Quaternions with applications

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Introduction

Quaternions may be a bit abstract to most people, or even unheard of. Therefore we start by looking at some elementary operations of quaternions. Then we will go into more complicated parts such as the exponential and the derivative of a quaternion and quaternion differential equations. In the end we look at some reasons to use quaternions as rotations, and numerical solution of quaternion differential equations.

Chapter 1

Elementary operations of quaternions

1.1 Introduction to quaternions

Quaternions were discovered by William R. Hamilton in 1843. Just as complex numbers can be represented in a plane, Hamilton wanted to come up with something similar for a three-dimensional space. After years of struggle, he came up with with quaternions, a four-dimensional space, which is an extension of complex numbers. We can write a quaternion, q, as

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

where $q_i \in \mathbb{R}$ for $i = 0, 1, 2, 3$ and $\mathbf{i} = (0, 1, 0, 0), \mathbf{j} = (0, 0, 1, 0), \mathbf{k} = (0, 0, 0, 1)$ are the fundamental quaternion units [5]. They are added like vectors, but their multiplication makes them special.

1.1.1 Multiplication of quaternion units

A quaternion unit multiplied with itself is just -1, i.e.

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

To see how multiplication between quaternion units work, we consider the Figure (A.3). Starting from the top, with the quaternion unit i, and going clockwise we find

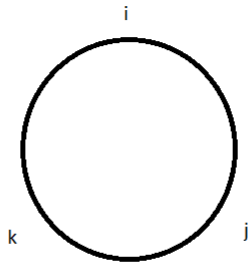

Figure 1.1

that

$$\mathbf{ij} = \mathbf{k},$$
$$\mathbf{jk} = \mathbf{i},$$
$$\mathbf{ki} = \mathbf{j}.$$

Going counter-clockwise we get

$$\mathbf{ji} = -\mathbf{k},$$
$$\mathbf{kj} = -\mathbf{i},$$
$$\mathbf{ik} = -\mathbf{j}.$$

We see clearly that the order of how two quaternions units matter. Thus quaternions are non-commutative.

We also notice that $\mathbf{ijk} = -1$ and $\mathbf{kji} = 1$.

1.1.2 The field $\mathbb{H}$

Just as we denote the set of real numbers with $\mathbb{R}$ and the set of complex numbers with $\mathbb{C}$, we use $\mathbb{H}$ for the set of quaternions. Thus the set of quaternions is defined as

$$\mathbb{H} = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \ : \ q_i \in \mathbb{R}, \ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\}$$

1.2 Addition and subtraction

Let us consider two quaternions $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ and $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$. Adding/subtracting p and q is straightforward and yields

$$q \pm p = (q_0 \pm p_0) + (q_1 \pm p_1)\mathbf{i} + (q_2 \pm p_2)\mathbf{j} + (q_3 \pm p_3)\mathbf{k}.$$

1.3 Multiplication

The product between two quaternions is called the Hamiltonian product (or quaternion produuct), and is usually denoted by $\star$. Given $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ and $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$, where we use formulas from section (1.1.1), the Hamiltonian product between q and p becomes

$$\begin{aligned} q \star p =& (q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3) + (q_0p_1 + q_1q_0 + q_2p_3 - q_3p_1)\mathbf{i} \\ &+ (q_0p_2 - q_1p_3 + q_3p_1 + q_3p_1)\mathbf{j} + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)\mathbf{k}. \end{aligned} \tag{1.1}$$

For convenience we will drop the $\star$.

1.3.1 Simpler notation

Writing a quaternion $q = r + v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ in the form

$$q = (r, \vec{v}) \quad \text{where } r \in \mathbb{R} \text{ and } \vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3,$$

one can simplify equation (1.1) using the dot product $\cdot$, and the cross product $\times$. We get

$$(r_1, \vec{v}_1)(r_2, \vec{v}_2) = (r_1r_2 - \vec{v}_1 \cdot \vec{v}_2, r_1\vec{v}_2 + r_2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2) \tag{1.2}$$

1.4 Norm, conjugate and inverse

The formulas for norm, conjugate and inverse of a quaternion, q, is:

$$||q|| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \qquad \text{(Norm)}$$

$$q^* = (r, \mathbf{v}^*) = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k} \qquad \text{(Conjugate)}$$

$$q^{-1} = \frac{q^*}{||q||^2} \qquad \text{(Inverse)}$$

Notice that $qq^{-1} = 1 = q^{-1}q$, as it should.

1.5 Division

Dealing with division for quaternions is similar to complex numbers. We multiply with the conjugate of the denominator. So for q/p we have

$$\frac{q}{p} = \frac{qp^*}{pp*} = \frac{qp^*}{||p||^2}$$

For example,

$$\begin{aligned}\frac{1 + \mathbf{i} + 2\mathbf{j}}{2\mathbf{j} + 3\mathbf{k}} &= \frac{(1 + \mathbf{i} + 2\mathbf{j})(-2\mathbf{j} - 3\mathbf{k})}{\sqrt{2^2 + 3^2}^2} \\ &= \frac{4 - 6\mathbf{i} + \mathbf{j} - 5\mathbf{k}}{13}\end{aligned}$$

Chapter 2

Quaternion differential equations

2.1 Introduction

According to Kou and Xia [3], QDEs (quaternion differential equations) are a new kind of differential equations. Some applications of QDEs can be found in fluid mechanics and quantum mechanics.

2.2 Solution of QDEs

A QDE has the form

$$\frac{dq(t)}{dt} = \frac{1}{2}\omega(t)q(t) \tag{2.1}$$

with some initial condition $q(t_0) = q_0$ and where $\omega(t) = \Big(0, \omega_x(t)), \omega_y(t), \omega_z(t)\Big)$ is the angular velocity. If we assume constant angular velocity, we may write (2.1) as

$$\begin{bmatrix} w' \\ x' \\ y' \\ z' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

We can think of this as an ODE on the form

$$q' = Aq \quad \Big(\text{where } ' = \frac{d}{dt}\Big)$$

with solution

$$q(t) = e^{A\Delta t} q_0 \tag{2.2}$$

where $\Delta t = t - t_0$.

There are two important questions that arises from this section:

- How are quaternions and exponentials related?
- How to solve QDEs when the angular velocity is not constant?

2.3 Exponential function

As we recall from complex analysis, the exponential of a complex number z could be written as

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^x(\cos y + i \sin y).$$

Is there any similar formula for quaternions?

We start by defining e^q as the power series,

$$e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!} = 1 + q + \frac{q^2}{2!} + \cdots .$$

Let $q = s + \vec{v}$, where $s \in \mathbb{R}$ and $\vec{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \in \mathbb{R}^3$. Since the real quaternion, i.e. the scalar part $s \in \mathbb{R}$, will commute with any quaternion we can write

$$e^q = e^{s+\vec{v}} = e^s e^{\vec{v}}$$

where

$$e^{\vec{v}} = \sum_{n=0}^{\infty} \frac{\vec{v}^n}{n!} = 1 + \vec{v} + \frac{\vec{v}^2}{2!} + \frac{\vec{v}^3}{3!} + \cdots .$$

If we calculate $\vec{v}^2, \vec{v}^3, \cdots$ we find

$$\begin{aligned}\vec{v}^2 &= -|\vec{v}|^2\\ \vec{v}^3 &= -\vec{v}|\vec{v}|^2\\ \vec{v}^4 &= |\vec{v}|^4\end{aligned}$$

In fact, for every even n (i.e. $n = 2k$) we have

$$\vec{v}^{2k} = (-1)^k|\vec{v}|^{2k}$$

and for every odd n (i.e. $n = 2k+1$) we have

$$\vec{v}^{2k+1} = (-1)^k\vec{v}|\vec{v}|^{2k}$$

Thus we end up with

$$\begin{aligned}e^{\vec{v}} =& 1 + \vec{v} - \frac{|\vec{v}|^2}{2!} - \frac{\vec{v}|\vec{v}|^2}{3!} + \frac{|\vec{v}|^4}{4!} + \frac{\vec{v}|\vec{v}|^4}{5!} + \cdots\\ =& \left(1 - \frac{|\vec{v}|^2}{2!} + \frac{|\vec{v}|^4}{4!} + \cdots\right) + \vec{v}\left(1 - \frac{|\vec{v}|^2}{3!} + \frac{|\vec{v}|^4}{5!} + \cdots\right)\frac{|\vec{v}|}{|\vec{v}|}\\ =& \left(1 - \frac{|\vec{v}|^2}{2!} + \frac{|\vec{v}|^4}{4!} + \cdots\right) + \frac{\vec{v}}{|\vec{v}|}\left(|\vec{v}| - \frac{|\vec{v}|^3}{3!} + \frac{|\vec{v}|^5}{5!} + \cdots\right)\\ =& \cos(|\vec{v}|) + \frac{\vec{v}}{|\vec{v}|}\sin(|\vec{v}|).\end{aligned}$$

So the exponential of a quaternion may be written as

$$e^q = e^{s+\vec{v}} = e^s\left(\cos(|\vec{v}|) + \frac{\vec{v}}{|\vec{v}|}\sin(|\vec{v}|)\right). \tag{2.3}$$

2.3.1 Constant angular velocity

If we go back to our formula for QDEs with constant angular velocity and replace the exponential function in (2.2) with (2.3) for $A\Delta t$ we obtain the formula

$$q(t) = \left(\cos\left(\frac{|\omega|\Delta t}{2}\right)I + \frac{2}{|\omega|}\sin\left(\frac{|\omega|\Delta t}{2}\right)A\right)q(t_0),$$

where I is the identity matrix of size 4×4.

2.4 Derivatives of quaternions

For a real function, $f(x) \in C^1$ (continuously differentiable), we have

$$\frac{d}{dx} f(x) = f'(x)$$

and (the chain rule)

$$\frac{d}{dx} f^2(x) = f'f + ff' = 2f'f.$$

Since quaternions are non-commutative, we cannot write

$$q'q + qq' = 2qq', \tag{2.4}$$

instead we introduce the commutator. The commutator of a and b is

$$\mathrm{ad}_a(b) = [a, b] = ab - ba$$

Thus we may write (2.4) as

$$q'q + qq' = 2q'q + (qq' - q'q) = 2q'q + \mathrm{ad}_q(q')$$

For $k = 3$, we get

$$q'q^2 + q'qq + q^2q' = 3q'q^2 + 3(\mathrm{ad}_q(q'))q + \mathrm{ad}_q^2(q').$$

So the power rule for quaternions is

$$\frac{dq^k}{dt} = \sum_{i=0}^{k-1} \binom{k}{i+1} \mathrm{ad}_q^i(q') q^{k-i-1} \tag{2.5}$$

Remark 1. Observe that

$$\mathrm{ad}_q^k(\mathrm{ad}_q^j(p)) = \mathrm{ad}_q^{k+j}(p)$$

2.4.1 Derivative of e^q

Using (2.5) we may find a formula for the derivative of the exponential. We recall that

$$e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!} = 1 + q + \frac{q^2}{2!} + \cdots,$$

then

$$\frac{d}{dt}e^q = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{q^n}{n!} = \frac{d}{dt}\left(1 + q + \frac{q^2}{2!} + \cdots\right)$$

We find that

$$\begin{aligned}
&\text{First term:} \quad \frac{d}{dt}1 = 0 \\
&\text{Second term:} \quad \frac{d}{dt}q = \mathrm{ad}_q^0(q') \\
&\text{Third term:} \quad \frac{d}{dt}\frac{q^2}{2} = q'q + \frac{1}{2}\mathrm{ad}_q(q') \\
&\text{Fourth term:} \quad \frac{d}{dt}\frac{q^3}{6} = \frac{1}{6}(3q'q^2 + 3(\mathrm{ad}_q(q'))q + \mathrm{ad}_q^2(q')) \\
&\qquad \vdots
\end{aligned}$$

Doing some simplifications we get

$$\begin{aligned}
\frac{d}{dt}e^q =& \left(\mathrm{ad}_q^0(q') + \frac{1}{2!}\mathrm{ad}_q(q') + \frac{1}{3!}\mathrm{ad}_q^2(q') + \cdots\right)\left(1 + q + \frac{q^2}{2!} + \frac{q^3}{3!} + \cdots\right) \\
=& \left(\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\mathrm{ad}_q^k(q')\right)e^q
\end{aligned}$$

where we define

$$\mathrm{dexp}_q(q') = \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\mathrm{ad}_q^k(q').$$

2.5 Inverse of dexp

The inverse of

$$\mathrm{dexp}_q(p) = \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\mathrm{ad}_q^k(p)$$

is

$$dexp_q^{-1}(p) = \sum_{k=0}^{\infty}\frac{B_k}{k!}\mathrm{ad}_q^k(p).$$

where B_k's are the Bernoulli numbers, as in Hairer et al. [1].

To prove this claim, we need to show that

$$\mathrm{dexp}_q^{-1}(\mathrm{dexp}_q(p)) = p.$$

We get

$$\begin{aligned}
\mathrm{dexp}_q^{-1}(\mathrm{dexp}_q(p)) =& \sum_{k=0}^{\infty} \frac{B_k}{k!} \mathrm{ad}_q^k \left(\sum_{j=0}^{\infty} \frac{1}{(j+1)!} \mathrm{ad}_q^j(p) \right) \\
=& \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{B_k}{k!} \frac{1}{(j+1)!} \mathrm{ad}_q^k(\mathrm{ad}_q^j(p)) \\
=& \sum_{k=0}^{\infty} \frac{B_k}{k!} \sum_{j=0}^{\infty} \frac{1}{(j+1)!} \mathrm{ad}_q^{k+j}(p).
\end{aligned}$$

Let

$$A_k = \frac{B_k}{k!} \sum_{j=0} \frac{1}{(j+1)!} \mathrm{ad}_q^{k+j}(p).$$

If we calculate $A_0,\ A_1, \cdots$ we get

$$\begin{aligned}
A_0 =& B_0(p + \frac{1}{2!}\mathrm{ad}_q(p) + \frac{1}{3!}\mathrm{ad}_q^2(p) + \frac{1}{4!}\mathrm{ad}_q^3(p) + \cdots) \\
A_1 =& B_1(\mathrm{ad}_q(p) + \frac{1}{2!}\mathrm{ad}_q^2(p) + \frac{1}{3}\mathrm{ad}_q^3(p) + \cdots) \\
A_2 =& \frac{B_2}{2}(\mathrm{ad}_q^2(p) + \frac{1}{2!}\mathrm{ad}_q^3(p) + \cdots) \\
& \vdots
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{k=0}^{\infty} A_k =& A_0 + A_1 + A_2 + \cdots && (2.6) \\
=& p + \left(\frac{B_0}{2} + B_1 \right) \mathrm{ad}_q(p) + \left(\frac{B_0}{3!} + \frac{B_1}{2!} + \frac{B_2}{2!} \right) \mathrm{ad}_q^2(p) + \cdots . && (2.7)
\end{aligned}$$

To get any further we must calculate the Bernoulli numbers.

The Bernoulli numbers are found by considering the equation

$$\sum_{k=0}^{\infty} \frac{B_k}{k!} z^k = \frac{z}{e^z - 1}. \tag{2.8}$$

If we multiply this equation with $e^z - 1$ we get

$$(e^z - 1) \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k = z. \tag{2.9}$$

Using that the Taylor series of $(e^z - 1)$ at $z = 0$ is

$$e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .$$

we can write (2.9) as

$$\begin{aligned} &\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\left(B_0 + B_1 z + \frac{B_2}{2!} z^2 + \frac{B_3}{3!} z^3 + \cdots\right) \\ =&B_0 z + z^2\left(\frac{B_0}{2} + B_1\right) + z^3\left(\frac{B_0}{3!} + \frac{B_1}{2!} + \frac{B_2}{2!}\right) \cdots = z. \end{aligned} \tag{2.10}$$

Comparing the LHS with the RHS of equation (2.10) we get

$$\begin{aligned} &B_0 = 1 \\ &\frac{B_0}{2!} + B_1 = 0 \iff B_1 = -\frac{1}{2} \\ &\frac{B_0}{3!} + \frac{B_1}{2!} + \frac{B_2}{2!} = 0 \iff B_2 = \frac{1}{6} \\ &\vdots \end{aligned}$$

This means that

$$\sum_{k=0}^{\infty} A_k = p,$$

and thus the inverse of dexp is going to be

$$\mathrm{dexp}_q^{-1}(p) = \sum_{k=0}^{\infty} \frac{B_k}{k!} \mathrm{ad}_q^k(p).$$

2.6 Magnus series expansion

The final part needed before we can start looking at QDEs with non-constant angular velocity is the Magnus series expansion.

Consider the matrix differential equation

$$Y' = A(t)Y \tag{2.11}$$

where $A(t)$ is a matrix dependent on t. From Hairer et al. [1] we have the theorem: **Theorem 2.** The solution of the differential equation (2.11) can be written as $Y(t) = \exp(\Omega(t))Y_0$ with $\Omega(t)$ defined by

$$\Omega' = \operatorname{dexp}_\Omega^{-1}(A(t)), \quad \Omega(0) = 0.$$

2.7 Non-constant angular velocity

Consider

$$\frac{dq}{dt} = \frac{1}{2}\omega(t)q(t) \tag{2.12}$$

with initial condition $q(t_0) = q_0$. Using Theorem (2) we look for a solution of the form

$$q(t) = e^{\Omega(t)}q_0. \tag{2.13}$$

To see that $\Omega' = \operatorname{dexp}_\Omega^{-1}(A(t))$ we start by taking the derivative of (2.13):

$$\frac{dq}{dt} = \frac{de^{\Omega(t)}}{dt}q_0. \tag{2.14}$$

From section 2.4.1 we had

$$\frac{de^{\Omega(t)}}{dt} = \left(\sum_{k=0}^{\infty} \frac{1}{(k+1)!}\operatorname{ad}_\Omega^k(\Omega')\right)e^q \tag{2.15}$$

Inserting (2.15) into (2.14) we get the following result

$$\frac{dq}{dt} = \left(\sum_{k=0}^{\infty} \operatorname{ad}_\Omega^k(\Omega')\right)e^\Omega q_0 = \operatorname{dexp}_\Omega(\Omega')q(t). \tag{2.16}$$

Comparing (2.12) with (2.16) we see that

$$\mathrm{dexp}_{\Omega}(\Omega') = \frac{1}{2}\omega(t). \tag{2.17}$$

If we take dexp^{-1} on both sides, we know from Section 2.5 that this is going to be

$$\Omega' = \mathrm{dexp}_{\Omega}^{-1}\Big(\frac{1}{2}\omega(t)\Big).$$

Using $A(t) = \frac{\omega(t)}{2}$ (for convenience) we get

$$\Omega' = \mathrm{dexp}_{\Omega}^{-1}(A(t)) = \sum_{k=0}^{\infty} \frac{B_k}{k!}\mathrm{ad}_{\Omega}^{k}(A(t)) \tag{2.18}$$

$$= \frac{1}{2}\left(\frac{B_0}{0!}A(t) + \frac{B_1}{1!}\mathrm{ad}_{\Omega}(A(t)) + \frac{B_2}{2!}\mathrm{ad}_{\Omega}^{2}(A(t)) + \cdots\right) \tag{2.19}$$

$$= A(t) - \frac{1}{2}[\Omega, A(t)] + \frac{1}{12}[\Omega, [\Omega, A(t)]] + \cdots \tag{2.20}$$

Using Picard's fixed point iteration we can solve (2.20) [1]. We get

$$\begin{aligned}
&\Omega_0(0) = 0 \quad \text{(from Theorem 2)}\\
&\Omega_1(t) = \int_0^t A(t_1)dt_1\\
&\Omega_2(t) = \frac{1}{2}\int_0^t \left[\int_0^{t_1} A(t_2)dt_2, A(t_1)\right]dt_1\\
&\vdots\\
&\Omega_n(t) = \int_0^t \left(A(t_1) - \frac{1}{2}[\Omega_{n-1}(t), A(t)] + \frac{1}{12}[\Omega_{n-1}[\Omega_{n-1}, A]] + \cdots\right)dt_1
\end{aligned}$$

Taking $\lim_{n\to\infty}\Omega_n$ we get

$$\begin{aligned}
\Omega(t) = &\int_0^t A(t_1)dt_1 - \frac{1}{2}\int_0^t \left[\int_0^{t_1} A(t_2)dt_2, A(t_1)\right]dt_1\\
&+ \frac{1}{4}\int_0^t \left[\int_0^{t_1}\left[\int_0^{t_2} A(t_3)dt_3, A(t_2)\right]dt_2, A(t_1)\right]dt_1 + \cdots .
\end{aligned} \tag{2.21}$$

So the solution of

$$\frac{dq}{dt} = \frac{1}{2}\omega(t)q(t)$$

is going to be

$$q(t) = e^{\Omega(t)} q_0.$$

where $\Omega(t)$ is found using (2.21).

Chapter 3

Applications of quaternions

3.1 Quaternions in computer graphics

There are many ways to represent rotations. Two common and easy methods are Euler angles and rotation matrices. Another way we can represent rotations is using quaternions. This method is more mathematically abstract, but gives much better results if you for example want to interpolate between two (arbitrary) orientations very smoothly. This is useful if you want rotations to look realistic games.

Quaternions also avoid the problem with gimbal lock [1], something the Euler angles does not.

[1] "Gimbal lock is the loss of one degree of freedom in a three-dimensional, three-gimbal mechanism that occurs when the axes of two of the three gimbals are driven into a parallel configuration, "locking" the system into rotation in a degenerate two-dimensional space" [4].

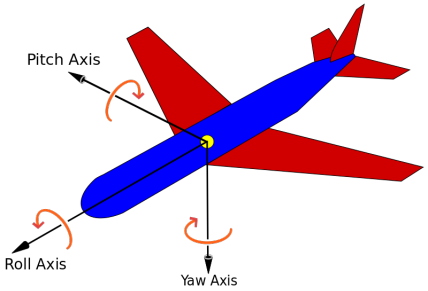

Figure 3.1: Image showing the roll, pitch and yaw axes. Picture taken from Wikipedia [6].

3.2 Rotation operator

To describe rotations we use unit quaternions. From Jia [2] we have the theorem

Theorem 3. For any unit quaternion

$$q = q_0 + \vec{q} = \cos\frac{\theta}{2} + \vec{u}\sin\frac{\theta}{2}$$

where $\vec{u}$ is the unit vector $\vec{u} = \frac{\vec{q}}{||\vec{q}||}$ and for any vector $\mathbb{R}^3$ the action operator

$$L_q(v) = q\vec{v}q^* \tag{3.1}$$

on $\vec{v}$ is equivalent to a rotation of the vector through an angle θ about $\vec{u}$ as the axis of rotation.

If we write

$$q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = \cos\frac{\theta}{2} + (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\sin\frac{\theta}{2},$$

we see that the angle θ can be found solving

$$\theta = 2\cos^{-1}(q_0).$$

3.3 Solution of QDEs

3.3.1 Finding $\Omega(t)$ analytically

In Section 2.7 we found that

$$\Omega(t) = \int_0^t A(t_1)dt_1 - \frac{1}{2}\int_0^t \Big[\int_0^{t_1} A(t_2)dt_2, A(t_1)\Big]dt_1 + \frac{1}{4}\int_0^t \Big[\int_0^{t_1}\Big[\int_0^{t_2} A(t_3)dt_3, A(t_2)\Big]dt_2, A(t_1)\Big]dt_1 + \cdots .$$

To see how we can find Ω analytically we will look at two examples.
Example 1. Consider the angular velocity $\omega(t) = 4t(0,1,1,1)$.

Calculating the first term of Ω we get

$$2\int_0^t (0, t_1, t_1, t_1)dt_1 = t^2(0,1,1,1)$$

Calculating the second term of (2.21) we get that

$$\int_0^t \Big[\int_0^{t_1} \frac{\omega(t_2)}{2}dt_2, \frac{\omega(t_2)}{2}\Big]dt_1 = (0,0,0,0)$$

since

$$[t^2(0,1,1,1), 2t(0,1,1,1)] = 2t^3\Big((0,1,1,1)(0,1,1,1) - (0,1,1,1)(0,1,1,1)\Big) = (0,0,0,0)$$

Thus the solution is

$$\Omega(t) = t^2(0,1,1,1).$$

Example 2. In our second example we look at the angular velocity $\omega(t) = 4t(0,1,1,1) + 2(0,1,2,3)$.

Calculating the first term of Ω we get

$$\int_0^t \Big(2t(0,1,1,1) + (0,1,2,3)\Big)dt = t^2(0,1,1,1) + (0,1,2,3)t. \tag{3.2}$$

To calculate the second term of (2.21) we start by taking the commutator of (3.2) and $\frac{\omega(t)}{2}$. We get

$$\begin{aligned}
&[t^2(0,1,1,1) + t(0,1,2,3), 2t(0,1,1,1) + (0,1,2,3)] = \\
&+ 2t^3\Big((0,1,1,1)(0,1,1,1) - (0,1,1,1)(0,1,1,1)\Big) \\
&+ t^2\Big(0,1,1,1)(0,1,2,3) - 2(0,1,1,1)(0,1,2,3)\Big) \\
&+ t^2\Big(2(0,1,2,3)(0,1,1,1) - (0,1,2,3)(0,1,1,1)\Big) \\
&+ t\Big(t(0,1,2,3)(0,1,2,3) - (0,1,2,3)(0,1,2,3)\Big)
\end{aligned}$$

Cancelling terms we end up with

$$t^2\Big((0,1,2,3) \times (0,1,1,1) - (0,1,1,1) \times (0,1,2,3)\Big) = t^2(0,-2,4,-2)$$

Integrating this yields

$$\int_0^t \tau^2(0,-2,4,-2))\ d\tau = \frac{1}{3}t^3(0,-2,4,-2).$$

Thus we get that

$$\begin{aligned}
\Omega(t) =& t^2(0,1,1,1) + (0,1,2,3)t + \frac{1}{3}t^3(0,1,-2,1) + \mathcal{O}(t^5) \\
=& \Big(0, t^2 + t + \frac{1}{3}t^3, t^2 + 2t - \frac{2}{3}t^3, t^2 + 3t + \frac{1}{3}t^3\Big)
\end{aligned} \tag{3.3}$$

3.3.2 Finding $\Omega(t)$ numerically

From Example 2 we see that the calculations are quite long even though $\omega(t)$ is only a polynomial of first order. Doing this by hand for higher order polynomials would be very time consuming. Therefore it would be more useful to find $\Omega(t)$ numerically. This is possible using for example Python and the library SymPy. In our code (see Appendix A) we also calculate the third term of $\Omega(t)$.

Comparison

From our code we find that the angular velocity given in Example 2 for $t = 0.3$ is going to be

$$\Omega(0.3) = (0, 0.3990, 0.6720, 0.9999), \quad \text{(two terms of } \Omega(t)) \tag{3.4}$$
$$\Omega(0.3) = (0, 0.4054, 0.6733, 0.9953). \quad \text{(three terms of } \Omega(t)) \tag{3.5}$$

We see that the (3.4) is exactly the same if insert $t = 0.3$ into (3.3).

3.3.3 The solution $q(t)$:

Now that we have found $\Omega(t_1) = \Big(0, \Omega_x(t_1), \Omega_y(t_1), \Omega_z(t_1)\Big)$ for some time t_1 numerically, we need to find the solution $q(t_1)$. Using that the exponential of a quaternion is

$$e^q = e^{s+\vec{v}} = e^s\bigg(\cos|\vec{v}|) + \frac{\vec{v}}{|\vec{v}|}\sin(|\vec{v}|)\bigg)$$

where $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$, we get

$$\begin{aligned} q(t_1) =& \bigg(\cos\left(\left|\sqrt{\Omega_x^2(t_1) + \Omega_y^2(t_1) + \Omega_z^2(t_1)}\right|\right) \\ &+ \frac{(\Omega_x(t), \Omega_y(t_1), \Omega_z(t_1)}{\sqrt{\Omega_x^2(t_1) + \Omega_y^2(t_1) + \Omega_z^2(t_1)}}\sin\left(\left|\sqrt{\Omega_x^2(t_1) + \Omega_y^2(t_1) + \Omega_z^2(t_1)}\right|\right)\bigg)q(0) \end{aligned}$$

For the angular velocity $\omega(t) = 4t(0, 1, 1, 1) + 2(0, 1, 2, 3)$ and the initial condition $q(0) = (0.5, 0.5, 0.5, 0.5)$ we get the following table for different times t.

$\omega(0.05) = (0, 0.0525, 0.1024, 0.1525)$ $q(0.05) = (0.3381, 0.4921, 0.5915, 0.5419)$
$\omega(0.10) = (0, 0.1104, 0.2094, 0.3102)$ $q(0.10) = (0.1553, 0.4670, 0.6619, 0.5654)$
$\omega(0.15) = (0, 0.1740, 0.3203, 0.4734)$ $q(0.15) = (-0.0422, 0.4232, 0.7051, 0.5674)$
$\omega(0.20) = (0, 02439, 0.4349, 0.6420)$ $q(0.20) = (-0.2463, 0.3601, 0.7159, 0.5451)$
$\omega(0.25) = (0, 0.3207, 0.5527, 0.8160)$ $q(0.25) = (-0.4467, 0.2785, 0.6897, 0.4970)$
$\omega(0.30) = (0, 0.4054, 0.6734, 0.9953)$ $q(0.30) = (-0.6316, 0.1804, 0.6244, 0.4223)$

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Appendix A

Python code

A.1 Quaternion product

```
def quatprod(q_0,q_1,q_2,q_3,r_0,r_1,r_2,r_3):
    return [r_0*q_0-r_1*q_1-r_2*q_2-r_3*q_3,
            r_0*q_1+r_1*q_0-r_2*q_3+r_3*q_2,
            r_0*q_2+r_1*q_3+r_2*q_0-r_3*q_1,
            r_0*q_3-r_1*q_2+r_2*q_1+r_3*q_0]
```

Figure A.1: Code for calculating the quaternion product

A.2 The angular velocity $\Omega(t)$

```
t = Symbol('t')
def Omega(f,g,h,j,time):
    def omega0(t): return 0
    def omega_x(t): return 0.5*g
    def omega_y(t): return 0.5*h
    def omega_z(t): return 0.5*j

    # Integrating between 0 and t:
    int_0 = [omega0(t),omega_x(t),omega_y(t),omega_z(t)] # this is just w(t)/2.
    int_1 = [integrate(omega0(t),t),integrate(omega_x(t),t),
             integrate(omega_y(t),t),integrate(omega_z(t),t)] # integral of w(t)/2

    # Commutator for the second term of Omega(t)
    diff1 = quatprod(int_1[0],int_1[1],int_1[2],int_1[3],int_0[0],int_0[1],int_0[2],int_0[3])
    diff2 = quatprod(int_0[0],int_0[1],int_0[2],int_0[3],int_1[0],int_1[1],int_1[2],int_1[3])

    int_2 = [integrate(diff1[0]-diff2[0],t),integrate(diff1[1]-diff2[1],t),
             integrate(diff1[2]-diff2[2],t),integrate(diff1[3]-diff2[3],t)]

    # Commutator for the third term of Omega(t)
    diff3 = quatprod(int_2[0],int_2[1],int_2[2],int_2[3],int_0[0],int_0[1],int_0[2],int_0[3])
    diff4 = quatprod(int_0[0],int_0[1],int_0[2],int_0[3],int_2[0],int_2[1],int_2[2],int_2[3])

    # Integrating between 0 and the given time t:
    integral1 = [integrate(omega0(t),(t,0,time)),integrate(omega_x(t),(t,0,time)),
                 integrate(omega_y(t),(t,0,time)),integrate(omega_z(t),(t,0,time))] # first term
    integral2 = [integrate(diff1[0]-diff2[0],(t,0,time)),
                 integrate(diff1[1]-diff2[1],(t,0,time)),
                 integrate(diff1[2]-diff2[2],(t,0,time)),
                 integrate(diff1[3]-diff2[3],(t,0,time))] # second term
    integral3 = [integrate(diff3[0]-diff4[0],(t,0,time)),
                 integrate(diff3[1]-diff4[1],(t,0,time)),
                 integrate(diff3[2]-diff4[2],(t,0,time)),
                 integrate(diff3[3]-diff4[3],(t,0,time))] # third term

    Omega_time = [0,0,0,0]
    for i in range(0,4):
        Omega_time[i] = integral1[i] - 0.5*integral2[i] + 0.25*integral3[i] # combine the terms
    return Omega_time
```

Figure A.2: Code for calculating $\Omega(t)$ using three terms for a given time $t = t_1$

A.3 The solution $q(t)$ of $q'(t) = \frac{1}{2}\omega(t)q(t)$

```
def qde(t,Omega_t,Q_initial):
    a,b,c,d = Omega_t[0],Omega_t[1],Omega_t[2],Omega_t[3] # The calculated Omega for a given time t
    Q0 = Q_initial # initial condition
    norm = sqrt(b**2 + c**2 + d**2)
    Q = [cos(norm),b*sin(norm)/norm,c*sin(norm)/norm,d*sin(norm)/norm]
    Quat = quatprod(Q[0],Q[1],Q[2],Q[3],Q0[0],Q0[1],Q0[2],Q0[3])
    return Quat
```

Figure A.3: Code for calculating $q(t)$ for a given time $t = t_1$