StudForsk-Project:
Algebraic Morse Theory

Background: Morse theory is a classical and important tool in differential topology for analyzing how spaces can be built out of smaller cells. Knowing the cell structure of a space makes the computation of its algebraic invariants extremely effective.

There is an analog of Morse theory in algebraic geometry, i.e. for the study of the zeros in $\mathbb{C}^n$ of polynomials in n variables. Assume that the nonzero complex numbers $\mathbb{C}^*$ act on the set $Z(f)$ of zeros of a polynomial f by multiplication in each coordinate. If the set of fixed points under this action is "nice", then $Z(f)$ has an algebraic cell structure and we can read off the dimension of the cells. Knowing the cell structure of $Z(f)$ makes it again much easier to compute its algebraic invariants, e.g. its Betti numbers.

The problem: For many important spaces in topology, e.g. certain Thom spaces of vector bundles, we know their algebraic invariants very well. But what we often do not know is whether they have an algebraic model. By an algebraic model of a space X we mean a set of zeros $Z(f)$ in $\mathbb{C}^n$ of a polynomial f which is homotopy equivalent to X. Knowing whether such algebraic models exist would be extremely interesting.

The task: A necessary condition for X and $Z(f)$ being equivalent is that their Betti numbers are equal. Remembering that finding Betti numbers is easy once we know the cell structure, the goal of this project is

- to understand how the cell structure of $Z(f)$ depends on f by looking at many examples; this is a purely combinatorial puzzle and does not require much prior knowledge, but teaches one a lot;
- to use these new insights to find candidates for algebraic models for interesting spaces, e.g. Thom spaces (not really necessary to know what these guys are).

There are many interesting known test cases out there to practise, e.g. Grassmannian varieties (learning about the cell structure of Grassmannians is anyway something on the to-do-list for life). And there are guesses for interesting $Z(f)$'s. So we do not have to fish in complete darkness. But in many important cases it is completely unknown if those guesses can work.

Prerequisites: Enthusiasm and passion for combinatorial puzzles. Having taken courses such as MA3201 - Rings and Modules or MA8202 - Commutative Algebra, MA3403 - Algebraic Topology, TMA4190 - Introduction to Topology, is desirable.

Time frame: Ca. 100 hours

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