Project. Peakon solutions to the Camassa–Holm equation I
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Background. The Camassa–Holm equation, given by [1]

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$$

serves as a model for shallow water. From a pure mathematical point of view this equation attracted considerable attention since a huge class of solutions enjoys wave breaking within finite time. By this we mean that there are a lot of solutions which describe the same phenomenon that you can observe when a wave breaks near a shore. A huge amount of energy accumulates in one point, while the wave tips over. This means roughly that $u_x(t, x)$, the spatial derivative of the wave profile $u(t, x)$, no longer exists for all (t, x). At the same time the function describing the energy of $u(t, x)$, turns into a delta function. Dependent on how the concentrated energy is manipulated

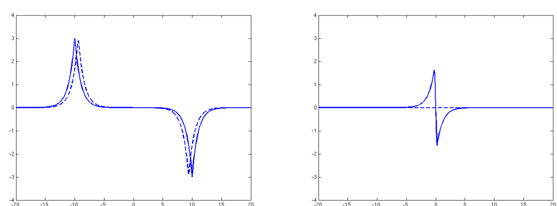

FIGURE 1. A solution which enjoys wave breaking.

one obtains different solution concepts. If you think, for example of a wave profile consisting of one peak traveling to the right and a second one to the left like in the pictures above, then the solution looks like the zero function when wave breaking occurs. By keeping the energy unchanged, one obtains that the two peaks pass through each other (= conservative solution). If we had chosen to continue the solution by being equal to zero, we would have dissipated all the accumulated energy (= dissipative solution).

Problem. Only very few solutions of the CH equation can be computed by hand, among them there is a certain class of solutions that serves as a source for inspiration, while being at the same time very illustrating [1]: the n-peakon solutions. These solutions can be written as

$$u(t, x) = \sum_{j=1}^{n} p_j(t) e^{-|x - q_j(t)|}, \tag{0.1}$$

[1]Here u_x, u_t .... denote the partial derivatives with respect to x, t, ...

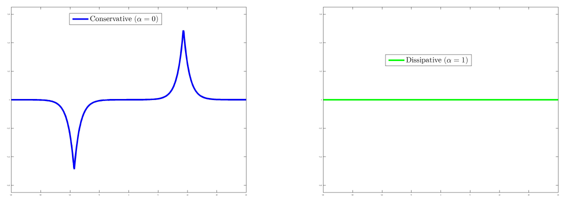

FIGURE 2. Two possible continuations of the above example beyond wave breaking.

where the $p_j(t)$ and $q_j(t)$ satisfy a system of differential equations. Within this framework we propose the following student project.

Let

$$u(0,x) = p_1(0)e^{-|x-q_1(0)|} + p_2(0)e^{-|x-q_2(0)|} + p_3(0)e^{-|x-q_3(0)|} + p_4(0)e^{-|x-q_4(0)|}$$

where $q_1(0) < q_2(0) < q_3(0) < q_4(0)$ and $p_1(0)$, $p_2(0)$, $p_3(0)$, $p_4(0) \in \mathbb{R}$.

1. Find all $u(0,x)$ such that

$$u(0,x) \begin{cases} = 0, & x \in [q_2(0), q_3(0)] \\ \neq 0, & x \in \mathbb{R} \backslash [q_2(0), q_3(0)] \end{cases}$$

.

2. Find the corresponding dissipative solution.

3. Compute the corresponding conservative solution (more advanced and therefore optional!).

Requirements. Matematikk 1, 2 and 4 or equivalent courses.

Workload. About 100 hours for 1–2 and about 150 hours for 1–3

REFERENCES

[1] K. Grunert and H. Holden. The general peakon-antipeakon solution for the Camassa–Holm equation. arXiv:1502.07686.

[2] H. Holden and X. Raynaud. A convergent numerical scheme for the Camassa–Holm equation based on multipeakons. *Discrete Cont. Dyn. Syst.* 14:505–523, 2006.

[3] H. Holden and X. Raynaud. Global conservative multipeakon solutions of the Camassa–Holm equation. *J. Hyperbolic Differ. Equ.* 4:39–64, 2007.

[4] H. Holden and X. Raynaud. Global dissipative multipeakon solutions of the Camassa–Holm equation. Comm. Partial Differential Equations 33:2040–2063, 2008.