Norm of the partial sum operator

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This project concerns finding the operator norm of the partial sum operator S_N over H^p. In particular we will look at the norm of S_1.

1 Introduction

We are looking at analytic functions on the unit disk in the space H^p for $0 < p \leq \infty$. Then every function can be associated with its boundary function on the unit circle, i.e. $H^p(\mathbb{T})$ which can be thought of as a subspace of $L^p(\mathbb{T})$. Given $f \in H^p$ it has a Taylor series which on $\mathbb{T}$ becomes a Fourier series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \rightsquigarrow \quad f(\mathrm{e}^{i\theta}) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\,\mathrm{e}^{in\theta}\,, \quad \hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{ix})\,\mathrm{e}^{-inx}\,\mathrm{d}x$$

Where $\hat{f}(n) = a_n$ for $n = 0, 1, 2, \ldots$ and zero otherwise. Now consider the partial sum operator S_N and in particular S_1:

$$S_N(f)(z) = \sum_{k=0}^{N} a_k z^k\,, \qquad S_1(f)(z) = a_0 + a_1 z$$

Our goal is then to determine the operator norm of S_1

1.1 Idea

The operator norm is given by:

$$\|S_1\|_p = \sup\left\{\frac{\|S_1(f)\|_p}{\|f\|_p} : f \in H^p, f \neq \mathbf{0}\right\}$$

Our idea is then to turn this maximization problem into a minimization problem. First we fix $S_1(f)$ and without loss of generality we assume it has the form $S_1(f)(z) = 1 + \lambda z$ where $\lambda \in \mathbb{R}^+$. Then our goal is to find a function $f \in H^p$ such that $f(z) = 1 + \lambda z + \mathcal{O}(z^2)$ and with minimal norm.

$$\|S_1\|_p = \sup_{0<\lambda<\infty} \frac{\|1 + \lambda z\|_p}{\inf \|1 + \lambda z + \mathcal{O}(z^2)\|_p}$$

2 Calculations

2.1 Norm of the projection

We will first calculate the norm of $1+\lambda z$ for $0<p<\infty$.

$$\begin{aligned}
\left\|1+\lambda z\right\|_p^p &= \frac{1}{2\pi}\int_0^{2\pi}\left|1+\lambda \mathrm{e}^{i\theta}\right|^p \,\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\left|1+\lambda \mathrm{e}^{i\theta}\right|^2\right)^{\frac{p}{2}} \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\left(1+\lambda \mathrm{e}^{i\theta}\right)\left(1+\lambda \mathrm{e}^{-i\theta}\right)\right]^{\frac{p}{2}} \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(1+2\lambda\cos\theta+\lambda^2\right)^{\frac{p}{2}} \,\mathrm{d}\theta \\
&= \frac{\left(1+\lambda^2\right)^{\frac{p}{2}}}{2\pi}\int_0^{2\pi}\left(1+\frac{2\lambda}{1+\lambda^2}\cos\theta\right)^{\frac{p}{2}} \mathrm{d}\theta \\
&= \frac{\left(1+\lambda^2\right)^{\frac{p}{2}}}{2\pi}\int_0^{2\pi}\sum_{n=0}^{\infty}\binom{p/2}{n}\left(\frac{2\lambda}{1+\lambda^2}\right)^n \cos^n\theta\,\mathrm{d}\theta \\
&= \left(1+\lambda^2\right)^{\frac{p}{2}}\sum_{n=0}^{\infty}\binom{p/2}{n}\left(\frac{\lambda}{1+\lambda^2}\right)^n \frac{1}{2\pi}\int_0^{2\pi} 2^n\cos^n\theta\,\mathrm{d}\theta \\
&\overset{*}{=} \left(1+\lambda^2\right)^{\frac{p}{2}}\sum_{k=0}^{\infty}\binom{p/2}{2k}\binom{2k}{k}\left(\frac{\lambda}{1+\lambda^2}\right)^{2k}
\end{aligned}$$

$$(*)\quad \frac{1}{2\pi}\int_0^{2\pi} 2^n\cos^n\theta\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\left(\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta}\right)^n \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{n}\binom{n}{k}\left(\mathrm{e}^{i\theta}\right)^{n-k}\left(\mathrm{e}^{-i\theta}\right)^k \mathrm{d}\theta$$

$$= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{i(n-2k)\theta}\,\mathrm{d}\theta = \binom{n}{n/2}$$

When n is even, and zero otherwise.

Observe that we may assume $\lambda\in(0,1]$ by the fact that $\|1+\lambda z\| = \lambda\left\|1+\frac{1}{\lambda}z\right\|$ Then we may proceed in a different manner:

$$\left\|1+\lambda z\right\|_p^p = \frac{1}{2\pi}\int_0^{2\pi}\left|1+\lambda \mathrm{e}^{i\theta}\right|^p \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\left|\left(1+\lambda \mathrm{e}^{i\theta}\right)^{\frac{p}{2}}\right|^2 \mathrm{d}\theta =$$

$$\left\|\left(1+\lambda \mathrm{e}^{i\theta}\right)^{\frac{p}{2}}\right\|_{L^2}^2 = \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=0}^{\infty}\binom{p/2}{n}\lambda^n \mathrm{e}^{in\theta}\right|^2 \mathrm{d}\theta = \sum_{n=0}^{\infty}\binom{p/2}{n}^2\lambda^{2n}$$

Putting everything together we have

$$\|1+\lambda z\|_p = \begin{cases} \sqrt{1+\lambda^2}\left(\sum_{k=0}^{\infty}\binom{p/2}{2k}\binom{2k}{k}\left(\frac{\lambda}{1+\lambda^2}\right)^{2k}\right)^{\frac{1}{p}} & \lambda\in(0,\infty) \\ \left(\sum_{n=0}^{\infty}\binom{p/2}{n}^2\lambda^{2n}\right)^{\frac{1}{p}} & \lambda\in(0,1] \end{cases} \tag{1}$$

Also note that for $p=2n$ both of these series turn in to finite sums valid for $\lambda\in(0,\infty)$

$$\|1+\lambda z\|_{2n} = \begin{cases} \sqrt{1+\lambda^2}\left(\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}\left(\frac{\lambda}{1+\lambda^2}\right)^{2k}\right)^{\frac{1}{2n}} \\ \left(\sum_{k=0}^{n}\binom{n}{k}^2\lambda^{2k}\right)^{\frac{1}{2n}} \end{cases}$$

2.2 Finding the projected function

We now want to find the function $f(z)=1+\lambda z+\mathcal{O}(z^2)$ with minimal norm. Let us call it g. First we have to establish that it exists. Let $A=\{f\in H^p(\mathbb{T}):\hat{f}(0)=1,\hat{f}(1)=\lambda\}$. This is a closed, convex subset of L^p. When $1<p<\infty$ we have by Shapiro [1] that $\exists\, g\in A$ such that $\|g\|<\|f\|\ \forall f\in A\setminus\{g\}$ and g is unique. For this reason assume $1<p<\infty$.

Our next goal is to find g and more importantly its norm. By rephrasing the properties of g we will be able to understand it better and find both the function and its norm.

Let $z\in\mathbb{T}$ and $Y=\{h\in H^p(\mathbb{T}):\hat{h}(0)=\hat{h}(1)=0\}$ be a closed subspace of H^p. By definition we say that $f\perp Y$ if and only if $\|f\|\le\|f+h\|$ for all $h\in Y$. Then by Shapiro [1] this is equivalent to saying that for all $h\in Y$ the following holds

$$\int |f|^{p-2}\bar{f}\cdot h = 0 \tag{2}$$

It is clear that $g\perp Y$ by definition of g. The fact that $Y\ni z^2,z^3,\ldots$ implies that the Fourier series of $|g|^{p-2}g$ is given by "almost only negative terms", i.e.

$$|g|^{p-2}g = \sum_{n=-\infty}^{1} a_n \mathrm{e}^{in\theta} = a_1\mathrm{e}^{i\theta} + \sum_{n=0}^{\infty} a_{-n}\mathrm{e}^{-in\theta} \tag{3}$$

Furthermore the general structure of g is given by

$$g(z) = 1+\lambda z+\sum_{n=2}^{\infty} b_n z^n = \sum_{n=0}^{\infty} b_n \mathrm{e}^{in\theta} \tag{4}$$

Then from (3) and (4) we get

$$|g|^p = |g|^{p-2} g\bar{g} = c_1 \mathrm{e}^{i\theta} + \sum_{n=0}^{\infty} c_{-n} \mathrm{e}^{-in\theta} \tag{5}$$

Since $|g|^p$ is a real function we may use the fact that for real functions $\hat{f}(-n) = \overline{\hat{f}(n)}$. Then (5) reduces to

$$|g(z)|^p = c_1 z + c_0 + \bar{c}_1 \bar{z} = c_0 + 2\,\mathrm{Re}(c_1 z) \tag{6}$$

From this we may conclude that $c_0 \in \mathbb{R}$ and $|c_1| \leq \frac{c_0}{2}$. Also since $|g|^p$ is a subharmonic function we have

$$1 = |g(0)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} \left| g(\mathrm{e}^{i\theta}) \right|^p \mathrm{d}\theta = c_0 \tag{7}$$

i.e. $1 \leq \|g\|_p = c_0{}^{1/p}$

From Duren [2] we have that we can factor g into an inner- and outer function such that $g(z) = I(z)G(z)$. Where $|I(z)| = 1$ on $\mathbb{T}$, $|I(z)| \leq 1$ and $|G| > 0$ in $\mathbb{D}$. And hence $|g| = |G|$ on $\mathbb{T}$. This implies that we can write $|g|^p = G^{p/2}\overline{G}^{p/2}$, and $G^{p/2}$ will be analytic in $\mathbb{D}$. From this we can easily see that $G^{p/2} = a + bz$ with $|a| \geq |b|$ is a possible solution, and since G is unique up to a unimodular constant we have that this is the only solution. Then $G(z) = (a + bz)^{2/p}$, and $\|g\|_p^p = |a|^2 + |b|^2$ from

$$|g|^p = G^{\frac{p}{2}} \overline{G}^{\frac{p}{2}} = (a + bz)\overline{(a + bz)} = |a|^2 + |b|^2 + 2\,\mathrm{Re}(\bar{a}bz)$$

To find the inner function observe that on $\mathbb{T}$ we have

$$\begin{aligned} a_1 + \sum_{n=0}^{\infty} a_{-n} \bar{z}^{n+1} &= \bar{z}|g|^{p-2} g = \bar{z}|g|^p \frac{g}{g\bar{g}} = \bar{z}\frac{|g|^p}{\bar{g}} = \bar{z}\frac{|g|^p}{\overline{G}} I \\ &= \bar{z}\frac{(a + bz)\overline{(a + bz)}}{\overline{(a + bz)}^{\frac{2}{p}}} I(z) = (a\bar{z} + b)(\bar{a} + \bar{b}\bar{z})^{1-\frac{2}{p}} I(z) \end{aligned}$$

Let $I(z) = \sum \alpha_n z^n$, and assume now $z \in \mathbb{D}$, then

$$(\bar{a} + \bar{b}\bar{z})^{\frac{2}{p}-1} \sum_{n=0}^{\infty} a_{-n+1} \bar{z}^n = (a\bar{z} + b) \sum_{n=0}^{\infty} \alpha_n z^n = a\alpha_0 \bar{z} + a\alpha_1 + b\alpha_0 + \sum_{n=1}^{\infty} (a\alpha_{n+1} + b\alpha_n) z^n$$

Since the left hand side is anti-analytic we get that

$$\begin{gathered} a\alpha_{n+1} + b\alpha_n = 0 \quad \text{for} \quad n = 1, 2, \ldots \\ \Rightarrow \alpha_{n+1} = -\frac{b}{a}\alpha_n = \cdots = \left(-\frac{b}{a}\right)^n \alpha_1 \quad \text{for} \quad n = 1, 2, \ldots \\ \Rightarrow I(z) = \alpha_0 + \alpha_1 \sum_{n=1}^{\infty} \left(-\frac{b}{a}\right)^{n-1} z^n = \alpha_0 + \alpha_1 z \sum_{n=0}^{\infty} \left(-\frac{b}{a}\right)^n z^n = \alpha_0 + \frac{\alpha_1 z}{1 + \frac{b}{a} z} = \frac{k + cz}{a + bz} \end{gathered}$$

Where $k = a\alpha_0$ and $c = a\alpha_1 + b\alpha_0$.

This is a linear fractional transformation and hence a conformal self-map of the unit disk. Therefore by Gamelin [3] it is either constant and unimodular or has the form

$$I(z) = \mathrm{e}^{i\varphi}\frac{z-\kappa}{1-\bar{\kappa}z} \qquad |\kappa| < 1 \tag{8}$$

$$I(z) = \frac{c}{a}\frac{z+\frac{k}{c}}{1+\frac{b}{a}z} \tag{9}$$

From this we get that $\left|\frac{c}{a}\right| = 1$, $\left|\frac{b}{a}\right| < 1 \Leftrightarrow |b| < |a|$ and $\frac{k}{c} = \frac{\bar{b}}{\bar{a}}$. Lets first assume I to be constant, i.e. $I(z) = \mathrm{e}^{i\varphi}$, then we have

$$1 + \lambda z + \mathcal{O}(z^2) = g(z) = I(z)G(z) = \mathrm{e}^{i\varphi}(a+bz)^{\frac{2}{p}} = \mathrm{e}^{i\varphi}a^{\frac{2}{p}}\left[1 + \frac{2}{p}\frac{b}{a}z + \mathcal{O}(z^2)\right]$$

$$\Rightarrow \quad a = \mathrm{e}^{-i\frac{p}{2}\varphi}, \qquad b = \frac{ap\lambda}{2} = \mathrm{e}^{-i\frac{p}{2}\varphi}\frac{p\lambda}{2}$$

$$g(z) = \mathrm{e}^{i\varphi}\left(\mathrm{e}^{-i\frac{p}{2}\varphi} + \mathrm{e}^{-i\frac{p}{2}\varphi}\frac{p\lambda}{2}z\right)^{\frac{2}{p}} = \left(1 + \frac{p\lambda}{2}z\right)^{\frac{2}{p}} \tag{10}$$

This is analytic only if $\frac{p\lambda}{2} \leq 1$. For this reason it can't possibly be the complete solution.

Now assume $I \neq$ constant. Then we can do the same thing we just did and get $k = a^{1-2/p}$ and $c = k\big(\lambda + \frac{b}{a}\big(1 - \frac{2}{p}\big)\big)$. Let $\frac{c}{a} = \mathrm{e}^{i\varphi}$ then

$$a^{1-\frac{2}{p}} = k = \frac{\bar{b}c}{\bar{a}}$$

$$a^{-\frac{2}{p}} = \frac{\bar{b}}{\bar{a}}\frac{c}{a} = \frac{\bar{b}}{\bar{a}}\mathrm{e}^{i\varphi}$$

$$b = \mathrm{e}^{i\varphi}\frac{a}{\bar{a}^{2/p}} \quad \Rightarrow \quad |b| = |a|^{1-\frac{2}{p}}$$

This implies that $|a| > 1$ to ensure $|b| < |a|$. Moreover we have

$$b = \mathrm{e}^{i\varphi}\frac{a}{\bar{a}^{2/p}} = \frac{c}{a}\frac{a}{\bar{a}^{2/p}} = \frac{c}{\bar{a}^{2/p}} = \frac{a\lambda + b\big(1-\frac{2}{p}\big)}{a^{2/p}\bar{a}^{2/p}}$$

$$b = a\,\frac{\lambda}{|a|^{4/p} - 1 + \frac{2}{p}}$$

Since the fraction on the right is real and positive we get that a and b have the same argument. This implies that $\frac{b}{a}$ is real and positive.

From Duren [2] we may factor the inner function $I(z)$ one step further into a Singular inner function and a Blaschke product. $I(z) = S(z)B(z)$ where both $|S|$ and $|B|$ are equal to 1 on $\mathbb{T}$, less than 1 in $\mathbb{D}$ and $S(z)$ has no zeros. $B(z)$ being a Blaschke product has the form

$$B(z) = z^m \prod_n \frac{|\kappa_n|}{\kappa_n} \frac{\kappa_n - z}{1 - \bar{\kappa}_n z}$$

Where m might be zero and $\{\kappa_n\}$ is the set of roots of the inner function. In our case $I(z)$ only has one root namely $-\frac{b}{a}$, hence

$$B(z) = \frac{z + \frac{b}{a}}{1 + \frac{b}{a} z} \tag{11}$$

From this it's immediately clear that $S(z) = \mathrm{e}^{i\varphi} = \frac{c}{a} = a^{-2/p}\big(\lambda + \frac{b}{a}\big(1 - \frac{2}{p}\big)\big)$. Then using the fact that $G(z) = (a + bz)^{2/p} = a^{2/p}(1 + \frac{b}{a}z)^{2/p}$ we see that we may assume a and b to be real and $S(z) = 1$.

$$1 = S(z) = a^{-\frac{2}{p}} \left(\lambda + \frac{b}{a}\left(1 - \frac{2}{p}\right)\right) = \lambda a^{-\frac{2}{p}} \left(1 + \frac{1 - \frac{2}{p}}{a^{4/p} - 1 + \frac{2}{p}}\right) = \lambda \left(\frac{a^{2/p}}{a^{4/p} - 1 + \frac{2}{p}}\right)$$

$$\big(a^{\frac{2}{p}}\big)^2 - \lambda a^{\frac{2}{p}} + \left(\frac{2}{p} - 1\right) = 0$$

$$a^{\frac{2}{p}} = \frac{1}{2}\left(\lambda \pm \sqrt{\lambda^2 - 4\left(\frac{2}{p} - 1\right)}\right)$$

It is clear that we need to have a plus sign in front of the square root and $\frac{p\lambda}{2} > 1$ in order to have $|a| > 1$. This also implies that for $\frac{p\lambda}{2} \leq 1$, g is given by (10). Denote $\Lambda = \frac{1}{2}\big(\lambda + \sqrt{\lambda^2 - 8/p + 4}\big)$, then $a = \Lambda^{p/2}$ and $b = \Lambda^{p/2 - 1}$. We now have an expression for g and its norm

$$g(z) = \begin{cases} \left(1 + \frac{p\lambda}{2} z\right)^{\frac{2}{p}} & \frac{p\lambda}{2} \leq 1 \\ \frac{z + \Lambda^{-1}}{1 + \Lambda^{-1} z} \left(1 + \Lambda^{-1} z\right)^{\frac{2}{p}} & \frac{p\lambda}{2} > 1 \end{cases} \tag{12}$$

$$\|g\|_p = \begin{cases} \left(1 + \left(\frac{p\lambda}{2}\right)^2\right)^{\frac{1}{p}} & \frac{p\lambda}{2} \leq 1 \\ \Lambda \left(1 + \Lambda^{-2}\right)^{\frac{1}{p}} & \frac{p\lambda}{2} > 1 \end{cases} \tag{13}$$

2.3 Extending the solution to $p = 1$

We would like our solution g to extend to the case $p = 1$ as well. We will in this subsection prove that it does.

First we wish to simplify the expression for g by substituting $p = 1$ in our formula. Then (12) becomes

$$g(z) = \begin{cases} 1 + \lambda z + \dfrac{\lambda^2}{4} z^2 & \lambda \leq 2 \\ 1 + \lambda z + z^2 & \lambda > 2 \end{cases} \tag{14}$$

Note that for $\lambda = 2$ the expressions are the same, we may therefore use the one most convenient at any given time. Also $1 + \lambda z + \frac{\lambda^2}{4} z^2 = \left(1 + \frac{\lambda}{2} z\right)^2$.

Recall that $g \perp Y$ iff $\|g\| \leq \|g+h\| \ \ \forall\, h \in Y$. This implies that g is the desired solution if and only if $g \perp Y$. This is what we will show.

From Shapiro [1] we have that $g \perp Y$ in L^1 iff $\exists\, f$ with $\|f\|_\infty = 1$ such that

$$\int g \cdot f = \int |g| = \|g\|_1 \quad \text{and} \quad \int f \cdot h = 0 \quad \forall\, h \in Y$$

Moreover if the set $\{z : g(z) = 0\}$ has measure zero this is equivalent to

$$\int \frac{\bar{g}}{|g|} h = 0 \quad \forall\, h \in Y$$

First assume $\lambda \geq 2$. Then for $z \in \mathbb{T}$ we have $\left|1 + \lambda z + z^2\right| = |\lambda + 2 \operatorname{Re} z|$, which implies

$$\|g\|_1 = \frac{1}{2\pi} \int_0^{2\pi} \left|1 + \lambda z + z^2\right| \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} \lambda + 2\cos\theta \, \mathrm{d}\theta = \lambda \tag{15}$$

Now observe that $\|\bar{z}\|_\infty = 1$ and clearly $\int g(z) \cdot \bar{z} = \lambda$ as well as $\int h(z) \cdot \bar{z} = 0 \ \ \forall\, h \in Y$.

Assume $\lambda < 2$. Since the set $\{z : g(z) = 0\}$ have measure zero, we get

$$\int_0^{2\pi} \frac{\left(1 + \frac{\lambda}{2}\bar{z}\right)^2}{\left|\left(1 + \frac{\lambda}{2} z\right)^2\right|} h(z)\, \mathrm{d}\theta = \int_0^{2\pi} \frac{\left(1 + \frac{\lambda}{2}\bar{z}\right)^2}{\left(1 + \frac{\lambda}{2} z\right)\left(1 + \frac{\lambda}{2}\bar{z}\right)} h(z)\, \mathrm{d}\theta = \int_0^{2\pi} \frac{1 + \frac{\lambda}{2}\bar{z}}{1 + \frac{\lambda}{2} z} h(z)\, \mathrm{d}\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} \cdot \frac{\lambda^2 - 4}{2 + \lambda z} + \frac{\lambda}{2}\bar{z}\right) h(z)\, \mathrm{d}\theta = \frac{1}{2} \int_0^{2\pi} \frac{\lambda^2 - 4}{2 + \lambda z} h(z)\, \mathrm{d}\theta + \frac{\lambda}{2} \int_0^{2\pi} \bar{z}\, h(z)\, \mathrm{d}\theta = 0$$

We have therefore shown $g \perp Y$ which means we have found a solution for $p = 1$. Note that we have no guarantee that it is unique so this is an interesting question. However we are more interested in its norm which we also get.

$$\|g\|_1 = \begin{cases} 1 + \dfrac{\lambda^2}{4} & \lambda \leq 2 \\ \lambda & \lambda > 2 \end{cases} \tag{16}$$

3 Conclusion

We now have all we need in order to find the operator norm of S_1 for $1 \leq p < \infty$. Our only problem is that we don't have a simple equation for $\|1 + \lambda z\|$ but instead an infinite sum. It is therefore out of reach to find the sup over λ. However the formulas in this report can be used to find numerical values of the norm and also estimates with the help of said numerics. The final results is as follows:

$$\|S_1\|_p = \sup_{0<\lambda<\infty} \frac{\|1+\lambda z\|_p}{\inf \|1+\lambda z+\mathcal{O}(z^2)\|_p} = \sup_{0<\lambda<\infty} \frac{\|1+\lambda z\|_p}{\|g\|_p} \tag{17}$$

$$\|1+\lambda z\|_p = \begin{cases} \sqrt{1+\lambda^2}\left(\sum_{k=0}^{\infty}\binom{p/2}{2k}\binom{2k}{k}\left(\frac{\lambda}{1+\lambda^2}\right)^{2k}\right)^{\frac{1}{p}} & \lambda \in (0,\infty) \\ \left(\sum_{n=0}^{\infty}\binom{p/2}{n}^2 \lambda^{2n}\right)^{\frac{1}{p}} & \lambda \in (0,1] \end{cases} \tag{18}$$

$$\|g\|_p = \begin{cases} \left(1+\left(\frac{p\lambda}{2}\right)^2\right)^{\frac{1}{p}} & \frac{p\lambda}{2} \le 1 \\ \Lambda\left(1+\Lambda^{-2}\right)^{\frac{1}{p}} & \frac{p\lambda}{2} > 1 \end{cases} \tag{19}$$

Where we may use both expressions for $\|1+\lambda z\|$ by the fact that $\|1+\lambda z\| = \lambda\left\|1+\frac{1}{\lambda}z\right\|$.

And g has the form:

$$g(z) = \begin{cases} \left(1+\frac{p\lambda}{2}z\right)^{\frac{2}{p}} & \frac{p\lambda}{2} \le 1 \\ \frac{z+\Lambda^{-1}}{1+\Lambda^{-1}z}\left(1+\Lambda^{-1}z\right)^{\frac{2}{p}} & \frac{p\lambda}{2} > 1 \end{cases} \tag{20}$$

References

[1] H. S. Shapiro, "Topics in Approximation Theory", Springer Lecture Notes 187, pp. 54–56, 1971.

[2] P. L. Duren, "Theory of H^p Spaces", Academic Press, pp. 18–24, 1970.

[3] T. W. Gamelin, "Complex Analysis", Springer, p. 263, 2001.