Means of some arithmetic functions

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Abstract

Let n be a positive integer. Denote by $\Omega(n)$ the number of prime divisors of n counting multiplicities. We propose to estimate the size of

$$\sum_{n=1}^{N} e^{r\Omega(n)} \tag{1}$$

for large N when r is a fixed real number and $r > \log 2$.

1. Introduction

Background. This problem is nicely solved for $0 < r < \log 2$. The solution can for example be found in a book written by Gerald Tenenbaum[4]. Looking at this result it can easily be observed that for any r , $0 < r < \log 2$ and large N there exists a constant $C = C(r)$ such that

$$\lim_{N\to\infty} \frac{\sum\limits_{n=1}^{N} e^{r\Omega(n)}}{CN(\log N)^{e^r-1}} = 1. \tag{2}$$

Theorem 1. For every real number $r > \log 2$ there exists a constant $C = C(r)$ such that for any positive integer N

$$e^{r\lfloor \log N/\log 2\rfloor} < \sum_{n=1}^{N} e^{r\Omega(n)} < Ce^{r\lfloor \log N/\log 2\rfloor}. \tag{3}$$

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[4]Gerald Tenenbaum, Introduction to analytic and probabilistic number theory

(English edition Cambridge University Press 1995), 202.

Theorem 2. If $r = \log 2$, then for any constant C there exist a positive integer N such that

$$\sum_{n=1}^{N} 2^{\Omega(n)} > C2^{\lfloor \log N/\log 2 \rfloor}. \tag{4}$$

Related problems. This problem is closely related to the corresponding problem for $\omega(n)$ where $\omega(n)$ is the number of prime divisors of n **not** counting multiplicities. This problem is also related to the following classical result:

Let $d(n)$ be the number of divisors of the positive integer n. Then for a fixed real number r

$$\lim_{N\to\infty} \frac{\sum\limits_{n=1}^{N} \big(d(n)\big)^r}{N(\log N)^{2^r-1}} = 1. \tag{5}$$

2. Proof

Theorem 1.

Main idea. For every positive integer N, let k_N be an integer such that $2^{k_N} \le N < 2^{k_N+1}$, and $F_N(m)$ be the number of integers $n \le N$ such that $\Omega(n) = k_N - m$. What is needed to be proven now is that the following sums have a constant upper bound for every real number r.

$$\sum_{n=1}^{N} \frac{e^{r\Omega(n)}}{e^{rk_N}} = \sum_{m=0}^{k_N} F_N(m)\frac{e^{r(k_N-m)}}{e^{rk_N}} = \sum_{m=0}^{k_N} \frac{F_N(m)}{e^{rm}}. \tag{6}$$

Let P_i be the $i-th$ prime number. For any positive integer m and real number ϵ, $0 < \epsilon < 1$, It is obvious that there is a positive integer $x(\epsilon) \le 2^{1/\epsilon}$ such that

$$\frac{P_x}{2} \le P_{x-1}^{1-\epsilon} \text{ and } \frac{P_{y+1}}{2} \ge P_y^{1-\epsilon}, \tag{7}$$

for any integer $y \ge x$. Now let $B(m,\epsilon)$ be the number of positive integers $\beta = \prod\limits_{i=1}^{\infty} P_i^{\beta_i}$ such that

$$\beta^{1-\epsilon} \le \left(2\prod_{i=1}^{x-1}(\frac{2P_i^{1-\epsilon}}{P_{i+1}})^{\beta_i}\right)2^m. \tag{8}$$

Put $C = 2\prod\limits_{i=1}^{x-1}(\frac{2P_i^{1-\epsilon}}{P_{i+1}})^{\beta_i}$.

Lemma 1. For any real number $l > 1$, there is a positive real number $\epsilon_{(l)} < 1$ and a constant $D = D_{(l)}$ such that for any integer $m \geq 0$

$$\frac{B(m,\epsilon)}{2^{lm}} \leq D. \tag{9}$$

Proof. We can rewrite Inequality 8 in the following way

$$\prod_{i=x}^{\infty} P_i^{\beta_i} \leq \left(2^m \times 2 \prod_{i=1}^{x-1} (\frac{2}{P_{i+1}})^{\beta_i}\right)^{\frac{1}{1-\epsilon}} = A. \tag{10}$$

For fixed $\beta_i, \beta_2, ..., \beta_{x-1}$, look at the left hand side of the inequality. There is a positive integer less than or equal to A, which is not divisible by any of the prime numbers $P_1, P_2, ..., P_{x-1}$. The number of such integers is a portion $q \leq 1$ of all integers less than or equal to A, and this is qA. Now sum A over all possible values of β for $\beta_i, \beta_2, ..., \beta_{x-1}$. This will be an upper bound for $B(m, \epsilon)$.

$$B(m,\epsilon) \leq q \sum_{\beta_1=0}^{\infty} \sum_{\beta_2=0}^{\infty} ... \sum_{\beta_{x-1}=0}^{\infty} \left(2^m \times 2 \prod_{i=1}^{x-1} (\frac{2}{P_{i+1}})^{\beta_i}\right)^{\frac{1}{1-\epsilon}}. \tag{11}$$

These are $x - 1$ geometric series which all are convergent because for any integer $i \geq 1 \Rightarrow \frac{2}{P_{i+1}} < 1$ and $\frac{1}{1-\epsilon} > 0$.

Notice: $q < 1$

$$\Rightarrow B(m,\epsilon) \leq 2^{\frac{m}{1-\epsilon}} \times 2^{\frac{1}{1-\epsilon}} \prod_{i=1}^{x-1} \frac{1}{1 - \left(\frac{2}{P_{i+1}}\right)^{\frac{1}{1-\epsilon}}} = D'_{(\epsilon)} 2^{\frac{m}{1-\epsilon}}, \tag{12}$$

where $D'_{(\epsilon)} = 2^{\frac{1}{1-\epsilon}} \prod_{i=1}^{x-1} \frac{1}{1-\left(\frac{2}{P_{i+1}}\right)^{\frac{1}{1-\epsilon}}}$ is a constant only dependent on ϵ.

Since $l > 1 \Rightarrow 0 < 1 - \frac{1}{l} < 1$. Now for any $l > 1$ put $\epsilon_{(l)} = 1 - \frac{1}{l}$. Then $\frac{2^{\frac{1}{1-\epsilon}}}{2^l} = 1$ and

$$\frac{B(m,\epsilon)}{2^{lm}} \leq \frac{D' 2^{\frac{m}{1-\epsilon}}}{2^{lm}} = D' \left(\frac{2^{\frac{1}{1-\epsilon}}}{2^l}\right)^m = D'_\epsilon = D'_{(1-\frac{1}{l})} = D_{(l)}, \tag{13}$$

and the conjecture in this lemma is proven. ■

Lemma 2. For any integer N and real number $l > 1$, there exists a constant $D = D_{(l)}$ such that for any integer $m \geq 0$

$$\frac{F_N(m)}{2^{lm}} < D. \tag{14}$$

Proof. The size of $F_N(m)$ should be estimated. The smallest integer n such that $\Omega(n) = k_N - m$ is $n_1 = 2^{k_N - m}$. Now in order to make some other integers n which satisfy the condition $\Omega(n) = k_N - m$, some $2 - factors$ of n_1 may be replaced by some other prime factors. This means that n_1 may be multiplied by $(\frac{P}{2})^\beta$ for some prime number $P > 2$ and some positive integer $\beta < k_N - m$. This means that new $n's$ can be made by multiplying n_1 by a number of the form $\prod_{i=1}^{\infty}(\frac{P_{i+1}}{2})^{\beta_i}$, where some β_i's are nonzero. Now $F_N(m)$ is the number of integers $n \leq N$ satisfying $\Omega(n) = k_N - m$, which means that n_1 can not be multiplied by a number bigger than 2×2^m, because if so, $n > n_1 \times 2^{m+1} = 2^{k_N+1} > N$. In addition, it is obvious that by multiplying n_1 by a number of the form $\prod_{i=1}^{\infty}(\frac{P_{i+1}}{2})^{\beta_i}$, all n's satisfying the requested condition are covered. Therefore looking away from all the conditions for integers β_i, an upper bound for $F_N(m)$ will be the number of numbers of the form $\prod_{i=1}^{\infty}(\frac{P_{i+1}}{2})^{\beta_i}$, which satisfy the condition

$$\prod_{i=1}^{\infty}(\frac{P_{i+1}}{2})^{\beta_i} < 2 \times 2^m. \tag{15}$$

Now compare $F_N(m)$ and $B(m,\epsilon)$ for any ϵ, $0 < \epsilon < 1$. Inequality 8 can be rewritten in the following way.

$$\frac{(\prod_{i=1}^{\infty} P_i^{\beta_i})^{1-\epsilon}}{\prod_{i=1}^{x-1}(\frac{2P_i^{1-\epsilon}}{P_{i+1}})^{\beta_i}} \leq 2 \times 2^m \Rightarrow \left(\prod_{i=1}^{x-1}\left(\frac{P_{i+1}}{2}\right)^{\beta_i}\right)\left(\prod_{i=x}^{\infty}\left(P_i^{1-\epsilon}\right)^{\beta_i}\right) \leq 2 \times 2^m. \tag{16}$$

$F_N(m) = \#\vec{b}$ where $\#\vec{b}$ is the number of vectors $\vec{b} = (\beta_1, \beta_2, ...)$ which satisfy Inequality 15. For all β_i's the corresponding base in Inequality 15 is less than or equal to the corresponding base in Inequality 16 by Assumption 7. This means that all the vectors $\vec{b}$ that satisfy Inequality 15, also satisfy Inequality 16. Therefore $F_N(m) = \#\vec{b} < B(m,\epsilon)$ for any ϵ, $0 < \epsilon < 1$. Now the conjecture in this lemma is proven by *Lemma* 1.

$$\frac{F_N(m)}{2^{lm}} < \frac{B(m,\epsilon)}{2^{lm}} \leq D_{(l)}. \blacksquare \tag{17}$$

Now put $l = \frac{\frac{r}{\log 2}+1}{2}$ in *Lemma* 2.

Notice: $r > \log 2 \Rightarrow l > 1$

$$\frac{F_N(m)}{2^{lm}} = \frac{F_N(m)}{\sqrt{e^{rm}}\sqrt{2^m}} < D_{(l)} = D_{\left(\frac{\frac{r}{\log 2}+1}{2}\right)} = E_{(r)}, \tag{18}$$

where $E = E_{(r)}$ is a constant only dependent on r

If $a_m = \frac{F_N(m)}{\sqrt{e^{rm}}\sqrt{2^m}}$, then $a_m < E$. In addition a_m is well-defined for all positive integers m and N. Then from Equation 6 , it can be observed that

$$\sum_{n=1}^{N} \frac{e^{r\Omega(n)}}{e^{rk_N}} = \sum_{m=0}^{k_N} \frac{F_N(m)}{e^{rm}} = \sum_{m=0}^{k_N} a_m (\frac{\sqrt{2}}{e^{r/2}})^m < \frac{E_{(r)}}{1 - \frac{\sqrt{2}}{e^{r/2}}} = C_{(r)} \tag{19}$$

since $r > \log 2 \Rightarrow \frac{\sqrt{2}}{e^{r/2}} < 1$.

Now one can see that $f(N) := \sum_{n=1}^{N} \frac{e^{r\Omega(n)}}{e^{rk_N}}$ has 1 as lower bound and a constant upper bound $C_{(r)}$ when r is fixed. In fact this function alternates between its bounds, and it is not convergent. ■

Theorem 2. By the same kind of argument as in the proof of *Lemma* 2, it can be observed that $F_N(m) > 2^{m-1}$ for any positive integers N and m. Therefore if $r = \log 2$,

$$\sum_{n=1}^{N} \frac{e^{r\Omega(n)}}{e^{rk_N}} = \sum_{m=0}^{k_N} \frac{F_N(m)}{2^m} > \frac{k_N}{2}. \tag{20}$$

Now for any constant $C = C_{(r)}$, put $N = 2^{2C}$. Then $k_N/2 = C$, and the theorem is proven. ■

References

[1] Gerald Tenenbaum,

Introduction to analytic and probabilistic number theory