aligned}
q_1(t) &= -q_4(t) \quad \Leftrightarrow \quad x_1(t) = -x_4(t) && \text{(24a)} \\
q_2(t) &= -q_3(t) \quad \Leftrightarrow \quad x_2(t) = -x_3(t) && \text{(24b)} \\
p_1(t) &= -p_4(t) \quad \Leftrightarrow \quad m_1(t) = -m_4(t) && \text{(24c)} \\
p_2(t) &= -p_3(t) \quad \Leftrightarrow \quad m_2(t) = -m_3(t). && \text{(24d)}
\end{aligned}$$

The hard part is to determine how these functions are defined. Given λ_1 and λ_2 one defines

$$a_j(t) = a_j(0)e^{-\frac{2t}{\lambda_j}} = a_je^{-\frac{2t}{\lambda_j}} \quad \text{where } a_0 = \frac{1}{2} \text{ and } \lambda_0 = 0$$

and subsequently

$$A_k = \sum_{j=0}^{4}(-\lambda_j)^k a_j. \tag{25}$$

In addition,

$$\Delta_k^l \ldots \text{ determinant of the } k \times k \text{ matrix whose } (i,j) \text{ entry is } A_{l+i+j-2}$$

and one defines $\Delta_0^l = 1$. Then one can represent $x_1(t)$ to $x_4(t)$ as follows

$$x_1(t) = \frac{1}{2}\ln\left(\frac{2\tilde{\Delta}_4^0}{\Delta_3^2}\right) \quad x_2(t) = \frac{1}{2}\ln\left(\frac{2\tilde{\Delta}_3^0}{\Delta_2^2}\right)$$
$$x_3(t) = \frac{1}{2}\ln\left(\frac{2\tilde{\Delta}_2^0}{\Delta_1^2}\right) \quad x_4(t) = \frac{1}{2}\ln\left(\frac{2\tilde{\Delta}_1^0}{\Delta_0^2}\right).$$

Here $\tilde{\Delta}_k^0 = \Delta_k^0 - \frac{1}{2}\Delta_{k-1}^2$. For $m_1(t)$ til $m_4(t)$ we have

$$m_1(t) = \frac{2\tilde{\Delta}_4^0\Delta_3^2}{\Delta_4^1\Delta_3^1} \quad m_2(t) = \frac{2\tilde{\Delta}_3^0\Delta_2^2}{\Delta_3^1\Delta_2^1}$$
$$m_3(t) = \frac{2\tilde{\Delta}_2^0\Delta_1^2}{\Delta_2^1\Delta_1^1} \quad m_4(t) = \frac{2\tilde{\Delta}_1^0\Delta_0^2}{\Delta_1^1\Delta_0^1}.$$

Note that $x_4(t)$ is of the form $x_4(t) = \frac{1}{2}\ln(2(a_1(t) + a_2(t) + a_3(t) + a_4(t)))$, which leads directly to (18), the ansatz for the CH equation. Thus the final task is to reduce the number of unknowns in (18) to 4 with the help of (24). This should be possible by following the same lines as in the 2 peakon case. However the computations are much more involved and several tries led to different results, which is why the project stopped at this point.

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