Piecewise linear solutions to the Hunter-Saxton equation

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1 Theory

Let $u = u(x,t)$ satisfy the partial differential equation

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2, \tag{1}$$

where u_t and u_x denote the partial derivative with respect to t and x, respectively. This equation was proposed in 1991 by Hunter and Saxton to model the dynamics of liquid crystals [2]. If we add a continuity constraint

$$\rho_t + (u\rho)_x = 0,$$

along with the term $\rho^2/2$ to the right-hand side of (1), we get the two-component Hunter-Saxton system, as studied by Wunsch [3]. A further generalisation of the described system is to add a constant $\varepsilon > 0$ to the term $\rho^2/2$, which is the one we are interested in.

There are several ways to integrate this system, and we choose the skew-symmetric integration operator

$$D^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right),$$

which applied to the system yields the system considered in this report,

$$\begin{aligned} u_t + uu_x &= \frac{1}{4}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right)(u_x^2 + \varepsilon\rho^2)(y,t)\,\mathrm{d}y \\ \rho_t + (u\rho)_x &= 0, \end{aligned} \tag{2}$$

along with the initial data $u_0(x)$ and $\rho_0(x)$.

We first define a scaling $\tilde{\rho} = \sqrt{\varepsilon}\rho$, and by inserting $\rho = \tilde{\rho}/\sqrt{\varepsilon}$ into (2), we get the system

$$\begin{aligned} u_t + uu_x &= \frac{1}{4}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right)(u_x^2 + \rho^2)(y,t)\,\mathrm{d}y \\ \rho_t + (u\rho)_x &= 0, \end{aligned}$$

studied in [1] by Anders Nordli. Thus we will rely solely on the results from [1] in this report, and adapt them to our system when needed. Furthermore we will state the results we are going to use, and refer to [1] for a more in-depth derivation.

First we introduce Lagrangian coordinates. From a physical perspective this can be interpreted as following a specific particle in the system, defined by the functions

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} q(\xi,t) &= u(q(\xi,t),t),\\
z(\xi,t) &= u(q(\xi,t),t),\\
v(\xi,t) &= u_x(q(\xi,t),t),\\
H(\xi,t) &= \int_{-\infty}^{q(\xi,t)} \left(u_x(y,t)^2 + \varepsilon\rho(y,t)^2\right)\,\mathrm{d}y,\\
\eta(\xi,t) &= \sqrt{\varepsilon}\rho(q(\xi,t),t).
\end{aligned} \tag{3}
$$

If (u,ρ) is a classical solution to (2), then the Lagrangian coordinates satisfy the system of ordinary differential equations

$$
\begin{aligned}
\dot{q} &= z,\\
\dot{z} &= \frac{1}{2}\left(H - \frac{1}{2}H_{tot}\right),\\
\dot{H} &= 0,\\
\dot{\eta} &= -v\eta,\\
\dot{v} &= \frac{1}{2}\left(\eta^2 - v^2\right)
\end{aligned} \tag{4}
$$

as in [1, Proposition 2.3]. This system can be solved, and has the analytic solution

$$
\begin{aligned}
q(\xi,t) &= \frac{1}{4}\left(H_0(\xi) - \frac{1}{2}H_{tot}\right)t^2 + z_0(\xi)t + q_0(\xi),\\
z(\xi,t) &= \frac{1}{2}\left(H_0(\xi) - \frac{1}{2}H_{tot}\right)t + z_0(\xi),\\
H(\xi,t) &= \int_{-\infty}^{\xi} \left(\frac{z_{0\xi}(y)^2}{q_{0\xi}(y)} + \eta_0(y)^2 q_{0\xi}(y)\right)\,\mathrm{d}y,\\
\eta(\xi,t) &= \frac{\eta_0(\xi)}{\left(1+\frac{1}{2}v_0(\xi)t\right)^2 + \left(\frac{1}{2}\eta_0(\xi)t\right)^2},\\
v(\xi,t) &= \frac{v_0(\xi) + \frac{1}{2}\left(\eta_0(\xi)^2 + v_0(\xi)^2\right)t}{\left(1+\frac{1}{2}v_0(\xi)t\right)^2 + \left(\frac{1}{2}\eta_0(\xi)t\right)^2}.
\end{aligned} \tag{5}
$$

Here q_0, z_0, H_0, η_0 and v_0 are the inital functions and

$$
H_{tot} = \int_{-\infty}^{\infty} \left(u_x(y,t)^2 + \varepsilon\rho(y,t)^2\right)\,\mathrm{d}y, \tag{6}
$$

which will be referred to as the total *energy* of the system. We will consider conservative solutions of (2), meaning that (6) is constant for all t, for a given ε. In addition we have $r = \eta q_\xi$, yielding the identity $q_\xi H_\xi = z_\xi^2 + r^2$, as shown in [1, Theorem 2.8].

We are now able to solve (3) given initial functions in the Lagrangian coordinates, however we still need to find these functions given the initial functions $u_0(x)$ and $\rho_0(x)$ of the original system. To that end we define a mapping from Eulerian to Lagrangian coordinates, given as

$$\begin{aligned} q(\xi) &= \sup\{x|\mu((-\infty, x)) + x < \xi\}, \quad \text{for all } \xi \text{ such that } q(\xi) = x \\ H(\xi) &= \xi - q(\xi), \\ z(\xi) &= u \circ q(\xi), \\ r(\xi) &= \sqrt{\varepsilon}(\rho \circ q(\xi))q_\xi(\xi), \end{aligned}$$

where $\mu = (u_x^2 + \varepsilon\rho^2)\,\mathrm{d}x$. In the same way we define the mapping from Lagrangian and back to Eulerian coordinates as

$$\begin{aligned} u(x) &= z(\xi), \\ \rho(x) &= \begin{cases} \frac{r(\xi)}{\sqrt{\varepsilon}q_\xi(\xi)}, & q_\xi(\xi) \neq 0 \\ 0, & q_\xi(\xi) = 0 \end{cases}, \\ \mu &= q_\#(H_\xi \mathrm{d}\xi), \end{aligned} \tag{7}$$

with $q_\#$ being the push-forward of $H_\xi \mathrm{d}\xi$, as defined in [1, Definition 2.11]. For completion we have included the whole part of both mappings, even though we are not interested in retrieving the original measure form (7). The reason for transforming the problem into Lagrangian is that constructing solutions of (2) for some given initial data is too difficult in the Eulerian space. Since u_0 and ρ_0 are known, it is possible to transform the initial Eulerian data to the initial functions of the Lagrangian system. We are then able to construct the variables for all times t using (5), which we then can transform back, acquiring the solution (u, ρ) in the original space.

In the next two subsections we will consider two concrete examples, which we solve using the method described above. We will then inspect what happens in the limit as ε vanishes. The examples are quite similar, but while the first example has an initial function u_0 which is monotonically increasing, u_0 is monotonically decreasing in the second example.

2 Examples

2.1 Example 1

Consider (2) with the initial functions given by

$$u_0(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 \leq x \leq 1 \\ 1, & 1 \leq x \end{cases} \quad \rho_0(x) = \begin{cases} 0, & x \leq 0 \\ 1, & 0 \leq x \leq 1 \\ 0, & 1 \leq x \end{cases}. \tag{8}$$

We start by finding H_{tot}. Since we consider conservative solutions of (2), we have that H_{tot} is constant for all t. Choosing $t = 0$ corresponds to using (8) in (6), and realising that the contribution from the integral is zero everywhere except for $0 \le x \le 1$, we get $H_{tot} = 1 + \varepsilon$. Next we find $q_0(\xi)$. Note that since $\mu((-\infty, x))$ is monotone and x is strictly monotone, then $\mu((-\infty, x)) + x$ is strictly montone. Thus we can find an expression of q_0 by solving $\mu((-\infty, x)) + x = \xi$. We have

$$\mu((-\infty, x)) + x = \begin{Bmatrix} 0, & x \le 0 \\ (1+\varepsilon)x, & 0 \le x \le 1 \\ 1+\varepsilon, & 1 \le x \end{Bmatrix} + x = \begin{Bmatrix} x, & x \le 0 \\ (2+\varepsilon)x, & 0 \le x \le 1 \\ x+1+\varepsilon, & 1 \le x \end{Bmatrix} = \xi,$$

and by inverting we get

$$q_0(\xi) = \begin{cases} \xi, & \xi \le 0 \\ \frac{1}{2+\varepsilon}\xi, & 0 \le \xi \le 2+\varepsilon \\ \xi - (1+\varepsilon), & 2+\varepsilon \le \xi \end{cases}.$$

Furthermore we have

$$H_0(\xi) = \begin{cases} 0, & \xi \le 0 \\ \frac{1+\varepsilon}{2+\varepsilon}\xi, & 0 \le \xi \le 2+\varepsilon \\ 1+\varepsilon, & 2+\varepsilon \le \xi \end{cases}$$

and

$$z_0(\xi) = \begin{cases} 0, & \xi \le 0 \\ \frac{1}{2+\varepsilon}\xi, & 0 \le \xi \le 2+\varepsilon \\ 1, & 2+\varepsilon \le \xi \end{cases}.$$

With these initial functions we can now find expressions for q and z in (5), yielding

$$q(\xi,t) = \begin{cases} -\frac{1+\varepsilon}{8}t^2 + \xi, & \xi \le 0 \\ \frac{1+\varepsilon}{4}\left(\frac{1}{2+\varepsilon}\xi - \frac{1}{2}\right)t^2 + \frac{1}{2+\varepsilon}\xi(t+1), & 0 \le \xi \le 2+\varepsilon \\ \frac{1+\varepsilon}{8}t^2 + t + \xi - (1+\varepsilon), & 2+\varepsilon \le \xi \end{cases},$$

$$z(\xi,t) = \begin{cases} -\frac{1+\varepsilon}{4}t, & \xi \le 0 \\ \frac{1+\varepsilon}{2}\left(\frac{1}{2+\varepsilon}\xi - \frac{1}{2}\right)t + \frac{1}{2+\varepsilon}\xi, & 0 \le \xi \le 2+\varepsilon \\ \frac{1+\varepsilon}{4}t + 1, & 2+\varepsilon \le \xi \end{cases}. \tag{9}$$

Finally, using (7), we transform back to Eulerian coordinates, arriving at the solution

$$u_\varepsilon(x,t) = \begin{cases} -\frac{1+\varepsilon}{4}t, & x \le -\frac{1+\varepsilon}{8}t^2 \\ \frac{1}{2}\frac{-(1+\varepsilon)t^2 + 2(1+\varepsilon)(2x-1)t + 8x}{(1+\varepsilon)t^2 + 4(1+t)}, & -\frac{1+\varepsilon}{8}t^2 \le x \le \frac{1+\varepsilon}{8}t^2 + t + 1 \\ \frac{1+\varepsilon}{4}t + 1, & \frac{1+\varepsilon}{8}t^2 + t + 1 \le x \end{cases}.$$

To find $\eta(\xi,t)$ we need η_0 and v_0. From (3) we have $v_0(\xi) = u_{0x}(q_0(\xi))$, and by observing that in this example $u_{0x} = \rho_0$, we have that $\eta_0(\xi) = \sqrt{\varepsilon}v_0(\xi)$. v_0 will be 1 when $0 \le \xi \le 2+\varepsilon$, and 0 everywhere else. Finding η_0 is then trivial, and inserting this in the formula for $\eta(\xi,t)$ in (5), we get

$$\eta(\xi,t) = \begin{cases} 0, & \xi \le 0 \\ \frac{4\sqrt{\varepsilon}}{(1+\varepsilon)t^2 + 4(1+t)}, & 0 \le \xi \le 2+\varepsilon \\ 0, & 2+\varepsilon \le \xi \end{cases}.$$

The function values are independent of ξ, so transforming back to x we just need to invert the intervals, as done in u_ε. We then get

$$\rho_\varepsilon(x,t) = \begin{cases} 0, & x \le -\frac{1+\varepsilon}{8}t^2 \\ \frac{4}{(1+\varepsilon)t^2+4(1+t)}, & -\frac{1+\varepsilon}{8}t^2 \le x \le \frac{1+\varepsilon}{8}t^2+t+1 \\ 0, & \frac{1+\varepsilon}{8}t^2+t+1 \le x \end{cases}.$$

Previously we calculated $H_{tot} = 1+\varepsilon$ using the initial functions, and we claimed that it should hold for all times. Inserting u_ε and ρ_ε in (6), we get

$$H_{tot} = \int_{-\frac{1+\varepsilon}{8}t^2}^{\frac{1+\varepsilon}{8}t^2+t+1} \frac{\left(2(1+\varepsilon)t+4\right)^2+\varepsilon 4^2}{\left((1+\varepsilon)t^2+4(1+t)\right)^2}\,\mathrm{d}y = 4\frac{(1+\varepsilon)^2t^2+4(1+\varepsilon)t+4+4\varepsilon}{\left((1+\varepsilon)t^2+4(1+t)\right)^2}\left(\frac{1+\varepsilon}{4}t^2+t+1\right)$$

$$= (1+\varepsilon)\frac{(1+\varepsilon)t^2+4(1+t)}{\left((1+\varepsilon)t^2+4(1+t)\right)^2}\Big((1+\varepsilon)t^2+4(1+t)\Big) = 1+\varepsilon.$$

That u_ε is a weak solution has been shown in [1]. It is essential that u_ε is not only piecewise linear, but also continuous. We want to look at the limit function $u(x,t) = \lim_{\varepsilon\to 0} u_\varepsilon(x,t)$, and how it relates to the energy conserving Hunter-Saxton equation, where ρ, and hence r, are identically zero. In this case the continuum equation is trivially satisfied, and we will consider weak solutions of (1).

Letting $\varepsilon \to 0$ in (9) yields

$$q(\xi,t) = \begin{cases} -\frac{1}{8}t^2+\xi, & \xi \le 0 \\ \frac{1}{8}\left(\xi-1\right)t^2+\frac{1}{2}\xi(t+1), & 0 \le \xi \le 2 \\ \frac{1}{8}t^2+t+\xi-1, & 2 \le \xi \end{cases},$$

$$z(\xi,t) = \begin{cases} -\frac{1}{4}t, & \xi \le 0 \\ \frac{1}{4}\left(\xi-1\right)t+\frac{1}{2}\xi, & 0 \le \xi \le 2 \\ \frac{1}{4}t+1, & 2 \le \xi \end{cases}. \tag{10}$$

We need to check whether the above equation satisfies (4). The first equation in (4) is satisfied, which is seen by comparing $\dot{q}$ and z piecewise. For the second equation we need to compute H, which we find using (5),

$$H(\xi,t) = \begin{cases} 0, & \xi \le 0 \\ \frac{1}{2}\xi, & 0 \le \xi \le 2 \\ 1, & 2 \le \xi \end{cases}.$$

In addition, since $\varepsilon = 0$, $H_{tot} = 1$. Using this and calculating $\dot{z}$ in (10), we see that this equation is also satisfied. H is independent of t, so the third equation in (10) is trivially satisfied. Lastly we need to check that $q_\xi H_\xi = z_\xi^2 + r^2$ holds. Since r is implicitly defined in terms of ρ, which is zero, then $r = 0$. $H_\xi = 1/2$ and $z_\xi = t/4 + 1/2$ for $0 \le \xi \le 2$ and zero everywhere else, so we need only to show that

$$\frac{1}{2}q_\xi = z_\xi^2 \text{ for } 0 \le \xi \le 2.$$

Calculating q_ξ in this interval gives

$$\frac{1}{2}q_\xi = \frac{1}{16}t^2 + \frac{1}{4}t + \frac{1}{4} = \frac{1}{4}\left(\frac{1}{2}t+1\right)^2 = z_\xi^2 \text{ for } 0 \le \xi \le 2,$$

where the last equality can be easily verified from (10). This proves the claim that the limit function u_ε as $\varepsilon \to 0$ is a weak conservative solution of (1). The expression for u_ε is in this case

$$\lim_{\varepsilon\to 0} u_\varepsilon(x,t) = \begin{cases} -\frac{1}{4}t, & x \le -\frac{1}{8}t^2 \\ \frac{1}{2}\frac{-t^2+2(2x-1)t+8x}{t^2+4(1+t)}, & -\frac{1}{8}t^2 \le x \le \frac{1}{8}t^2+t+1 \\ \frac{1}{4}t+1, & \frac{1}{8}t^2+t+1 \le x \end{cases}.$$

2.2 Example 2

We now look at another example, where we consider (2) with the initial functions

$$u_0(x) = \begin{cases} 1, & x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & 1 \le x \end{cases} \quad \rho_0(x) = \begin{cases} 0, & x \le 0 \\ 1, & 0 \le x \le 1 \\ 0, & 1 \le x \end{cases}.$$

First note that ρ_0 and u_{0x}^2 are the same as in the previous example, which means that we can reuse q_0, H_0 and H_{tot}. We only need to compute z_0 for the Lagrangian transformation of the initial functions, this now being

$$z_0(\xi) = \begin{cases} 1, & \xi \le 0 \\ 1 - \frac{1}{2+\varepsilon}\xi, & 0 \le \xi \le 2+\varepsilon \\ 0, & 2+\varepsilon \le \xi \end{cases}.$$

Proceeding as before results in

$$q(\xi,t) = \begin{cases} -\frac{1+\varepsilon}{8}t^2 + t + \xi, & \xi \le 0 \\ -\frac{1+\varepsilon}{4}\left(\frac{1}{2} - \frac{1}{2+\varepsilon}\xi\right)t^2 + \left(1 - \frac{1}{2+\varepsilon}\xi\right)t + \frac{1}{2+\varepsilon}\xi, & 0 \le \xi \le 2+\varepsilon \\ \frac{1+\varepsilon}{8}t^2 + \xi - (1+\varepsilon), & 2+\varepsilon \le \xi \end{cases},$$

$$z(\xi,t) = \begin{cases} -\frac{1+\varepsilon}{4}t + 1, & \xi \le 0 \\ -\frac{1+\varepsilon}{2}\left(\frac{1}{2} - \frac{1}{2+\varepsilon}\xi\right)t + 1 - \frac{1}{2+\varepsilon}\xi, & 0 \le \xi \le 2+\varepsilon \\ \frac{1+\varepsilon}{4}t, & 2+\varepsilon \le \xi \end{cases}. \tag{11}$$

By using $u(x) = z(\xi)$ for all x such that $x = q(\xi)$, we arrive at the solution

$$u_\varepsilon(x,t) = \begin{cases} -\frac{1+\varepsilon}{4}t + 1, & x \le -\frac{1+\varepsilon}{8}t^2 + t \\ \frac{1}{2}\frac{-(1+\varepsilon)t^2+2(1+\varepsilon)(2x-1)t+8(1-x)}{(1+\varepsilon)t^2+4(1-t)}, & -\frac{1+\varepsilon}{8}t^2 + t \le x \le \frac{1+\varepsilon}{8}t^2+1 \\ \frac{1+\varepsilon}{4}t, & \frac{1+\varepsilon}{8}t^2+1 \le x \end{cases}.$$

Next we find ρ_ε. The calculations are almost identical as for the previous example, except now we have $\rho_0 = -u_{0x}$, so

$$\rho_\varepsilon(x,t) = \begin{cases} 0, & x \le -\frac{1+\varepsilon}{8}t^2 + t \\ \frac{4}{(1+\varepsilon)t^2+4(1-t)}, & -\frac{1+\varepsilon}{8}t^2 + t \le x \le \frac{1+\varepsilon}{8}t^2+1 \\ 0, & \frac{1+\varepsilon}{8}t^2+1 \le x \end{cases}.$$

In the same way as before, H_{tot} can be shown to be independent of t also in this case.

We take $\varepsilon \to 0$ in (11) and end up with

$$q(\xi,t) = \begin{cases} -\frac{1}{8}t^2 + t + \xi, & \xi \leq 0 \\ \frac{1}{8}(\xi - 1)t^2 + \left(1 - \frac{1}{2}\xi\right)t + \frac{1}{2}\xi, & 0 \leq \xi \leq 2 \\ \frac{1}{8}t^2 + \xi - 1, & 2 \leq \xi \end{cases},$$

$$z(\xi,t) = \begin{cases} -\frac{1}{4}t + 1, & \xi \leq 0 \\ \frac{1}{4}(\xi - 1)t + 1 - \frac{1}{2}\xi, & 0 \leq \xi \leq 2 \\ \frac{1}{4}t, & 2 \leq \xi \end{cases}.$$

Again, by comparing $\dot{q}$ and z interval-wise, we easily see that they are equal. As before we need H in order confirm that the second equation in (4) holds. Inserting $q_{0\xi}$ and $z_{0\xi}$ into H in (3) we get that H is the same in the previous example. Furthermore, $z_{0\xi}^2$ is also equal to the previous example. Lastly we check that $q_\xi H_\xi = z_\xi^2 + r^2$ is satisfied. The equation $\dot{H} = 0$ is trivially satisfied, in addition to $r = 0$, so we need to check that

$$\frac{1}{2}q_\xi = z_\xi^2, \text{ for } 0 \leq \xi \leq 2$$

hold. Inserting for q_ξ yields

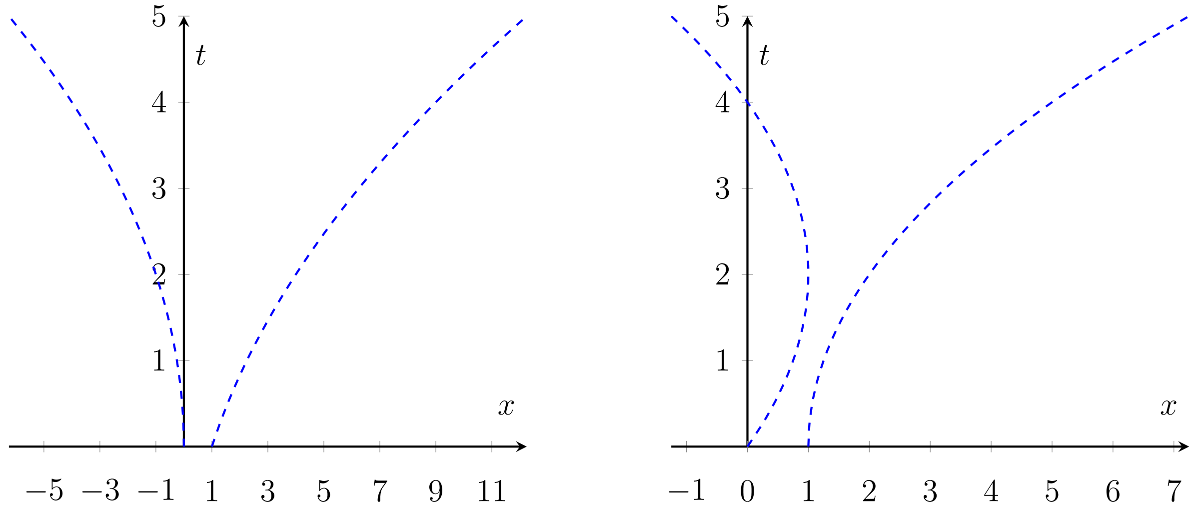
$$\frac{1}{2}q_\xi = \frac{1}{2}\left(\frac{1}{8}t^2 - \frac{1}{2}t + \frac{1}{2}\right) = \frac{1}{4}\left(\frac{1}{2}t - 1\right)^2 = z_\xi^2,$$

proving that the limit function u_ε as $\varepsilon \to 0$ is a weak conservative solution of (1). Furthermore, in the limit we see that

$$\lim_{\varepsilon \to 0} u_\varepsilon(x,t) = \begin{cases} -\frac{1}{4}t + 1, & x \leq -\frac{1}{8}t^2 + t \\ \frac{1}{2}\frac{-t^2 + 2(2x-1)t + 8(1-x)}{t^2 + 4(1-t)}, & -\frac{1}{8}t^2 + t \leq x \leq \frac{1}{8}t^2 + 1 \\ \frac{1}{4}t, & \frac{1}{8}t^2 + 1 \leq x \end{cases}.$$

2.3 Comparison of the two examples

In this section we look at the differences between the two examples. We start by plotting the curves separating the intervals for the two examples, shown in the figure below.



(a) Example 1

(b) Example 2

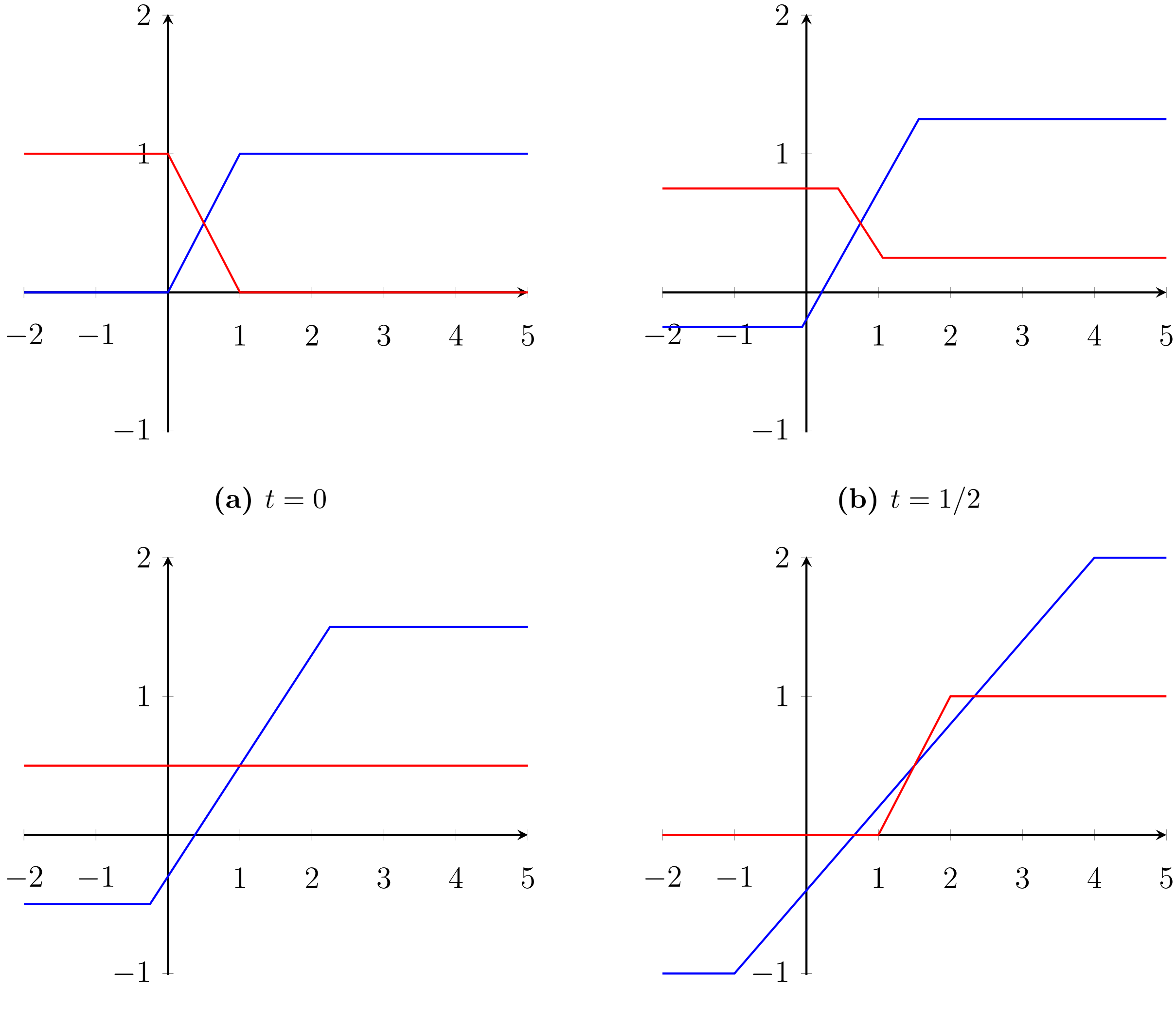
Figure 1: The curves separating the three intervals of the piecewise solution of (2) for $\varepsilon = 1$.

Here we have chosen $\varepsilon = 1$. We see that for Example 1 the curves are closest to each other at $t = 0$, meaning that the middle interval will take up larger and larger parts of our domain as time goes by. For Example 2, however, the curves get closer to each, other before they separate. Intuitively they should not be able to cross, but we study how close they can be. We subtract the upper curve from the lower curve, getting

$$\frac{1+\varepsilon}{8}t^2 + 1 - \left(-\frac{1+\varepsilon}{8}t^2 + t\right) = \frac{1+\varepsilon}{4}t^2 - t + 1.$$

The minimum of the expression above occurs at $t = 2/(1+\varepsilon)$, corresponding to a distance $1 - 1/(1+\varepsilon)$. The intergrand of H_{tot} corresponds to the pointwise energy of system, and this shows that all the energy is located between the curves in Figure 1. Furthermore, in Example 2 we see that all the energy is located in $0 \le x \le 1$ initially, and for smaller ε the energy will accumulate in a smaller interval, before being spread out again. Taking the limit $\varepsilon \to 0$, the curves will become infinitely close at $t = 2$, and the energy function will simply be the delta distribution $\delta(x - 3/2)$.

We plot u_ε for the two examples at four different times below.



(a) $t = 0$ **(b)** $t = 1/2$ **(c)** $t = 1$ **(d)** $t = 2$

Figure 2: u_ε as a function x for different times for the two examples, setting $\varepsilon = 1$. Example 1 and 2 are the red the blue and read graphs, respectively.

As we see, both functions tend to become monotonically increasing after a while, regardless of whether or not the initial function is. In addition, since the solutions are piecewise linear and continuous, it is necessary for the solution of Example 2 to become constant before it can become monotonically increasing, as depicted in Figure 2c.

References

[1] Anders Samuelsen Nordli. On the Hunter-Saxton equation, 2012.

[2] John K. Hunter, Ralph Saxton, and Boris Becker. Dynamics of director fields, 1991.

[3] Marcus Wunsch. The generalized hunter–saxton system. *SIAM Journal on Mathematical Analysis*, 42(3):1286–1304, 2010.