Studforsk: Curvature of zero curves of harmonic functions

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Abstract

If a real 2-dimensional harmonic function has a zero set that goes through the origin and splits the unit circle into into two complimentary caps with opposite sign, one can prove that the curvature of the zero set is bounded by $k \leq 8$ near the origin. Building on this idea, one can prove a domain where the sign of the harmonic functions won't change (up to symmetries): $\{(1+r^2)cos(\theta)/2r > \sqrt{2}(cos(\frac{\theta}{2}) + |sin(\frac{\theta}{2})|)\}$.

1. Introduction

1.1. Background

In a paper by Û. Kuran [1] the curvature of the zero set of harmonic functions is discovered to have an upper bound in some situations. Let $h(x)$ be a 2-dimensional harmonic function where $x = (x_1, x_2) = (rcos(\theta), rsin(\theta))$. Three requirements are put on the zero set: $h(0) = 0$, $\frac{\partial h}{\partial x_2}(0,0) = 0$, and the zero set goes through the unit circle only two times. This means the unit circle is divided into two complementary caps with opposite signs. In this case the tangent in the origin is vertical. We can represent the zero set locally near the origin by $x_1 = \frac{1}{2}kx_2^2 + \tau(x_2)$. τ is here rather uninteresting, the point is that the zero set can be a somewhat complex curve, as long as it goes through the origin and only two points on the unit circle. An example of this is shown in figure (1).

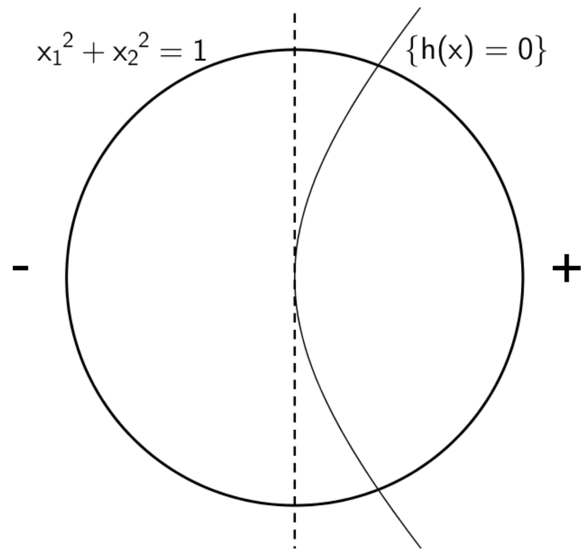

Figure 1: An example of how the zero set splits the unit circle into two complimentary caps: Positive on the left, negative on the right. In this example, the function is $h(x) = x_1^2 - x_2^2 + 2x_1$.

Here, Kuran proves that the the principal curvature of of the zero set in the origin satisfies $k \leq 8$. This means that the zero set cannot 'bend' as much as it wants, and therefore it cannot go wherever it wants near the origin. From this, an interesting question is raised: Where can we be sure that the zero set won't go? In what area near the origin can we be sure of the sign of the function?

1.2. The statement

This text will prove that for 2-dimensional harmonic functions that have zero sets that go through the origin and the unit circle twice, splitting it in two complimentary caps, the domain near the origin where the function does not change signs can be written in polar coordinates as the set that satisfies:

$$(1+r^2)cos(\theta)/2r > \sqrt{2}(cos(\frac{\theta}{2}) + |sin(\frac{\theta}{2})|) \tag{1}$$

2. The Proof

2.1. Setting up the proof

The method used is very similar to the one in Kuran's paper. Let $y = (y_1, y_2)$ be the coordinates of the unit circle, so $y_1 = cos(\phi)$ and $y_2 = sin(\phi)$, $0 < \phi < 2\pi$. x can be represented in polar coordinates by $x = re^{i\theta}$. The Poisson kernel [2] will be used to represent the harmonic function inside the unit circle:

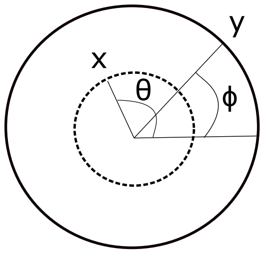

Figure 2: The coordinates used in the proof. The solid circle is the unit circle.

$$\begin{aligned} h(x) &= \int_0^{2\pi} P(x,\phi)h(\phi)d\phi, \ x = re^{i\theta}, \ r<1 \\ P(x,\phi) &= \frac{1-x_1^2-x_2^2}{(x_1-y_2)^2+(x_2-y_2)^2} = \frac{1-r^2}{1-2rcos(\theta-\phi)+r^2}, \ y = e^{i\phi} \end{aligned} \tag{2}$$

Note here that x, and therefore r and θ, are fixed. They are the coordinates of the point inside the unit circle we would like to evaluate. Also note that in (2), it's actually supposed to be $h(e^{i\phi})$ instead of $h(\phi)$, but for our purpose it is okay to 'cheat' as the outcome of the integral is the same no matter what notation we use. Using the three properties mentioned in the introduction, and assuming $h(x) = 0$, three things are known:

$$\int_0^{2\pi} h(\phi)d\phi = 0 \tag{3}$$

$$\int_0^{2\pi} 2sin(\phi)h(\phi)d\phi = 0 \tag{4}$$

$$\int_{0}^{2\pi} \frac{(1-r^2)h(\phi)}{1-2rcos(\theta-\phi)+r^2}d\phi = 0 \tag{5}$$

(3) comes from the fact that $h(0) = 0$. Putting $x = 0$ into the Poisson integral returns 0. (4) comes from the fact that the tangent of the zero set in the origin is assumed to be vertical, so the derivative in the x_2 direction is zero at the origin. Using Lebesgue's dominated convergence theorem:

$$\frac{dh(x_1,x_2)}{dx_2} = \int_{0}^{2\pi} \frac{\partial P(x,\phi)}{\partial x_2}h(\phi)d\phi \tag{6}$$

Using that $\frac{\partial P}{\partial x_2}(0,0) = 2y_2 = 2sin(\phi)$. (5) and $h(x) = 0$ are equiavalent. This means that what is to be proved only holds if $h(x) = 0$. A linear combination $L(y)$ of the three functions: $1, sin(\phi), \frac{1-r^2}{1-2rcos(\theta-\phi)+r^2}$ clearly satisfies:

$$\int_{0}^{2\pi} L(\phi)h(\phi)d\phi = 0 \tag{7}$$

The three functions can be multiplied by $1-2rcos(\theta-\phi)+r^2$ to make them easier to work with. Using $cos(\theta-\phi) = cos(\theta)cos(\phi) + sin(\theta)sin(\phi)$ one obtains:

$$1-2r(cos(\theta)y_1 + sin(\theta)y_2) + r^2, y_1(1-2r(cos(\theta)y_1 + sin(\theta)y_2) + r^2), 1-r^2$$

Note that $1-2rcos(\theta-\phi)+r^2 \geq 0$, so the 'new' integrand still changes sign only two times around the unit circle. Remember that θ and r are fixed. Seeing as the first and third function contains 1 and r^2, which are constants, the terms containing them can be removed from the first function, and the third function is constant, so it can be rewritten to 1. The 'new' functions are:

$$1,\ r(cos(\theta)y_1 + sin(\theta)y_2),\ y_2(1+r^2-2r(cos(\theta)y_1 + sin(\theta)y_2))$$

So the 'new' linear combination $\tilde{L}(y)$ is on the form:

$$\tilde{L}(y) = \alpha + \beta(rcos(\theta)y_1 + rsin(\theta)y_2) + \gamma y_2(1+r^2-2rcos(\theta)y_1 - 2rsin(\theta)y_2 \tag{8}$$

And:

$$\int_{0}^{2\pi} \tilde{L}(\phi)\tilde{h}(\phi)d\phi = 0,\ \tilde{L}(\phi) = L(\phi)(1-2rcos(\theta)+r^2),\ \tilde{h} = \frac{h}{(1-2rcos(\theta)+r^2)} \tag{9}$$

Note that the linear combination contains y_1y_2 and y_2^2. Following Kuran's idea, the goal is now to find a linear combination:

$$\tilde{L}(y) = (ay_1 + by_2 + c)(Ay_1 + By_2 + C) + (y_1^2 + y_2^2 - 1) \tag{10}$$

That is equal to the linear combination (8) on the unit circle. The function has a few properties that is important: Firstly, note that it contains quadratic terms, which are present in (8). $ax_1 + bx_2 + c = 0$ is the line that goes through the two points on the unit circle where $h(x) = 0$. Therefore $a^2 + b^2 = 1$ and $|c| < 1$. This is called the first line. Note that this means the function is zero at the same points as $h(x)$ on the unit circle. This also means that the integrand in (7) is either bigger than or equal to zero, or smaller than or equal to zero. This means, in return, that for (9) to hold, the integrand has to be zero. This sets up a contradiction. In other words, if the two linear combinations are equal to each other for some x, there cannot exist a harmonic function that isn't simply $h(x) = 0$ for that x. So, if we know for which x the two linear combinations (8) and (10) can be equal to each other, we know where the zero set cannot go, because no valid harmonic function can then satisfy the integral (7). The second line, described by $Ax_1 + Bx_2 + C = 0$ must therefore not cross the unit circle. Remember the point of this linear combination is that it has the property of being zero only at two points on the unit circle, while splitting the unit circle into two caps of opposite signs. We need the function $Ax_1 + Bx_2 + C$ to carry the same sign over the whole unit circle, so that the linear combination (10) in fact splits the unit circle into two caps, meaning that $Ax_1 + Bx_2 + C = 0$ must not cross the unit circle, and we can write an important condition we will use later:

$$\frac{|C|}{\sqrt{A^2+B^2}} \geq 1 \tag{11}$$

Everything is in place for the proof. The question is, for what x_1 and x_2 can the linear combinations be equal to each other?

2.2. Solving the equations

Setting (8) and 10) equal to each other gives:

$$\alpha+\beta(rcos(\theta)y_1+rsin(\theta)y_2)+\gamma y_2(1+r^2-2rcos(\theta)y_1-2rsin(\theta)y_2) = (ay_1+by_2+c)(Ay_1+By_2+C)+(y_1^2+y_2^2-1) \tag{12}$$

Six equations arise:

$$\begin{aligned} \alpha &= Cc-1 \\ \beta rcos(\theta) &= aC+Ac \\ \beta rsin(\theta)+\gamma(1+r^2) &= bC+Bc \\ -2rcos(\theta)\gamma &= aB+Ab \\ 0 &= aA+1 \\ -2rsin(\theta)\gamma &= Bb+1 \end{aligned}$$

The goal is to find A, B and C, so the inequality (11) can be used. Solving the equations give:

$$\begin{aligned} A &= -\frac{1}{a} \\ \frac{aB+Ab}{cos(\theta)} = \frac{Bb+1}{sin(\theta)} &\Rightarrow B = \frac{1}{a}\frac{acos(\theta)+bsin(\theta)}{asin(\theta)-bcos(\theta)} \\ \gamma = \frac{Bb+1}{-2rsin(\theta)} &= -\frac{1}{a}\frac{1}{2r(asin(\theta)-bcos(\theta)} \\ \frac{aC+Ac}{cos(\theta)} = \frac{bC+cB-\gamma(1+r^2)}{sin(\theta)} &\Rightarrow C = \left(\frac{1}{a}\frac{ac}{(asin(\theta)-bcos(\theta))^2}+\frac{(1+r^2)cos(\theta)}{2r(asin(\theta)-bcos(\theta))^2}\right) \end{aligned}$$

The equation of the second line can be written as:

$$Ax_1+Bx_2+C=0$$
$$-\frac{1}{a}x_1+\frac{1}{a}\frac{acos(\theta)+bsin(\theta)}{asin(\theta)-bcos(\theta)}x_2+\frac{1}{a}\left(\frac{ac}{(asin(\theta)-bcos(\theta))^2}+\frac{(1+r^2)cos(\theta)}{2r(asin(\theta)-bcos(\theta))^2}\right)=0$$

Multiplting by $a(asin(\theta)-bcos(\theta))$:

$$-(asin(\theta)-bcos(\theta))x_1+(acos(\theta)+bsin(\theta))x_2+\frac{ac+(1+r^2)cos(\theta)/2r}{asin(\theta)-bcos(\theta)}=0$$

We now have our 'new' A, B and C. Now we can use the important inequality (11), mentioned earlier, and the fact that $(asin(\theta)+bcos(\theta))^2+(acos(\theta)-bsin(\theta))^2=1$ to write:

$$\frac{|\frac{ac+(1+r^2)cos(\theta)/2r)}{asin(\theta)-bcos(\theta)}|}{\sqrt{(asin(\theta)-bcos(\theta))^2+(acos(\theta)+bsin(\theta))^2}} = |\frac{ac+(1+r^2)cos(\theta)/2r)}{asin(\theta)-bcos(\theta)}| \geq 1$$

Here, $-\pi/2 < \theta < \pi/2$ and $0 < r < 1$, assumin the domain of interest is in the x_1 direction. Because of this: $(1+r^2)cos(\theta)/2r$ is always positive, $|asin(\theta) - bcos(\theta)| \leq 1$, and $|ac| < 1$. The only way the left side is smaller than one, would be if ac is 'sufficiently' negative. The inequality can be rewritten to:

$$(1+r^2)cos(\theta)/2r - ac \geq |asin(\theta) - bcos(\theta)| \tag{13}$$

Now $a > 0$ and $c = 1$ (or $a < 0$ and $c = -1$) is the only interesting case. c can't actually be 1, but it can be arbitrarily close to 1. This means the inequality turns into a strict inequality. Note that the inequality always holds if $a \leq 0$ and $c \approx 1$. Remembering that $a^2 + b^2 = 1$, we introduce: $a = cos(\omega)$ and $b = sin(\omega)$, $-\pi/2 < \omega < \pi/2$. The inequality can then be written as:

$$(1+r^2)cos(\theta)/2r > cos(\omega) + |sin(\theta - \omega)|$$

The larger the right side of this inequality is, the smaller our resulting domain is. Therefore, we are interested in the maximum value of $cos(\omega) + |sin(\theta - \omega)|$, the 'worst case'. We must derivate and set the result equal to zero. There are two cases here. One where $\theta - \omega > 0$ and one where $\theta - \omega < 0$. Assuming the first case, one can remove the absolute value, derivate and set the derivative equal to zero:

$$\begin{aligned}
-sin(\omega) - cos(\theta - \omega) &= 0 \\
sin(\omega) &= -cos(\theta - \omega) \\
&= -sin(\frac{\pi}{2} - \theta + \omega) \\
&= sin(-\frac{\pi}{2} + \theta - \omega)
\end{aligned}$$

Therefore:

$$\begin{aligned}
\omega &= -\frac{\pi}{2} + \theta - \omega + 2\pi k \\
\omega &= \frac{\theta}{2} - \frac{\pi}{4} + \pi k
\end{aligned}$$

We are only interested in $-\pi/2 < \theta < \pi/2$, so the term with πk is irrelevant. Assuming the second case, $\theta - \omega < 0$, one arrives at $\omega = \frac{\theta}{2} + \frac{\pi}{4}$. In other words, the maximum value is described by two functions:

$$cos(\frac{\theta}{2} - \frac{\pi}{4}) + |sin(\frac{\theta}{2} - \frac{\pi}{4})| = \frac{1}{\sqrt{2}}(cos(\frac{\theta}{2}) - sin(\frac{\theta}{2}) + |sin(\frac{\theta}{2}) + cos(\frac{\theta}{2})|)$$

$$and$$

$$cos(\frac{\theta}{2} + \frac{\pi}{4}) + |sin(\frac{\theta}{2} - \frac{\pi}{4})| = \frac{1}{\sqrt{2}}(cos(\frac{\theta}{2}) + sin(\frac{\theta}{2}) + |sin(\frac{\theta}{2}) + cos(\frac{\theta}{2})|)$$

For the first one, the sine terms cancel out for $\theta > 0$, and for the second one, they cancel out for $\theta < 0$. If they don't cancel, they are positive and added together. Obviously, we are not interested in the case where they cancel out. In other words: the first function holds for $\theta < 0$ and the second one hold for $\theta > 0$. The easiest way of writing this is simply:

$$\sqrt{2}(cos(\frac{\theta}{2}) + |sin(\frac{\theta}{2})|)$$

The inequality (13) can in the end be written as:

$$(1+r^2)cos(\theta)/2r > \sqrt{2}(cos(\frac{\theta}{2}) + |sin(\frac{\theta}{2})|) \tag{14}$$

Which is the domain we set out to prove. This inequality describes the domain where the harmonic function must carry the same sign. Of course, this is in the worst case scenario, so the actual domain where a function has the same sign may be larger, but not smaller! The case studied uses the x_2-axis as the vertical tangent and a positive derivative in the x_1 direction. The domain is rotation symmetric, as long as it is perpendicular to the tangent line of the zero set at the origin.

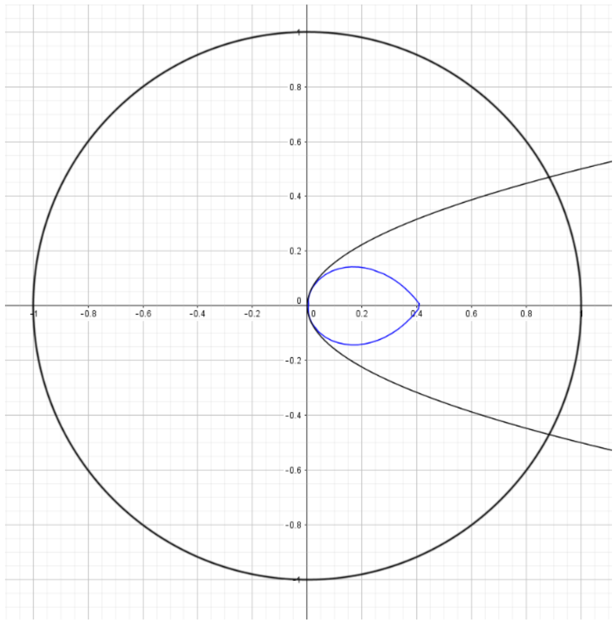

Figure 3: The blue 'teardrop' is the domain where the harmonic function surely has the same sign. The curve $x_1 = 4x_2^2$ is also drawn in. Notice the curve does not intersect with the domain.

2.3. What does this mean?

The resulting domain is shown in figure (3). If we consider the case where $x_2 = 0$, we can be sure of the sign of the function for $x_1 < \sqrt{2} - 1 \approx 0.414$. Harmonic functions satisfies the Laplace equation, which somewhat limits how 'crazy' they can be. However, even harmonic functions can look ugly if you 'zoom out', or if you make an ugly function on purpose. On the contrary, in the example studied, the 'scale' of the function (or rather, the zero set of the function) is limited to be of the same 'scale' as the unit circle. In other words, what we have attempted to prove is at what sort of scale harmonic functions change signs, and where the zero set is prohibited from going.

3. Further notes

3.1. Searchinig for an example

It would be referable to find a good example that displays the result. The first thing one might try is a linear combination of the harmonic polynomials, which are the real and imaginary part of $(x_1 + ix_2)^n$ for all positive integer n. One quickly finds that it's hard to find a domain that is as small as the one proved. One also finds that the domain gets smaller and smaller the closer one comes to a zero set that goes through the unit circle more than two times. This is logical considering the discussion above.

An interesting case is in a paper by S. Steinerberger [3]. Here he finds a function that 'maximizes' how close the zero set is to going through the unit circle more than two times. The function is:

$$w(x, y) = \frac{(x_1^2 + x_2^2 - 1)(x_1 - 2x_1^2 + x_1^3 - 4x_2^2 + x_1 x_2^2)}{(1 - 2x_1 + x_1^2 + x_2^2)^2} \tag{15}$$

However, one can easily set up a counter example. Let's look at the function:

$$h(x_1, x_2) = P(x, \phi) - 1 - 2x_2 y_2 = \frac{(1 - x_1^2 - x_2^2)}{(x_1 - y_1)^2 + (x_2 - y_2)^2} - 1 - 2x_2 y_2 \tag{16}$$

Notice that this is just the Poisson kernel with a few extra terms so that $h(0) = 0$ and the derivative in the x_2 direction is always zero. Remember that $\frac{\partial P}{\partial x_2}(0, 0) = 2y_2 = 2sin(\phi)$. One can vary y to get the wanted results. The problem with this function is that it is not continuous on the unit circle. $h(y_1, y_2)$ does not exist. The solution to this is to say that the 'unit circle' is just barely smaller than the actual unit circle, so that the discontinuous part is outside of the circle and the zero set is indeed equal to zero on two places on the unit circle. This limits y so that $|y_2| \leq 1/2$. If not, the zero set would be zero for more than two points. One can then take the intersection of two of these functions with $y_2 = 1/2$ and $y_2 = -1/2$ and one finds a domain that does two things: disproves the domain in Steinerberger's paper and proves that the domain found is this text is correct for this function. A plot of this is shown in figure (4). Although, with this example, the zero set intersects the x_1-axis at about $x_1 \approx 0.87$, compared to the domain proved in this text which only allows the zero set to intersect the x_1-axis for $x_1 \approx 0.414$. As discussed, this example clearly isn't the 'worst case'.

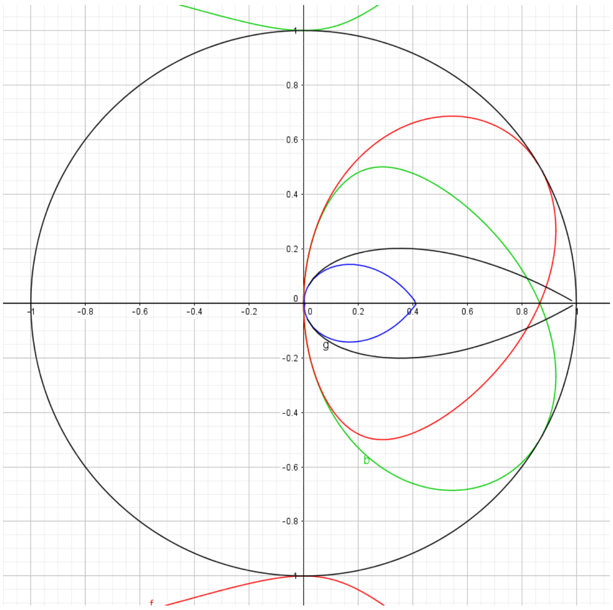

Figure 4: A plot of the zero set of (16) with $y_2 = -1/2$ (green), $y_2 = 1/2$ (red), (15) (black) and the domain proved in this text (blue)

3.2. Can we go further?

One starts to wonder if one can prove a domain bigger than the one proved in this text. Maybe the 'worst case' example doesn't exist? One also starts to wonder if it is possible to construct a better example with a smaller domain. When considering examples, (15) would be a good place to look. If one could tweak the function slightly so that the domain 'bends' away from the x_1-axis, one could construct a good example. Another place to look is (16). The extra terms added can possibly be tweaked. One can construct examples on the form:

$$h(x_1, x_2) = P(x, \phi) + f(x) \tag{17}$$

Where $f(0) = -1$, $\frac{\partial f}{\partial x_2}(0, 0) = -2y_2$ and of course, h is still harmonic. Then one can tweak y to find a good example.

Then there is the case of higher dimensions. Can a similar domain be proved for higher dimensions using the same procedure? Note that Kuran proves the curvature is also bounded for higher dimensions, though he uses some very specific conditions.

One thing we know for certain: We can say more about the zero set of harmonic functions near the origin than just the curvature being bounded.

4. Referencencess

[1] Û. Kuran, "On the zeros of harmonic functions", J. London Math Soc., 44(1969), 303-309
[2] S. Axler. (2000), Harmonic Function Theory, http://www.axler.net/HFT.html
[3] S. Steinerberger, "On local properties of harmonic functions", arXiv:1307.2069v3 [math.CA]