Normalizations, Constants and Fundamental solutions

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1 Introduction

The Evolutionary p-Laplace equation,

$$\frac{\partial u}{\partial t} = \nabla \cdot (|\nabla u|^{p-2} \nabla u) \tag{1}$$

is a parabolic partial differential equation which has an explicit fundamental solution, the so called Barenblatt solution, found in 1951,

$$B_p(x,t) = t^{-\frac{n}{\lambda}} \left\{ c - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \left(\frac{|x|}{t^{\frac{1}{\lambda}}} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}}, \tag{2}$$

where $\lambda := n(p-2) + p$ and $p > 2$. We would like to look at some of the properties of the Barenblatt solution, and in particular determine the constant, c, which normalizes (2).

Firstly it is beneficial to introduce some notation that are used. The notation used in the Barenblatt solution is given by

$$\{a\}_+ = \begin{cases} a & : a \geq 0 \\ 0 & : a < 0 \end{cases}$$

meaning the solution is identically zero when the expression inside the bracket is negative.

The Laplace operator is defined as followed

$$\Delta = \nabla \cdot \nabla = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

We call two functions, f and g, asymptotically equal

$$f \sim g \quad \text{for } x \to \infty, \quad \text{if } \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

We also need to define the gamma function, as it is frequently used throughout the calculations. The gamma function is defined as [1]

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t,$$

and the closely related beta function, is given by

$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} \mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

2 Verifying the Barenblatt solution

Before continuing with the properties of the Barenblatt solution, we would like to confirm that (2) is a solution to (1).

Proof. We'll start by finding the derivative with respect to t, which is given by

$$\frac{\partial B_p}{\partial t} = -B_p\Big(\frac{n}{\lambda t} - \lambda^{\frac{p}{1-p}} \frac{|x|}{t^{\frac{1-\lambda}{\lambda}}} \Big(\frac{|x|}{t^{\frac{1}{\lambda}}}\Big)^{\frac{1}{p-1}} \Big\{c - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \Big(\frac{|x|}{t^{\frac{1}{\lambda}}}\Big)^{\frac{p}{p-1}}\Big\}_+^{-1}\Big). \tag{3}$$

Now let us first take look at the case $n = 1$, thereby obtaining the following derivative with respect to x

$$\frac{\partial B_p}{\partial x} = -\Big(t^{-\frac{1}{\lambda}} \lambda^{\frac{1}{1-p}} \Big(\frac{|x|}{t^{\frac{1}{\lambda}}}\Big)^{\frac{1}{p-1}} \frac{\operatorname{sign}(x)}{t^{\frac{1}{\lambda}}} \Big\{c - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \Big(\frac{|x|}{t^{\frac{1}{\lambda}}}\Big)^{\frac{p}{p-1}}\Big\}_+^{\frac{1}{p-2}}\Big) \tag{4}$$

from this it is quite easy to see

$$\Big|\frac{\partial B_p}{\partial x}\Big| = \Big(t^{-\frac{1}{\lambda}} \lambda^{\frac{1}{1-p}} \Big(\frac{|x|}{t^{\frac{1}{\lambda}}}\Big)^{\frac{1}{p-1}} \frac{1}{t^{\frac{1}{\lambda}}} \Big\{c - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \Big(\frac{|x|}{t^{\frac{1}{\lambda}}}\Big)^{\frac{p}{p-1}}\Big\}_+^{\frac{1}{p-2}}\Big). \tag{5}$$

which gives us the following expression

$$\Big|\frac{\partial B_p}{\partial x}\Big|^{p-2} \frac{\partial B_p}{\partial x} = -\lambda^{-1} (t^{-\frac{p-1}{\lambda}})^2 x B_p.$$

From the product rule of differentiation and the fact that $n = 1$ and $\lambda = 2(p-1)$, we are left with

$$\frac{\partial}{\partial x}\Big(\Big|\frac{\partial B_p}{\partial x}\Big|^{p-2} \frac{\partial B_p}{\partial x}\Big) = -\lambda^{-1} t^{-1} \Big(B_p + x\frac{\partial B_p}{\partial x}\Big). \tag{6}$$

By comparing (3) and (6) we are left with

$$\frac{\partial B_p}{\partial t} = \frac{\partial}{\partial x}\Big(\Big|\frac{\partial B_p}{\partial x}\Big|^{p-2} \frac{\partial B_p}{\partial x}\Big), \tag{7}$$

which means (2) is a solution to (1) for $n = 1$.

For $n > 1$, meaning $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. We define $|x| := \|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$, as the standard norm in $\mathbb{R}^n$.

The gradient of (2) is given by

$$\nabla B_p = \Big(\frac{\partial B_p}{\partial x_1}, \dots, \frac{\partial B_p}{\partial x_n}\Big), \tag{8}$$

where

$$\frac{\partial B_p}{\partial x_i} = -t^{-\frac{n}{\lambda}} \lambda^{\frac{1}{1-p}} \Big(\frac{|x|}{t^{\frac{1}{\lambda}}}\Big)^{\frac{1}{p-1}} \frac{1}{t^{\frac{1}{\lambda}}} \Big\{c - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \Big(\frac{|x|}{t^{\frac{1}{\lambda}}}\Big)^{\frac{p}{p-1}}\Big\}_+^{\frac{1}{p-2}} \frac{x_i}{|x|} \tag{9}$$

which means that (8) can be written as

$$\nabla B_p = t^{-\frac{n}{\lambda}} \lambda^{\frac{1}{1-p}} \Big(\frac{|x|}{t^{\frac{1}{\lambda}}}\Big)^{\frac{1}{p-1}} \frac{1}{t^{\frac{1}{\lambda}}} \Big\{c - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \Big(\frac{|x|}{t^{\frac{1}{\lambda}}}\Big)^{\frac{p}{p-1}}\Big\}_+^{\frac{1}{p-2}} \frac{x}{|x|}, \tag{10}$$

and from (10) it is clear that

$$|\nabla B_p| = t^{-\frac{n}{\lambda}} \lambda^{\frac{1}{1-p}} \Big(\frac{|x|}{t^{\frac{1}{\lambda}}}\Big)^{\frac{1}{p-1}} \frac{1}{t^{\frac{1}{\lambda}}} \Big\{c - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \Big(\frac{|x|}{t^{\frac{1}{\lambda}}}\Big)^{\frac{p}{p-1}}\Big\}_+^{\frac{1}{p-2}}, \tag{11}$$

From combining (10) and (11), we end up with the following expression

$$|\nabla B_p|^{p-2} \nabla B_p = \frac{x B_p}{\lambda t}. \tag{12}$$

This means, by using the product rule for a scalar and a vector, we are left with

$$\nabla \cdot \Big(\frac{x B_p}{\lambda t}\Big) = \frac{1}{\lambda t}\big(B_p \nabla \cdot x + x \cdot \nabla B_p\big). \tag{13}$$

By comparing (3) and (12) we have that

$$\frac{\partial u}{\partial t} = \nabla \cdot (|\nabla u|^{p-2} \nabla u),$$

which shows that (2) is a solution to (1).

□

3 Determining the constant

3.1 Introduction

Let the domain $\Omega = \{x \in \mathbb{R}^n | B_p(x,t) \neq 0\}$, i.e. $\Omega = \mathrm{supp}(B_p)$. Now notice that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^n} B_p \mathrm{d}x &= \int_{\mathbb{R}^n} \frac{\partial B_p}{\partial t} \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \nabla \cdot (|\nabla B_p|^{p-2} \nabla B_p) \mathrm{d}x \\ &= \oint_{\partial \Omega} |\nabla B_p|^{p-2} \nabla B_p \cdot \hat{n} \mathrm{d}S \end{aligned}$$

In section 2, we found an expression for the integrand, given by (12), which results in the following integral

$$\oint_{\partial\Omega} \frac{xB_p}{\lambda t} \cdot \hat{n}\mathrm{d}S = 0,$$

since B_p is zero at the boundary. Therefore we can conclude that.

$$\int_{\mathbb{R}^n} B_p \mathrm{d}x = \text{Constant}.$$

We now want to determine the constant, c, which normalize the Barenblatt solution, i.e.

$$\int_{\mathbb{R}^n} B_p \mathrm{d}x = 1.$$

3.2 Determining the constant in $\mathbb{R}$

Let us first look at case $n = 1$, which means the integral is given as

$$\int_{\mathbb{R}^n} t^{-\frac{n}{\lambda}} \left\{ c - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \left(\frac{|x|}{t^{\frac{1}{\lambda}}} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}} \mathrm{d}x = 1, \tag{14}$$

notice that the integral is symmetric around the origin. Therefore we can rewrite the integral as

$$\int_{-\infty}^{\infty} t^{-\frac{1}{\lambda}} \left\{ c - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \left(\frac{|x|}{t^{\frac{1}{\lambda}}} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}} \mathrm{d}x = 2 \int_0^{\infty} t^{-\frac{1}{t}} \left\{ c - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \left(\frac{|x|}{t^{\frac{1}{\lambda}}} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}} \mathrm{d}x.$$

Since (2) is zero for every value given by

$$c < \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \left(\frac{|x|}{t^{\frac{1}{\lambda}}} \right)^{\frac{p}{p-1}},$$

we have that the limit of the integral is given by

$$\bar{x} = t^{\frac{1}{\lambda}} \left(\frac{p\lambda^{\frac{1}{p-1}}}{p-2} c \right)^{\frac{p-1}{p}}, \tag{15}$$

and are left with the following equation

$$2t^{-\frac{1}{\lambda}} c^{\frac{p-1}{p-2}} \int_0^{\bar{x}} \left\{ 1 - \frac{1}{c} \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \left(\frac{x}{t^{\frac{1}{\lambda}}} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}} \mathrm{d}x = 1. \tag{16}$$

From here we can use the following transformation

$$y = \frac{1}{c} \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \left(\frac{x}{t^{\frac{1}{\lambda}}} \right)^{\frac{p}{p-1}},$$
$$\mathrm{d}y = t^{-\frac{1}{\lambda}} y \frac{p}{p-1} \left(\frac{x}{t^{\frac{1}{\lambda}}} \right)^{-1} \mathrm{d}x,$$

leaving us with

$$2c^{\frac{p-1}{p-2}}\frac{p-1}{p}\left(\frac{p}{p-2}\frac{c}{\lambda^{\frac{1}{1-\lambda}}}\right)^{\frac{p-1}{p}}\int_0^1(1-y)^{\frac{p-1}{p-2}}y^{\frac{p-1}{p}-1}\mathrm{d}y=1.$$

which means that

$$c^{\frac{2(p-1)^2}{p(p-2)}}=\frac{p}{2\lambda^{\frac{1}{p}}(p-1)B(\frac{2p-3}{p-2},\frac{p-1}{p})}\left(\frac{p-2}{p}\right)^{\frac{p-1}{p}},\tag{17}$$

and the constant is given by

$$c=\left\{\frac{p}{2\lambda^{\frac{1}{p}}(p-1)B(\frac{2p-3}{p-2},\frac{p-1}{p})}\left(\frac{p-2}{p}\right)^{\frac{p-1}{p}}\right\}^{\frac{p(p-2)}{2(p-1)^2}}.\tag{18}$$

3.3 Determining the constant in $\mathbb{R}^n$

We will now determine the constant for $n>1$, by looking at the integral

$$\int_{\mathbb{R}^n}B_p(x,t)\mathrm{d}x=1,\tag{19}$$

now where $x\in\mathbb{R}^n$. First we can rewrite the integral in spherical coordinates and notice the angular independence of the integrand, leaving us with

$$t^{-\frac{n}{\lambda}}\int_0^{\bar{r}}\left\{c-\frac{p-2}{p}\lambda^{\frac{1}{1-p}}\left(\frac{r}{t^{\frac{1}{\lambda}}}\right)^{\frac{p}{p-1}}\right\}_+^{\frac{p-1}{p-2}}r^{n-1}\mathrm{Area}(S_n)\mathrm{d}r=1,\tag{20}$$

where $\mathrm{Area}(S_n)=2\pi^{n/2}/\Gamma(n/2)$, is the area of the unit sphere in $\mathbb{R}^n$. See Appendix A for the calculation. We can use the same transformation as for $n=1$, i.e.

$$y=\frac{1}{c}\frac{p-2}{p}\lambda^{\frac{1}{1-p}}\left(\frac{r}{t^{\frac{1}{\lambda}}}\right)^{\frac{p}{p-1}},$$
$$\mathrm{d}y=t^{-\frac{1}{\lambda}}y\frac{p}{p-1}\left(\frac{r}{t^{\frac{1}{\lambda}}}\right)^{-1}\mathrm{d}r,$$

and we obtain the following equation,

$$\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\frac{p-1}{p}\left(\frac{p}{p-2}\frac{c}{\lambda^{\frac{1}{1-p}}}\right)^{\frac{n(p-1)}{p}}c^{\frac{p-1}{p-2}}\int_0^1(1-y)^{\frac{p-1}{p-2}}y^{\frac{n(p-1)}{p}-1}\mathrm{d}y=1.$$

From here we are left with

$$c^{\frac{p-1}{p-2}+\frac{n(p-1)}{p}}=\frac{p\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}\lambda^{\frac{n}{p}}(p-1)B(\frac{2p-3}{p-2},\frac{n(p-1)}{p})}\left(\frac{p-2}{p}\right)^{\frac{n(p-1)}{p}},\tag{21}$$

which can be rewrite as

$$c=\left\{\frac{p\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}\lambda^{\frac{n}{p}}(p-1)B(\frac{2p-3}{p-2},\frac{n(p-1)}{p})}\left(\frac{p-2}{p}\right)^{\frac{n(p-1)}{p}}\right\}^{\frac{1}{\frac{p-1}{p-2}+\frac{n(p-1)}{p}}}.\tag{22}$$

Notice that (22) goes to (18) as $n\to 1$.

4 Verifying weak solution property

We want to verify the weak solution property of function (2). Given a test function $\phi \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$, which is defined in a closed bounded domain $M \subset \mathbb{R}^n \times (0,\infty)$, and vanishes outside [2]. Then if

$$\int_0^\infty \int_{\mathbb{R}^n} B_p(x,t)\frac{\partial \phi}{\partial t}\mathrm{d}x\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^n} |\nabla B_p(x,t)|^{p-2}\nabla B_p(x,t)\cdot \nabla\phi \mathrm{d}x\mathrm{d}t, \quad (23)$$

B_p is called a weak solution.

Proof. By multiplying ϕ to (1), and integrate, we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} \phi\frac{\partial B_p}{\partial t}\mathrm{d}x\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^n} \phi\nabla\cdot(|\nabla B_p|^{p-2}\nabla B_p)\mathrm{d}x\mathrm{d}t. \quad (24)$$

Let us first look at the left-hand side of (24). By integration by parts, one obtains

$$\int_0^\infty \int_{\mathbb{R}^n} \phi\frac{\partial B_p}{\partial t}\mathrm{d}x\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^n} \frac{\partial(B_p\phi)}{\partial t}\mathrm{d}x\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^n} B_p\frac{\partial \phi}{\partial t}\mathrm{d}x\mathrm{d}t.$$

Since both B_p and ϕ have continuous partial derivatives in M, we can rewrite the first integral on the right-hand side,

$$\int_0^\infty \int_{\mathbb{R}^n} \frac{\partial(B_p\phi)}{\partial t}\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^n}\int_0^\infty \frac{\partial(B_p\phi)}{\partial t}\mathrm{d}t\mathrm{d}x = 0,$$

since ϕ is zero on ∂M. From this we obtain the following identity

$$\int_0^\infty \int_{\mathbb{R}^n} \phi\frac{\partial B_p}{\partial t}\mathrm{d}x\mathrm{d}t = -\int_0^\infty \int_{\mathbb{R}^n} B_p\frac{\partial \phi}{\partial t}\mathrm{d}x\mathrm{d}t. \quad (25)$$

Let us now look at the right-hand side of (24). Once again by integration by parts one obtain

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} \phi\nabla\cdot(|\nabla B_p|^{p-2}\nabla B_p)\mathrm{d}x\mathrm{d}t = &\int_0^\infty \int_{\mathbb{R}^n} \nabla\cdot(\phi|\nabla B_p|^{p-2}\nabla B_p)\mathrm{d}x\mathrm{d}t \\ &- \int_0^\infty \int_{\mathbb{R}^n} |\nabla B_p|^{p-2}\nabla B_p\cdot\nabla\phi\mathrm{d}x\mathrm{d}t.\end{aligned}$$

From the divergent theorem applied on first integral on the right-hand side, and the fact that ϕ is zero on ∂M, we are left with

$$\int_0^\infty \int_{\mathbb{R}^n} \phi\nabla\cdot(|\nabla B_p|^{p-2}\nabla B_p)\mathrm{d}x\mathrm{d}t = -\int_0^\infty \int_{\mathbb{R}^n} |\nabla B_p|^{p-2}\nabla B_p\cdot\nabla\phi\mathrm{d}x\mathrm{d}t, \quad (26)$$

and by combining (24), (25) and (26), we obtain (23). Therefore (2) satisfy the weak solution condition. □

5 Special case

5.1 The heat equation

For the special case $p = 2$, we can see that (1) becomes

$$\frac{\partial u}{\partial t} = \Delta u, \tag{27}$$

which is known as the heat equation. A known solution to the heat equation is

$$u(x,t) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}, \tag{28}$$

where $x \in \mathbb{R}^n$ and $t \in (0,\infty)$.

Proof. The proof is a simple calculation.

First we'll find that

$$\frac{\partial u}{\partial t} = u\Big(\frac{|x|^2}{4t^2} - \frac{n}{2t}\Big), \tag{29}$$

and that

$$\frac{\partial u}{\partial x_i} = -\frac{x_i}{2t}u,$$

hence the gradient of (28) is given by

$$\nabla u = -\frac{x}{2t}u. \tag{30}$$

From the definition of the Laplace operator, we have the following result

$$\Delta u = \nabla \cdot \nabla u = \nabla \cdot \Big(-\frac{x}{2t}u\Big).$$

By using the product rule for divergence we are left with

$$-\frac{1}{2t}(\nabla u \cdot x + u\nabla \cdot x) = u\Big(\frac{|x|^2}{4t^2} - \frac{n}{2t}\Big). \tag{31}$$

By comparing (29) and (31) we can see that (28) is a solution to the heat equation. □

Lemma 5.1. Let the solution to the heat equation, u, be given as in (28), then u is normalized, i.e.

$$\int_{\mathbb{R}^n} u \mathrm{d}x = 1$$

Proof. Let u be given as in (28), then

$$\begin{aligned}\int_{\mathbb{R}^n} u\mathrm{d}x &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{e^{-\frac{x_1^2+\cdots+x_n^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}\mathrm{d}x_1\ldots\mathrm{d}x_n\\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}}\prod_{i=1}^{n}\int_{-\infty}^{\infty} e^{-\frac{x_i^2}{4t}}\mathrm{d}x_i\\ &= (\pi)^{-\frac{n}{2}}\prod_{i=1}^{n}\int_{-\infty}^{\infty} e^{-y_i^2}\mathrm{d}y_i\\ &= 1,\end{aligned}$$

which show that u is normalized. □

From lemma 5.1 we can see that both the solution to the heat equation and the Barenblatt solution are normalized[1]. As $p \to 2$ and the Evolutionary p-Laplace equation goes to the heat equation, we would therefore assume that the Barenblatt solution would go towards (28).

To show that that the Barenblatt solution goes to the heat solution, we will use the property of Stirling's formula [3],

$$\frac{\Gamma(n+\eta)}{\Gamma(n+\theta)} \sim n^{\eta-\theta},$$

which means that we have the asymptotic behavior of the Beta function given by

$$B(\alpha,\beta) \sim \Gamma(\beta)\alpha^{-\beta},$$

for $\alpha \gg \beta$, and therefore we are left with

$$c \sim \left\{\frac{p\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}\lambda^{\frac{n}{p}}(p-1)\Gamma(\frac{n(p-1)}{p})}\left(\frac{p-2}{p}\frac{2p-3}{p-2}\right)^{\frac{n(p-1)}{p}}\right\}^{\frac{1}{\frac{p-1}{p-2}+\frac{n(p-1)}{p}}}, \tag{32}$$

and notice that the expression inside the bracket converges to $(4\pi)^{-n/2}$, and we have

$$\lim_{p\longrightarrow 2} c = 1. \tag{33}$$

If we now look at the Barenblatt solution, we have

$$B_p(x,t) = t^{-\frac{n}{\lambda}} c^{\frac{p-1}{p-2}}\left\{1 - \frac{p-2}{cp}\lambda^{\frac{1}{1-p}}\left(\frac{|x|}{t^{\frac{1}{\lambda}}}\right)^{\frac{p}{p-1}}\right\}_+^{\frac{p-1}{p-2}}. \tag{34}$$

If we focus on the constant outside the bracket, we have

$$c^{\frac{p-1}{p-2}} \sim \left\{\frac{p\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}\lambda^{\frac{n}{p}}(p-1)\Gamma(\frac{n(p-1)}{p})}\left(\frac{p-2}{p}\frac{2p-3}{p-2}\right)^{\frac{n(p-1)}{p}}\right\}^{\frac{p-1}{(p-1)+\frac{n(p-1)(p-2)}{p}}}, \tag{35}$$

[1] At least for the constant given by (22).

and notice that

$$\lim_{p\longrightarrow 2} c^{\frac{p-1}{p-2}} = 2^{-n}\pi^{-\frac{n}{2}} = (4\pi)^{-\frac{n}{2}}, \tag{36}$$

This means that we have[2]

$$\lim_{p\longrightarrow 2} B_p = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, \tag{37}$$

which show that the Barenblatt solution goes towards the solution of the heat equation as $p \to 2$.

6 Conclusion

We have shown that (2) is a solution to (1), and the weak solution property. When we normalized the Barenblatt solution, we found that the constant was given by

$$c = \left\{ \frac{p\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}\lambda^{\frac{n}{p}}(p-1)B(\frac{2p-3}{p-2}, \frac{n(p-1)}{p})} \left(\frac{p-2}{p}\right)^{\frac{n(p-1)}{p}} \right\}^{\frac{1}{\frac{p-1}{p-2}+\frac{n(p-1)}{p}}}. \tag{38}$$

for $x \in \mathbb{R}^n$. With this constant, we also showed that Barenblatt solution goes towards the fundamental solution of the heat equation as $p \to 2$, which is what we would expect as (1) goes to the heat equation.

A Area of the n-dimensional unit sphere

We want to calculate the area of the n-dimensional unit sphere, $S_n = \{x \in \mathbb{R}^n : |x| \leq 1\}$. First let us look at the integral

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_1^2+\cdots+x_n^2)} \mathrm{d}x_1 \ldots \mathrm{d}x_n = \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{-x_i^2} \mathrm{d}x_i = (\sqrt{\pi})^n. \tag{39}$$

On the other hand, we can also rewrite the integral in equation (39) in spherical coordinates, and notice that the integrand is angular independent. Hence we have

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_1^2+\cdots+x_n^2)} \mathrm{d}x_1 \ldots \mathrm{d}x_n = \int_0^{\infty} e^{-r^2} r^{n-1} \mathrm{d}r \int \cdots \int \mathrm{d}S_n, \tag{40}$$

where the last integrals are the area of the unit sphere. By changing variables from $r^2 = t$, we have

$$\frac{\mathrm{Area}(S_n)}{2} \int_0^{\infty} e^{-t} t^{\frac{n}{2}-1} \mathrm{d}t = \frac{\mathrm{Area}(S_n)}{2} \Gamma(\frac{n}{2}),$$

Therefore by combining equation (39) and (40), we can write the area of the unit sphere as

$$\mathrm{Area}(S_n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}. \tag{41}$$

[2]From the definition, $e^x := \lim_{n\to 0}(1+xn)^{\frac{1}{n}}$

B Visualization of the Barenblatt solution in $\mathbb{R}^2$

Here we would like to give a small visualization of the Barenblatt solution for different values of p, as well as the solution to the heat equation from (28). All of the plots consists of four subplots, each one at a different value of t. This is to show the propagation of (2) and (28). Notice the similarity between the Barenblatt solution for $p = 2.01$ found in figure 5 and the solution to the heat equation as seen in figure 6. This makes sense from what we showed in section 5.

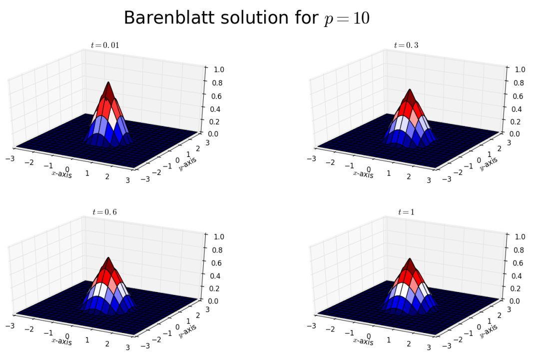

Figure 1: The Barenblatt solution with $p = 10$, shown at $t = 0.01$ in the top left, $t = 0.3$ in the top right, $t = 0.6$ in the bottom left and $t = 1$ in the bottom right

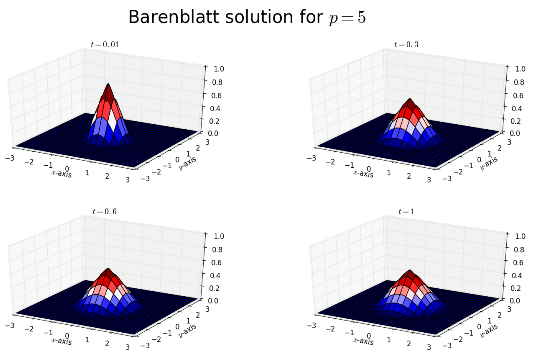

Figure 2: The Barenblatt solution with $p = 5$, shown at $t = 0.01$ in the top left, $t = 0.3$ in the top right, $t = 0.6$ in the bottom left and $t = 1$ in the bottom right

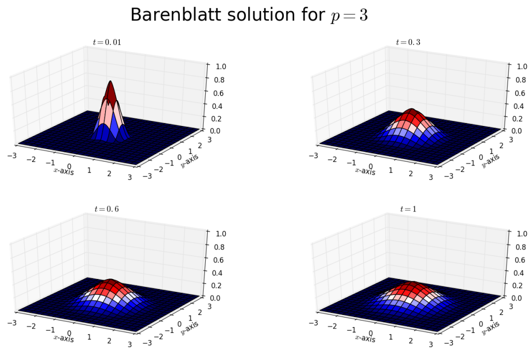

Figure 3: The Barenblatt solution with $p = 3$, shown at $t = 0.01$ in the top left, $t = 0.3$ in the top right, $t = 0.6$ in the bottom left and $t = 1$ in the bottom right

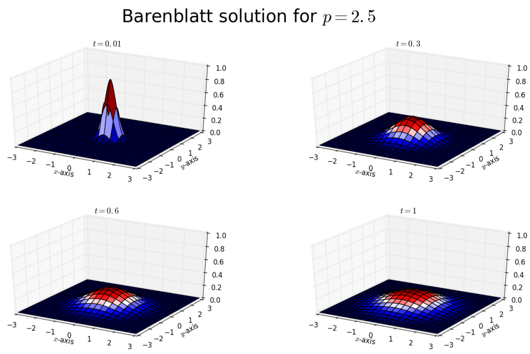

Figure 4: The Barenblatt solution with $p = 2.5$, shown at $t = 0.01$ in the top left, $t = 0.3$ in the top right, $t = 0.6$ in the bottom left and $t = 1$ in the bottom right

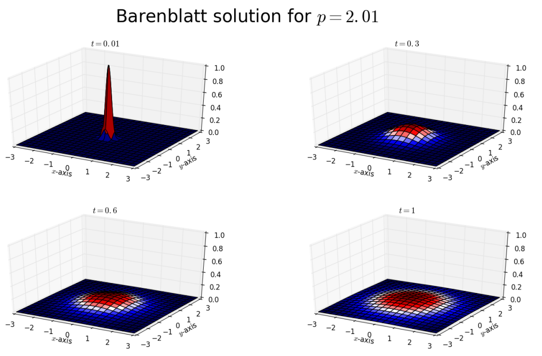

Figure 5: The Barenblatt solution with $p = 2.01$, shown at $t = 0.01$ in the top left, $t = 0.3$ in the top right, $t = 0.6$ in the bottom left and $t = 1$ in the bottom right

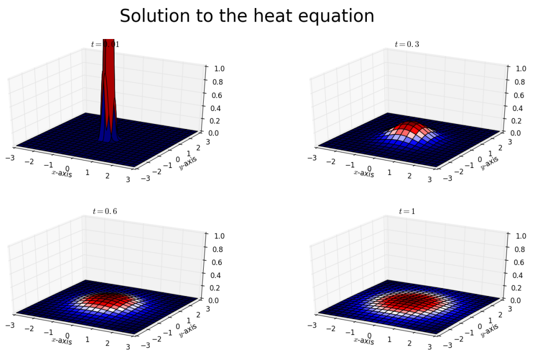

Figure 6: The solution to the heat equation, shown at $t = 0.01$ in the top left, $t = 0.3$ in the top right, $t = 0.6$ in the bottom left and $t = 1$ in the bottom right

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