Computing almost split sequences

A StudForsk project supervised by Øyvind Solberg

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1 Introduction

Almost split sequences is an important concept in the representation theory of finite dimensional algebras. These sequences are often difficult to compute by hand, and it is necessary to develop computational tools for this purpose.

One attempt to construct an algorithm to compute almost split sequences was made by Lian in her Master's Thesis from 2012. Her aim was to compute almost split sequences ending in an indecomposable and non-projective finitely generated module over a specified ring Λ. By [Lia12, Proposition 61] all almost split sequences in $\mathrm{mod}(\Lambda)$ will be of this form. When Marek Trunkat tried to implement Lian's work in the GAP package QPA (Quivers and Path Algebras)[GS16], it turned out that the algorithm did not always give the correct result. The aim for this project has been to figure out why this is the case, and to investigate if it is possible to make the algorithm work as intended.

We start by explaining the idea behind Lian's algorithm, while referring to her thesis. We then proceed to pointing out why the algorithm does not work.

2 Lian's Algorithm

Throughout this section,

1. let k be a field,
2. let Λ be a finite dimensional algebra over k,

3. let X be an indecomposable, non-projective and finitely generated left Λ-module,
4. and let $\Gamma := \underline{\mathrm{End}}_\Lambda(X)$ be the quotient of $\mathrm{End}_\Lambda(X)$ by the submodule of morphisms that factor through projective modules. This is a local ring by [Lia12, Lemma 64].

2.1 Rough sketch of Lian's algorithm

By [Lia12, Theorem 69(ii)], the set of almost split exact sequences

$$0 \longrightarrow D\,\mathrm{Tr}(X) \longrightarrow Y \longrightarrow X \longrightarrow 0$$

over Λ is a finitely generated Γ-module isomorphic to $E := \mathrm{Soc}_\Gamma\left(\mathrm{Ext}^1_\Lambda(X, D\,\mathrm{Tr}(X))\right)$. This is the module we want to compute.

As a special case of the Auslander-Reiten formula there is a Γ-isomorphism

$$\omega_X\colon D\Gamma \to \mathrm{Ext}^1_\Lambda(X, D\,\mathrm{Tr}(X)).$$

This isomorphism is explicitly constructed in Chapter 3 of Lian's thesis. Consequently, there is a Γ-isomorphism

$$\mathrm{Soc}_\Gamma(\omega_X)\colon\ \mathrm{Soc}_\Gamma(D\Gamma) \to E.$$

The Γ-module $\mathrm{Soc}_\Gamma(D\Gamma)$ is simple by [Lia12, Lemma 67]. The idea behind Lian's algorithm is to find a non-zero element $y \in \mathrm{Soc}_\Gamma(D\Gamma)$ and apply ω_X, because then $\omega_X(y)$ is a generator of E.

2.2 The mistake in Lian's thesis

We will now outline Lian's approach to finding a non-zero element in $\mathrm{Soc}_\Gamma(D\Gamma)$. It is this argument that contains the subtle mistake in Lian's thesis. Our outline is not completely faithful to Lian, but the essence of the argument remains the same.

By [Lia12, Lemma 67] there is a natural Γ-isomorphism

$$\mathrm{Soc}_\Gamma(D\Gamma) \cong D\,\mathrm{Top}_{\Gamma^{\mathrm{op}}}(\Gamma),$$

so any non-zero element of $\mathrm{Soc}_\Gamma(D\Gamma)$ corresponds to a non-zero element of $D\,\mathrm{Top}_{\Gamma^{\mathrm{op}}}(\Gamma)$.

Clearly the equivalence class $\overline{1_X} \in \operatorname{Top}_{\Gamma^{\mathrm{op}}}(\Gamma)$ is non-zero. Any choice of k-basis for $\operatorname{Top}_{\Gamma^{\mathrm{op}}}(\Gamma)$ yields a k-isomorphism

$$d\colon \operatorname{Top}_{\Gamma^{\mathrm{op}}}(\Gamma) \to D\operatorname{Top}_{\Gamma^{\mathrm{op}}}(\Gamma),$$

and then $d(\overline{1_X})$ is a non-zero element of $D\operatorname{Top}_{\Gamma^{\mathrm{op}}}(\Gamma)$.

Up to this point everything is logically sound. However, what we have done is computationally expensive because we have computed $\operatorname{Top}_{\Gamma^{\mathrm{op}}}(\Gamma)$, which involves computing the radical of Γ.

To make the algorithm more efficient, Lian did the following. Extend $\{\overline{1_X}\}$ to a basis $\{\overline{1_X}, v_2, \ldots, v_n\}$ for Γ. This yields a k-isomorphism

$$d\colon \Gamma \to D\Gamma$$

Lian claims that $d(\overline{1_X}) \in \operatorname{Soc}_\Gamma(D\Gamma)$. However, this is not true in general:

Proposition 2.1. *We have $d(\overline{1_X}) \in \operatorname{Soc}_\Gamma(D\Gamma)$ if and only if the radical $\underline{r}$ of Γ is contained in* $\operatorname{span}\{v_2, \ldots, v_n\}$.

Proof. By construction of d, we have

$$\ker d(\overline{1_X}) = \operatorname{span}\{v_2, \ldots, v_n\}.$$

And from the proof of [Lia12, Lemma 67] it is clear that

$$\operatorname{Soc}_\Gamma D\Gamma = \{f \in D\Gamma : \underline{r} \subseteq \ker f\}.$$

Combining these facts yields the result. □

Remark 2.2. If $\underline{r} = 0$, then $\underline{r} \subseteq \operatorname{span}\{v_2, \ldots, v_n\}$ trivially. So Lian's algorithm works in this case. In particular, Lian's algorithm works if $\dim_k \Gamma = 1$.

2.3 There is no way around computing the radical

Computing the radical $\underline{r}$ of Γ is expensive, and one advantage of Lian's algorithm is that it does not require the computation of $\underline{r}$. But, as we have seen, Lian's algorithm does not work in general. One might hope that the mistake in Lian's thesis could be "fixed", yielding an efficient algorithm that does not require the radical to be computed. Unfortunately, it turns out that knowing $\operatorname{Soc}_\Gamma D\Gamma$ is equivalent to knowing $\underline{r}$ up to problems of linear complexity. We are now going to describe how to construct $\operatorname{Soc}_\Gamma D\Gamma$ given $\underline{r}$, and the other way around.

Suppose that we know $\underline{r}$. Then we also know $\mathrm{Top}_{\Gamma^{\mathrm{op}}}(\Gamma)$, and we have a canonical projection

$$p\colon \Gamma \to \mathrm{Top}_{\Gamma^{\mathrm{op}}}(\Gamma).$$

Applying D we get a map

$$p^*\colon D\,\mathrm{Top}_{\Gamma^{\mathrm{op}}}(\Gamma) \to D\Gamma,$$

and $\operatorname{im} p^* = \mathrm{Soc}_\Gamma D\Gamma$ (see [Lia12, Lemma 67]).

Conversely, suppose that we know $\mathrm{Soc}_\Gamma D\Gamma$. This is a simple left Γ-module. So if $f \in \mathrm{Soc}_\Gamma D\Gamma$ is any non-zero element, the Γ-homomorphism

$$\begin{aligned} \phi\colon \Gamma &\to \mathrm{Soc}_\Gamma D \\ a &\mapsto a \cdot f \end{aligned}$$

is an epimorphism. Hence we have an isomorphism

$$\Gamma/\ker\phi \cong \mathrm{Soc}_\Gamma D.$$

Since $\mathrm{Soc}_\Gamma D$ is simple, $\ker\phi$ is a maximal left ideal. But Γ has a unique maximal left ideal since it is a local ring, so $\ker\phi = \underline{r}$ is the only possibility.

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References

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[Lia12] Tea Sormbroen Lian. "Computing Almost Split Sequences". MA thesis. Norwegian University of Science and Technology, 2012.